Optimal Control of a Fishery under Critical Depensation

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Abstract: This study focuses the optimal control of a fishery under critical depensation. The effort has been considered as a dynamic variable and taxation as a control instrument. Criteria for the existence of different equilibrium points and their local stability are derived. It is proved that the interior equilibrium point is globally asymptotically stable provided it is locally stable. Finally, the optimal taxation policy is discussed with the help of control theory.

Key words: Critical depensation, limit cycles, global stability, optimal taxation

Introduction

Governments and other authorities manage fisheries because the biological, social and economic consequences of an unregulated fishery are undesirable. These consequences could include fishery collapse, economic inefficiency, loss of employment, habitat loss or decreases in the abundance of rare species. Management is usually imposed by an external regulator rather than the fishers themselves. Their management objectives may be intended to ensure the economic and social well being of future generations or to protect habitats and species of conservation concern. Effective fisheries management requires clear objectives supported by the best scientific advice and appropriate management actions.

The traditional, continuous time model of pure open access, based on logistic net growth, does not allow for extinction (Clark, 1990). A model where net growth exhibits critical depensation is formulated. Clark (1990) categorized the dynamics of continuous deterministic models as depensation and compensation models. The growth function is called a depensation curve if it is convex for small levels of the stock. If not it is called a compensation curve. Further, if the growth rate is negative for small stock levels, we say that the resource possess critical depensation. For such process the stock will go extinct if it is reduced below a certain level. Here extreme care must be taken by the practitioner.

In a critical depensation model under open access or sole ownership exploitation it is possible that the fish population level will be driven to extinction. All that is required is that the optimal population level is below the minimum viable population size. If the depensation exists, fishery managers should be extremely nervous because fished stocks may not recover after being fished to very low abundance, even when fishing is stopped.

Where renewable resource management is practiced it is generally based on the concept of Maximum Sustainable Yield (MSY), which itself is based on models of biological growth (Clark, 1990). This is not necessarily the best management method, because the long-run consumption profile does not coincide with that of utility maximization. The resource stock under the MSY is not necessarily optimal with respect to production due to the positive relationship between productivity in harvesting activities and the resource stock size.

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Exploitation of marine fisheries naturally involves the problems of law enforcement. Various aspects of law enforcement in regulating fisheries have been discussed by Sutinen and Anderson (1985), Anderson and Lee (1986) and others. Taxation, license fees, lease of property rights, seasonal harvesting, etc., are usually considered as regulatory mechanisms to keep the damage to the ecosystem minimal. Out of these several regulating options, taxation is considered to be superior because of its economic flexibility (Clark, 1990). Harvesting problems with or without tax have also been studied by Chaudhuri and Johnson (1990), Chaudhuri (1988), Ganguly and Chaudhuri (1995), Krishna et al. (1998), Dubey et al. (2003), Kar and Chaudhuri (2003a, b), Kar et al. (2004), Pradhan and Chaudhuri (1999), Chattopadhyay et al. (1999). We assume that exploitation of the fish species is regulated by an authority by imposing a tax \( t \) per unit biomass of the fish harvested. The imposition of tax acts as a deterrent to the fisherman to control harvesting of fish and in turn, helps it to grow.

The introduction of an appropriate tax can reduce the social damage caused by an externality. In fishery economics, the most crucial externality is the dynamic externality (the externality induced by harvesters on a resource stock). If the tax is used to reduce the dynamic externality the harvesters will face a tax on harvest (e.g., landings or effort levels). It is not the scope of this study to describe the optimal policy for a specified fishery resource. We try to illuminate some important facts a real world stock managers must keep in mind.

**The Basic Model**

The resource grows due to reproduction, i.e., natural growth. It is common in the literature to assume that the natural growth function of the resource takes the form \( F(x) = rx(1-x/K) \), where \( K \), \( r \) and \( x \) denote the carrying capacity, the intrinsic growth rate and the resource's stock respectively. Our aim here is to generalize this function in order to capture the effect of critical depensation on the natural growth of the resource.

Taking into account the critical depensation, the net growth of the resource or population dynamics will be described by the following equation:

\[
x = F(x) = rx(1-x/K)(x/L-1)
\]

where \( L \) is the minimum viable population level.

The harvest function is assumed to be linear in the rate of its fishing effort \( E(t) \) and in the stock \( x(t) \), that is:

\[
h(t) = qE(t)x(t),
\]

where \( q \) is the catchability coefficient which is supposed to be constant.

Using this harvesting term, we rewrite the equation for \( x \) as

\[
\dot{x} = F(x) - rx(1-x/K)(x/L-1) - qE(t)x(t).
\]

The growth potential of the resource stock is limited: although the growing environment of the resource stock can be improved to some extent, thereby improving its natural growth rate in the ecosystem, the renewable resource stock has an upper limit. To control the over exploitation of this limited resource, governments or agencies may be called for to achieve an equilibrium closer to the efficient outcome. The most direct instruments available are implementation of property rights or tax schemes.

The controlling agency levies a tax on the harvesting, which is the most direct policy instrument. The implemented tax, \( t \), is a quantity tax on harvest, such that the profit \((pq-x-c)E\) changes to \((p-t)q(x-c)E\), the purpose of the tax is to regulate the harvesting effort. Introduction of tax makes harvesting effort \( E \) a dynamic variable governed by the equation:
\[ E = \lambda [(p - \tau)qx - cE] \]

where \( p \) is the price of unit catch, \( c \) is the cost of the unit effort, \( \lambda \) is a conversion factor. Thus in our problem the state variables \( x \) and \( E \) are governed by the equations

\[ \dot{x} = rx(1 - x/K)(x/L - 1) - qE(t)x(t) \]
\[ \dot{E} = \lambda [(p - \tau)qx - cE] \]  

(1)

these are supplemented by the initial conditions

\[ x(0) = x_0 \text{ and } E(0) = E_0. \]

**Qualitative Analysis**

In this section, we discuss the existence and stability of non-negative equilibria of system (1). The points \((0, 0), (L, 0)\) and \((K, 0)\) are nonnegative equilibria for all permissible parameters. The system (1) has one possible interior equilibrium \((x^*, E^*)\), where

\[ x^* = \frac{c}{(p - \tau)q} \]
\[ E^* = \frac{r}{KLq} \frac{[c - (p - \tau)qL][(p - \tau)qK - c]}{(p - \tau)^2} \]

In the case of taxation, it is natural to assume that \( p > \tau \), this implies that \( x^* \) is positive. For \( E^* \) to be positive we must have

\[ \left\{ c - (p - \tau)qK \right\} \left\{ (p - \tau)qK - c \right\} > 0 \text{ and which implies} \]
\[ p - \frac{c}{qK} < \tau < p - \frac{c}{qL} \]  

(2)

since, \( K > L \).

Combining all these results we have the following theorem.

**Theorem 1**

The system with harvesting described by Eq. (1) has a unique interior equilibrium for any tax level \( \tau \) with

\[ p - \frac{c}{qL} < \tau < p - \frac{c}{qK} \]

We shall now discuss the stability of all the equilibria.

The Jacobian of the system (1) is

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\[ V(x, y) = \begin{bmatrix} \frac{L}{x} \left[ 1 - \frac{x}{k_1} \right] \left( 1 - \frac{x}{k_2} \right) - \frac{x}{K} \nabla \cdot \left( \lambda \left( p - \tau \right) qE \right) - \lambda \left( p - \tau \right) q \left( x - c \right) \end{bmatrix} \]

Now we shall study the nature of each of these equilibrium points. First we consider the trivial equilibrium point \((0, 0)\). The eigenvalues of \(V(0, 0)\) are not \(0\) and \(-c\). So it is unstable.

Consider the second equilibrium point \((L, 0)\). Clearly the eigenvalues of \(V(L, 0)\) are \(\pi(1 - L/K)\) and \(\lambda((p - \tau)qL - c)\). Since we assume that \(K < L\), therefore it is also unstable.

For the third equilibrium point \((K, 0)\), the eigenvalues of \(V(K, 0)\) are \(\pi(1 - L/K)\) and \(\lambda((p - \tau)qK - c)\). Therefore, \((K, 0)\) is stable if

\[ \tau > \frac{p - c}{qK} \]

Comparing Eq. (2) and (3) it is clear that the existence of an interior equilibrium \(\left( x^*, E^* \right)\), in the positive quadrant of \((x, E)\) plane eliminate the possibilities of \((K, 0)\) being stable.

Now we come to the interior equilibrium point \((x^*, E^*)\). The characteristic equation of \(V(x^*, E^*)\) is

\[ \lambda_1 + m_1 \lambda + m_2 = 0 \]

where

\[ m_1 = -[\pi \left( \frac{1}{L} - \frac{x^*}{K} \right) \left( 1 - \frac{x^*}{L} \right)] \]

\[ m_2 = \lambda((p - \tau)q)^2x^* \]

Sum of the roots of Eq. (4) is:

\[ \frac{\pi \left( \frac{1}{L} - \frac{x^*}{K} \right) \left( 1 - \frac{x^*}{L} \right)}{L \left( p - \tau \right)^2 q^2} = \frac{\pi \left( p - \tau \right) q \left( K + L \right) - 2c}{L \left( p - \tau \right)^2 q^2} \]

It is negative if

\[ \left( p - \tau \right) q \left( K + L \right) - 2c < 0 \]

i.e., if

\[ \frac{p - c}{q \left( K + L \right)} < \tau \]

Product of the roots is

\[ \alpha \left( p - \tau \right) q^2E^*x^* = \frac{\alpha \pi \left( p - \tau \right) q \left( K + L \right) - 2c}{L \left( p - \tau \right)^2 q^2} \left\{ \frac{\left( p - \tau \right) q \left( K - c \right)}{\left( p - \tau \right) q \left( K - c \right)} \right\} \geq 0 \quad \text{[Using (2)]} \]

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Therefore the interior equilibrium point is stable if condition (5) is satisfied. Thus the constraints on $\tau$ for the existence of a stable positive equilibrium are

$$p - \frac{c}{q \frac{K + L}{2}} < \tau < p - \frac{c}{qK}$$

(6)

Hence we may state the following theorem.

**Theorem 2**

The equilibrium points $(0, 0)$, $(L, 0)$ are unstable and $(K, 0)$ is stable with $\tau > p - c/qK$. The interior equilibrium points is locally stable if

$$p - c/(q(K + L)/2) < \tau < p - c/qK.$$  

From the point of view of ecological managers, it may be desirable to have a unique positive equilibrium which is globally asymptotic stable, in order to plan harvesting and keep sustainable development of ecosystem.

**Theorem 3**

The interior equilibrium point $(x^*, E^*)$ is globally asymptotically stable.

**Proof**

We shall prove this theorem based on the criterion provided by Cheng et al. (1981). For this global stability analysis we recall our model system (1) in the following form

$$\dot{x} = x\alpha(x) - E\beta(x)$$

$$E = E[\lambda(p - \tau)\beta(x) - \lambda c]$$

(7)

where,

$$\alpha(x) = \tau(1 - x/K)(x/L - 1), \beta(x) = qx$$

(8)

We have shown that for,

$$p - \frac{c}{q \frac{K + L}{2}} < \tau < p - \frac{c}{qK},$$

$(x^*, E^*)$ is locally asymptotically stable. Cheng et al. (1981) proved that the model systems of the form Eq. (7) will be globally stable around the locally asymptotically stable positive interior equilibrium point, if and only if the following condition holds,

$$\frac{d}{dx} \left[ \frac{Q(x) - Q(x^*)}{\beta(x) - \beta(x^*)} \right] \leq 0$$

(9)

$Q(x)$ involved in the above expression is defined by

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\[
Q(x) = \frac{d}{dx} \left( x \alpha(x) - \frac{\alpha(x)}{\beta(x)} \frac{d}{dx} \beta(x) \right) \tag{10}
\]

From Eq. (8), we get

\[
\beta(x) - \beta(x') = q(x-x') \tag{11}
\]

From Eq. (10), we get

\[
Q(x) = r \left[ \frac{x}{L} + \frac{x}{K} - 2 \frac{x}{KL} \right]
\]

Therefore,

\[
Q(x) - Q(x') = r(x - x') \left[ \frac{1}{L} + \frac{1}{K} \frac{2(x + x')}{KL} \right]
\]

Hence,

\[
\frac{d}{dx} \left[ \frac{Q(x) - Q(x')}{\beta(x) - \beta(x')} \right] = - \frac{2r}{qKL} < 0
\]

By the help of the results provided by Cheng et al. (1981), we can conclude that the positive interior equilibrium \((x', E')\) is also globally stable. Hence the proof.

**Optimal Tax Policy**

After discussing the nature of the dynamics of the system (1), now we are in a position to apply maximum principles (Pontryagin et al. 1962) to obtain the optimal tax policy. We need to find the path traced out by \((x(t), E(t))\) with this optimal tax policy so that if the population are kept along this path, we are assured of achieving the objective of the harvesting agency. As the system (1) has globally asymptotically stable interior equilibrium corresponding to each tax belonging to

\[
(p - \frac{c}{K+L}, p - \frac{c}{qK})
\]

if

\[
\tau' \in (p - \frac{c}{K+L}, p - \frac{c}{qK})
\]

represents optimal tax level then the optimal paths can be the optimal tax level \(\tau'\).

Now we consider the optimal harvesting problem with the view to maximize the objective functional

\[
J = \int_{0}^{T} e^{-\lambda t} (px - c) dt
\]

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subject to constraints Eq. (1) and (6). As per the maximum principle, the Hamiltonian is given by

\[
H = e^{-\lambda t} (pqx - c)E + \mu_1 \left[ \frac{rx}{L} \left( 1 - \frac{x}{K} \right) \left( \frac{x}{L} - 1 \right) - qxE \right] + \mu_2 \left[ \lambda \left( p - \tau \right) qxE - cE \right].
\]

Clearly this is a linear control problem on infinite horizon. Hence the solution will be a combination of bang-bang and singular controls. First we study the singular solution for the optimization problem. The associated adjoint differential equations for the above problem are given by

\[\frac{d\mu_1}{dt} = -\frac{\partial H}{\partial x} = -\left[ e^{-\lambda t} pqE + \mu_1 \left\{ \frac{rx}{L} \left( 1 - \frac{x}{K} \right) + \frac{rx}{L} \left( \frac{x}{L} - 1 \right) - qxE \right\} + \mu_2 \lambda (p - \tau) qxE \right] \tag{12}\]

\[\frac{d\mu_2}{dt} = -\frac{\partial H}{\partial E} = -\left[ e^{-\lambda t} (pqx - c) - qx \mu_1 + \mu_2 \left\{ \lambda \left( p - \tau \right) qx - c \right\} \right] \tag{13}\]

where \(\mu_1\) and \(\mu_2\) are costate functions.

Using the maximum principle

\[\frac{\partial H}{\partial \tau} = -\mu_2 \lambda qxE = 0\]

gives \(\mu_2 = 0\)

Now,

\[\mu_1 = 0 \Rightarrow \mu_1 = \frac{e^{-\lambda t} (pqx - c)}{qx} \tag{14}\]

Putting the values of \(\mu_1\) and \(\mu_2\) in Eq. (12), we get:

\[\frac{e^{-\lambda t} (pqx - c)}{qx} = \left[ e^{-\lambda t} pqE + \frac{e^{-\lambda t} (pqx - c)}{qx} \left( \frac{rx}{L} \left( 1 - \frac{x}{K} \right) \left( \frac{x}{L} - 1 \right) + \lambda \frac{rx}{L} \left( \frac{x}{K} \left( \frac{x}{L} - 1 \right) - qxE \right) \right) \right] \tag{15}\]

Substituting the interior steady state of system (1) in Eq. (15) we obtain the following third order polynomial in \(\tau\) given by

\[\tau^3 + P \tau^2 + Q \tau + R = 0 \tag{16}\]

where

\[P = \left[ prLq^2 - 2 pqLq^2 + rq(K + L) \right] \delta \]
\[
Q = \frac{p \cdot KLq' \delta - 2p' rKLq' + 2c}{r}
\]

\[
R = \frac{p \cdot KLq' - p' r(K + L) + pr^2}{\delta}
\]

Let \( \tau^* \) be a solution (if it exists) of Eq. (16). Using this value of \( \tau^* \) in the expression of \( x' \) and \( E' \), we get the optimal equilibrium.

Now our job is to reach this optimal solution optimally from the initial state \( (x_0, E_0) \). This is achieved by applying bang-bang control to the system as follows. Let us define

\[
\tau = \begin{cases} 
\tau_{max} \text{ when } \mu_x(t) > 0, \\
\tau_{min} \text{ when } \mu_x(t) < 0,
\end{cases}
\]

Let \( T \) be the time at which the path \((x(t), E(t))\) generated by the bang-bang control \( \tau = \bar{\tau} \) reaches the state \((x(\bar{\tau}), E(\bar{\tau}))\). Then the optimal control policy will be

\[
\tau(t) = \begin{cases} 
\bar{\tau} \text{ for } 0 < t < T, \\
\tau^* \text{ for } t > T.
\end{cases}
\]

**Simulation**

For simulation let us take \( \tau = 0.12 \), \( L = 50 \), \( K = 200 \), \( q = 0.01 \), \( p = 15 \), \( c = 9 \), \( \delta = 0.05 \) in appropriate units.

For the above values of parameters we obtain from (6), \( 7.8 < \tau < 10.5 \). It is also found that Eq. (16) has only one real root \( \tau = 9.0 \) which is acceptable because it falls in the range \( 7.8 < \tau < 10.5 \). For this optimal tax \( \tau^* = 9.0 \), the optimal equilibrium solution \((x(\tau^*), E(\tau^*))\) is \((150, 0)\), which is found to be stable because the condition (6) holds in this case.

**Discussion**

In this study, a nonlinear mathematical model to study the dynamics of a fishery under critical depreciation has been proposed and analyzed. The harvesting effort has been considered as a dynamic variable and taxation as a control instrument to protect the fish population from over exploitation. The harvesting term has been assumed to follow the catch-per-unit effort hypothesis. For the existence and stability of the interior equilibrium points, a finite range of tax has been obtained and it is found that any regulatory agency must follow these parametric conditions at the time of tax formulation per unit biomass of the landed fish. The application of the control theory enabled us to find the optimal taxation policy.

**References**