New Formulations for Vendor Managed Inventory Problem

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ABSTRACT

This study, presented two vendor managed inventory models for: (1) one buyer and one supplier problem and (2) two buyers and one supplier problem. The primary assumption of this study is that the shortage is allowed in the studied problems. The proposed methods provide a simple condition, which makes it easy to decide when and how the costs of a vendor managed inventory model are less than those of traditional one. The study is supported by some numerical examples to illustrate the implementation of the proposed methods.

Key words: Vendor managed inventory, supply chain management, inventory shortage, one buyer one supplier problem, two buyers one supplier problem

INTRODUCTION

During the past decade, there have been some evidences indicating that Vendor Managed Inventory (VMI) could improve the performance of supply chain by decreasing inventory levels and increasing fill rates (Emigh, 1999). Retailers and manufacturers have recognized that their profitability and revenue growth is directly linked to supply chain efficiency due to a highly competitive and volatile market (Asgekar and Suleski, 2003). As a result, many consumers, manufacturers and retailers have looked for collaborative partnerships as an avenue for supply chain optimization. VMI is believed to be one of the most successful of these collaborative partnerships. Using the idea of VMI, retailers can shift the responsibility for planning and replenishment operations to the manufacturers.

VMI offers great benefits for manufacturers as well, since they can respond more quickly to unexpected changes in consumer demand, increase customer service levels and decrease stock-outs. VMI is a process in which the supplier generates orders for the customer based on demand information sent by the customer. VMI can provide numerous business benefits such as increased sales, reduction in lead-time, reduction in inventory and retail in-stock improvement (Waller et al., 1999).

Fry et al. (2001) consider a (z, Z)-type VMI contract in a one-supplier one-retailer supply chain. The retailer sets a minimum inventory level z and a maximum inventory level Z and the supplier pays a penalty to the retailer for every unit of retailer inventory that is outside this band. Both parties know the retailer’s demand distribution. The supplier produces all demands with no capacity limit. He also has the option of outsourcing in order to maintain the desired retailer’s inventory levels. These authors mention that in all the VMI agreements they observed in practice, the penalties are not incurred immediately (on a daily basis for example), they are practically based on long-term (approximately yearly) performance, often as part of balanced score card evaluation.
Cachon (2001) studies how to achieve channel coordination in a one-supplier multi-retailer competitive supply chain using VMI. Both the supplier and the retailers incur inventory and backorder costs. Cachon shows that VMI is not guaranteed to coordinate the chain partners unless all members are willing to accept or pay fixed transfer payments. His numerical study shows that VMI provides no improvement in supply chain costs when fixed transfer payments are forbidden. Kraiselburd et al. (2004) examine one retailer and one supplier problem under a newsvendor setting. Bernstein and Federgruen (2003) analyze the constant-demand-rate case and consider a model of VMI where the replenishment decision-making is transferred to the supplier, but the retailer is able to make his/her own pricing decisions. Some other studies on VMI consider logistics issues; Fry et al. (2001) provided a review of the literature regarding this subject. Darwish and Odah (2010) present a model for VMI problem with single vendor and multiple retailers. Their proposed model can easily describe supply chains with capacity constraints by selecting high penalty cost.

This study focuses on another aspect of VMI which is different from aspects considered in previous studies. In this study, we first investigate a special case of VMI in which there are one supplier and one buyer and shortage is allowed. Yao et al. (2007) study a simpler problem where there are one buyer and one supplier without considering inventory shortage. We then consider the case of two buyers one supplier in which shortage is again allowed. For each problem, we first present the notations and assumptions and the problem formulation is then described along with a numerical example. Finally, conclusion remarks are given to summarize the contribution of the study.

**General assumptions:** The following assumptions hold throughout the study.

- The demand is constant
- Each order is delivered at once
- Transportation times are negligible
- Shortage is allowed
- The setup, holding and shortage costs in VMI model are paid by the supplier

**ONE BUYER ONE SUPPLIER PROBLEM**

We use the following notations for the problem.

- $K_0, K_1$ : Total inventory costs for the supply chain including the costs of the supplier and the buyer in traditional and VMI models, respectively
- $K_0^*, K_1^*$ : The optimal values of $K_0, K_1$, respectively
- $K_{S0}, K_{S1}$ : Total inventory costs of the supplier in traditional and VMI models, respectively
- $K_{B0}, K_{B1}$ : Total inventory costs of the buyer in traditional and VMI models, respectively
- $Q$ : The size of each order
- $Q_0^*, Q_1^*$ : The optimal value of $Q$ in traditional and VMI models, respectively
- $B$ : The amount of shortage in each period
- $B_0^*, B_1^*$ : The optimal value of $B$ in traditional and VMI models, respectively
- $b$ : The cost of shortage per unit
- $A_s$ : The cost of each order for the supplier
- $A$ : The cost of each order for the buyer
- $R$ : The demand of buyer in each period
- $H$ : The inventory holding cost per unit of product and per period
In a traditional inventory model when shortage is allowed, total inventory cost is calculated as follows:

\[ K_o = K_{S_o} + K_{B_o} \]  

(1)

where,

\[ K_{S_o} = \frac{A_o * R}{Q} \]

and

\[ K_{B_o} = \frac{A * R}{Q} + \frac{H_o}{2} \frac{(Q-B)^2}{Q} + \frac{bB^2}{2Q} \]

In a traditional system we also have:

\[ Q_o^* = \sqrt{\frac{2RA}{H}} \sqrt{\frac{H+b}{b}} \]  
\[ B_o^* = \frac{HQ}{H+b} = \sqrt{\frac{2RA}{b}} \sqrt{\frac{H}{H+b}} \]

(2)

Using Eq. 1 and 2, the optimal value of \( K_o \) is calculated as follows:

\[ K_{o}^* = \frac{R(2A + A_o)\sqrt{H+b}}{\sqrt{2RA(H+b)}} \]

(3)

Total inventory cost of VMI model with shortage is also calculated as follows:

where,

\[ K_i = \frac{A_o * R}{Q} + \frac{A * R}{Q} + \frac{H_o}{2} \frac{(Q-B)^2}{Q} + \frac{bB^2}{2Q} \]

(4)

In order to find the optimal value of \( K_i \), we need to take the derivative of \( K_i \) with respect to \( Q \) and \( B \) and set:

\[ \frac{\partial K_i}{\partial Q} = 0 \] and \[ \frac{\partial K_i}{\partial B} = 0 \]

this yields the following results:

\[ Q_i^* = \sqrt{\frac{2R(A_o + A)}{H}} \sqrt{\frac{H+b}{b}} \]

(5)

and
$B_i' = \frac{HQ}{H+b} = \sqrt{\frac{2R(A_a + A)}{b}} \sqrt{\frac{H}{H+b}}$

Using Eq. 4 and 5, the optimal value of $K_i$ is calculated as follows:

$$K_i' = \frac{2R(A_a + A)\sqrt{Hb}}{\sqrt{2R(A_a + A)(H+b)}} \quad (5)$$

The VMI model is preferred to the traditional one if:

$$K_i' \leq K_o' \quad (7)$$

or

$$\frac{2R(A_a + A)\sqrt{Hb}}{\sqrt{2R(A_a + A)(H+b)}} \leq \frac{R(2A + A_a)\sqrt{Hb}}{\sqrt{2R(2A + A_a)(H+b)}}$$

$$\frac{2(A_a + A)}{\sqrt{A}} \leq \frac{(2A + A_a)}{\sqrt{A}} \Rightarrow 4(A_a + A) \leq \frac{(2A + A_a)^2}{A}, \quad A_a, A > 0$$

which yields $X^2 \geq 0$, where $\frac{A_a}{A} = X \quad (8)$

Equation 8 shows that the cost of VMI is always less than that of traditional method. This result has already been found for single buyer and single supplier model when shortage is not allowed and this study proves this point for a more general case, i.e., the case in which the shortage is allowed. Now, we demonstrate the implementation of the proposed method using a numerical example.

**Numerical example:** Consider the following data:

$$A = 10, A_a = 100, R = 100, H = 5, b = 2$$

Using Eq. 3 and 6, the optimal values of $K_0$ and $K_i$ are $K_0^* = 320.76$ and $K_i^* = 177.295$. Therefore, the cost of the VMI model is less than that of the traditional one. Now suppose that the value of $A/A$ is changed to 20. Therefore, we have:

$$A = 10, A_a = 200, R = 100, H = 5, b = 2$$

Again, the optimal values of $K_0$ and $K_i$ are $K_0^* = 587.66$ and $K_i^* = 244.78$, therefore, the VMI model beats the traditional one.
TWO BUYERS ONE SUPPLIER PROBLEM

Here, we study three different models. We first need to introduce some additional notations used in all three models which are as follows:

\(K_0, K_1\) : Total inventory costs for the supply chain which includes the costs of the supplier and the buyers in traditional and VMI models, respectively

\(K_0^*, K_1^*\) : The optimal values of \(K_0, K_1\), respectively

\(Q_1, Q_2\) : The size of each order in traditional model for buyers 1 and 2, respectively

\(Q_1^*, Q_2^*\) : The optimal values of \(Q_1, Q_2\), respectively

\(B_1, B_2\) : The amount of shortage per period in traditional model for buyers 1 and 2, respectively

\(B\) : The amount of shortage per period for the supplier in VMI model

\(B_1^*, B_2^*\) : The optimal values of \(B_1, B_2\)

\(B^*\) : The optimal value of \(B\) (we assume that \(B_1 = B_2 = B\) for VMI model)

\(b\) : The cost of shortage per unit for the supplier in VMI model

\(b_1, b_2\) : The cost of shortage per unit in traditional model for buyers 1 and 2, respectively

\(A_0\) : The cost of each order for the supplier.

\(A_1, A_2\) : The cost of each order for the buyers 1 and 2, respectively.

\(R_1, R_2\) : The demand of buyers 1 and 2, respectively in each period

\(H_1, H_2\) : The inventory holding cost per unit of product and per period for buyers 1 and 2, respectively

\(T\) : Duration of each period in VMI model

\(T^*\) : The optimal value of \(T\)

We assume that \(B\) is total shortage in VMI model such that \(B_2 = \max(B_1, B_2)\).

**Model 1:** We now assume that the shortage is allowed only in traditional model and no shortage is allowed in VMI model. In a traditional inventory model, total inventory cost of buyers 1 and 2 is calculated as follows:

\[
K_0 = \frac{A_1 R_1}{Q_1} + \frac{A_2 R_2}{Q_2} + \frac{A_1 R_1}{Q_1} + \frac{A_2 R_2}{Q_2} + \frac{(Q_1 - B_1)^2}{2Q_1} + \frac{b_1 B_1^2}{2Q_1} + \frac{(Q_2 - B_2)^2}{2Q_2} + \frac{b_2 B_2^2}{2Q_2} \tag{9}
\]

The above equation is used for all three models discussed in this section. In a traditional system, we also have:

\[
Q_{1^*} = \sqrt{\frac{2R_1 A_1}{H_1} \frac{(H_1 + b_1)}{b_1}} \quad B_{1^*} = \sqrt{\frac{2R_2 A_2}{b_1} \frac{H_1}{(H_1 + b_1)}} \tag{10}
\]

Using Eq. 9 and 10, the optimal value of \(K_0\) is calculated as follows:

\[
K_0^* = \frac{R_1 (2A_1 + A_2) \sqrt{H_1 b_1}}{\sqrt{2R_1 A_1 (H_1 + b_1)}} + \frac{R_2 (2A_2 + A_1) \sqrt{H_2 b_2}}{\sqrt{2R_2 A_2 (H_2 + b_2)}} + \sum_{i=1}^{\infty} \frac{R_i (2A_i + A_{i+1}) \sqrt{H_i b_i}}{\sqrt{2R_i A_i (H_i + b_i)}} \tag{11}
\]
The total costs of inventory in VMI model without considering shortage is calculated as follows:

\[ K_i = \frac{(A_i + A_z + A_e)}{T} + \frac{1}{2} R_i T H_i + \frac{1}{2} R_e T H_e \]  

(12)

Equation 12 is a pseudoconvex function. In order to find the optimal value of \( K_i \), we need to take the derivative of \( K_i \) with respect to \( T \), i.e.,

\[ \frac{\partial K_i}{\partial T} = 0 \]

which yields the following equation:

\[ T^* = \frac{2(A_i + A_z + A_e)}{(R_i H_i + R_e H_e)} \]

(13)

Using Eq. 12 and 13, the optimal value of \( K_i \) is calculated as follows:

\[ K_i^* = \sqrt{2(A_i + A_z + A_e)(R_i H_i + R_e H_e)} \]

(14)

Obviously, the cost of VMI model is less than that of traditional one if:

\[ K_i^* < K_0^* \]

or

\[ \sqrt{2(A_i + A_z + A_e)(R_i H_i + R_e H_e)} < \frac{R_i (2A_i + A_e) H_i b_i + R_e (2A_e + A_z) H_e b_e}{\sqrt{2 R_i A_i (H_i + b_i)} + \sqrt{2 R_e A_e (H_e + b_e)}} \]

(15)

A simple modification of Eq. 15 yields the following quadratic function:

\[
    (R_i \sqrt{H_i b_i G} + R_e \sqrt{H_e b_e F})^2 A_e^2 + 4[R_i R_e (A_i + A_e) \sqrt{H_i b_i G} b_i F + R_i^2 A_i G - R_e^2 A_e G] A_e + \\
    2R_i A_i \sqrt{H_i b_i G} + 2R_e A_e \sqrt{H_e b_e F})^2 - 2(A_i + A_e) (R_i H_i + R_e H_e) FG > 0
\]

(16)

where:

\[ \begin{align*}
    F &= 2R_i A_i (H_i + b_i) \\
    G &= 2R_e A_e (H_e + b_e)
\end{align*} \]

Therefore, we have:

\[ \alpha A_i^2 + \beta A_i + \gamma > 0 \]
where:

\[
\alpha = (R_i \sqrt{H_i b_i G} + R_j \sqrt{H_j b_j F})^2
\]

\[
\beta = 4[R_i R_j (A_i + A_j) \sqrt{H_i b_i G b_j F} - R_i^2 H_i A_i G - R_j^2 H_j A_j F]
\]

\[
\gamma = (2R_i A_i \sqrt{H_i b_i G} + 2R_j A_j \sqrt{H_j b_j F})^2 - 2(A_i + A_j)(R_i H_i + R_j H_j)FG
\]

The resulting quadratic function has two real roots:

\[
x_i = \frac{-\beta + \sqrt{\Delta}}{2\alpha}
\]

and

\[
x_2 = \frac{-\beta - \sqrt{\Delta}}{2\alpha}
\]

such that \( \Delta = \beta^2 - 4\alpha\gamma \). Obviously, whenever the condition \( X_2 < A_i < X_1 \) is met, the cost of the traditional model is less than that of the VMI model \( (K_0 - K_1 > 0) \) and if \( A_i > X_1 \) or \( A_i < X_2 \) and \( A_i^* > 0 \) then \( K_0 - K_1 > 0 \). We now demonstrate the implementation of the proposed method using a numerical example.

**Numerical example**

**Consider the following data:**

\[
A_i = 10, \ A_j = 8, \ R_i = 100, \ R_j = 200, \ H_i = 5, \ H_j = 3, \ b_i = 6, \ b_j = 4
\]

\[\Rightarrow \ X_i = 14.21, \ X_j = -17.998\]

\( A_i = 10 \), Using Eq. 11 and 14, the optimal values of \( K_0 \) and \( K_1 \) are \( K_0^* = 231.14 \) and \( K_1^* = 248.19 \), respectively. Therefore, the traditional model works better than the VMI model. Now suppose that the value of \( A_i \) is changed to 20. In this case, we have:

\[
A_i = 10, \ A_j = 8, \ R_i = 100, \ R_j = 200, \ H_i = 5, \ H_j = 3, \ b_i = 6, \ b_j = 4
\]

\[\Rightarrow \ X_i = 14.21, \ X_j = -17.998\]

Using Eq. 11 and 14, the optimal values of \( K_0 \) and \( K_1 \) are \( K_0^* = 314.36 \) and \( K_1^* = 289.14 \), respectively. Therefore, the costs of VMI model are less than those of the traditional one. Table 1 summarizes the numerical results of example 1 for some different values of \( A_i \).

**Model 2:** In this model, the shortage is allowed in both traditional and VMI models. The assumptions of the traditional model for this case are the same as those for the first model. The total inventory cost in VMI model with shortage is calculated as follows:

444
Table 1: Numerical results for example 1

<table>
<thead>
<tr>
<th>( A_i )</th>
<th>( K_0 )</th>
<th>( K_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>180.53</td>
<td>224.944</td>
</tr>
<tr>
<td>9</td>
<td>222.817</td>
<td>243.721</td>
</tr>
<tr>
<td>12</td>
<td>247.783</td>
<td>256.906</td>
</tr>
<tr>
<td>14</td>
<td>264.429</td>
<td>265.33</td>
</tr>
<tr>
<td>14.2</td>
<td>266.691</td>
<td>266.158</td>
</tr>
<tr>
<td>14.3</td>
<td>266.923</td>
<td>266.571</td>
</tr>
<tr>
<td>14.5</td>
<td>268.587</td>
<td>267.395</td>
</tr>
<tr>
<td>15</td>
<td>272.748</td>
<td>269.444</td>
</tr>
</tbody>
</table>

\[
K_1 = \frac{A_2 + A_1 + A_i}{T} + \frac{(R_i T - B)^2}{2R_i T} H_i + b B^2 + \frac{(R_i T - B)^2}{2R_i T} H_j + b B^2
\]  

(17)

Problem Eq. 17 is a pseudoconvex function and its proof is shown in the appendix. Differentiating \( K_1 \) with respect to \( T \) and \( B \) and equating to zero, i.e.,

\[
\frac{\partial K_1}{\partial T} = 0 \quad \text{and} \quad \frac{\partial K_1}{\partial B} = 0
\]

yields the following results:

\[
B^* = \frac{R_i R_j (H_i + H_j)}{(R_i + R_j) b + R_i H_j + R_j H_i}
\]  

(18)

\[
T^* = \frac{2(A_2 + A_1 + A_i)\left[b(R_i + R_j) + R_i H_j + R_j H_i\right]}{(R_i H_j + R_i H_j)\left[b(R_i + R_j) + R_i H_j + R_j H_i\right] - R_i R_j (H_i + H_j)^2}
\]

Using Eq. 17 and 18, the optimal value of \( K_1 \) is calculated as follows:

\[
K_1^* = \sqrt{\frac{2(A_2 + A_1 + A_i)\left[R_i H_j + R_j H_i\right]\left[b(R_i + R_j) + R_i H_j + R_j H_i\right] - R_i R_j (H_i + H_j)^2}{b(R_i + R_j) + R_i H_i + R_j H_i}}
\]  

(19)

The costs of VMI model are less than those of the traditional model as long as the following condition holds.

\[
K_1^* \leq K_0^*
\]  

(20)
\[ \frac{R_i (2A_i + A_{ij}) \sqrt{H_{ij}}}{\sqrt{2R_i A_i (H_{ij} + b_j)}} + \frac{R_j (2A_j + A_{ij}) \sqrt{H_{ij}}}{\sqrt{2R_j A_j (H_{ij} + b_j)}} \]

A simple modification of Eq. 20 yields the following quadratic function:

\[
M \left[ R_i^2 b_i G_i + R_j^2 b_j G_j + 2R_i R_j \sqrt{P} \right] (A_{ij}^2) + 4 \left[ (R_i^2 b_i G_i A_i + R_j^2 b_j G_j A_j + R_i R_j \sqrt{P} (A_i + A_j)) M - 2FG_i G_j \right] A_i^2 + 4 \left[ (R_i^2 b_i G_i A_i^2 + R_j^2 b_j G_j A_j^2 + 2R_i R_j A_i A_j \sqrt{P}) M - 2FG_i G_j (A_i + A_j) \right] \geq 0
\]  

(21)

Where:

- \( M = b(R_i + R_j) + R_i H_i + R_j H_j \)
- \( G_1 = 2R_i A_i (H_i + b_j) \)
- \( G_2 = 2R_j A_j (H_j + b_j) \)
- \( P = (R_i + R_j) M - R_i R_j (H_i + H_j)^2 \)
- \( P = H_i b_i b_j G_i G_j \)

The resulting quadratic function Eq. 21 has two real roots:

\[
x_1 = \frac{-\beta + \sqrt{\Delta}}{2\alpha} \quad \text{and} \quad x_2 = \frac{-\beta - \sqrt{\Delta}}{2\alpha}
\]

such that \( \Delta = \beta^2 - 4\alpha\gamma \). Obviously, if \( X_1 < A_x < X_2 \) then the cost of the traditional model is less than that of the VMI model, i.e., \( K_0 - K_i < 0 \) and when \( (A_x > X_1) \) or \( A_x < X_2 \) and \( A_x > 0 \), the VMI model beats the traditional one \( (K_0 - K_i < 0) \).

**Numerical example**

Consider the following data:

- \( A_i = 10, A_j = 15, R_i = 100, R_j = 250, H_i = 5, H_j = 3, b_1 = 6, b_2 = 4, b = 10 \)

\[ X_1 = 8.3, X_2 = -24.99 \]

\( A_x = 5 \), using Eq. 11 and 14, the optimal values of \( K_0 \) and \( K_i \) are \( K_0^* = 224.6 \) and \( K_i^* = 236.6 \), respectively. Therefore, the traditional model works better than the VMI model. Now suppose that the value of \( A_x \) is changed to 10. Therefore, we have:

- \( A_i = 10, A_j = 15, A_x = 10, R_i = 100, R_j = 250, H_i = 5, H_j = 3, b_1 = 6, b_2 = 4, b = 10 \)

\[ X_1 = 8.3, X_2 = -24.99 \]

Using Eq. 11 and 14, the optimal values of \( K_0 \) and \( K_i \) are \( K_0^* = 261.97 \) and \( K_i^* = 255.999 \), respectively. Therefore, the cost of VMI model is less than that of the traditional one. Table 2 shows the numerical results of example 2 for some different values of \( A_x \).
Model 3: In this model, the shortage is allowed in both traditional and VMI models. In VMI model the supplier has two shortages called B₁ and B₂ for buyers 1 and 2, respectively. The total cost of inventory in VMI model with shortage is computed as follows:

\[ K_i = \frac{A_i + A_2}{T} + \frac{(R_iT - B_i)^2}{2R_iT} H_i + b_i B_i + \frac{(R_iT - B_i)^2}{2R_iT} H_2 + b_2 B_2 \]  

(22)

Problem Eq. 22 is a pseudoconvex function. Taking the derivative of \( K_i \) with respect to \( T \) and \( B_1 \) and \( B_2 \) and letting them be equal to zero, i.e.,

\[ \frac{\partial K_i}{\partial T} = 0, \quad \frac{\partial K_i}{\partial B_1} = 0 \quad \text{and} \quad \frac{\partial K_i}{\partial B_2} = 0 \]

yields the optimal value of \( K_i \) as follows:

\[ B_1^* = \frac{H_i R_i T}{H_i + b_i}, \]

\[ B_2^* = \frac{H_2 R_i T}{H_2 + b_i} \]

(23)

\[ T^* = \frac{2(A_i + A_2)(H_i + b_i)(H_2 + b_2)}{R_i H_1 b_1 (H_2 + b_2) + R_i H_2 b_2 (H_1 + b_1)} \]

Using Eq. 22 and 23, the optimal value of \( K_i \) is calculated as follows:

\[ K_i^* = \frac{\sqrt{2(A_i + A_2)(R_i H_1 b_1 (H_2 + b_2) + R_i H_2 b_2 (H_1 + b_1))}}{(H_i + b_i)(H_2 + b_2)} \]

(24)

The VMI model is preferred to the traditional one if:

\[ K_i^* \leq K_0^* \]
\[ \sqrt{2(A_1 + A_2 + A_3)R_1b_1(H_1 + b_1) + 2(A_1 + A_2 + A_3)R_2b_1(H_2 + b_1) \leq \sqrt{(H_1 + b_1)(H_2 + b_1)} \]
\[ \frac{R_1(2A_1 + A_2)\sqrt{H_1b_1}}{2R_1a_1(H_1 + b_1)} + \frac{R_2(2A_1 + A_2)\sqrt{H_2b_2}}{2R_2a_2(H_2 + b_2)} \]

A simple modification of Eq. 25 yields the following quadratic function:

\[ \frac{(R_1\sqrt{H_1b_1}G + R_2\sqrt{H_2b_2}F)^2A_1^2}{(4R_1a_1(A_1 + A_2)\sqrt{H_1b_1}a_1G + 4R_2a_2(\sqrt{H_2b_2}F - 8R_1A_1A_2(M + N))A_2^2} + (2R_1a_1\sqrt{H_1b_1}G + 2R_2a_2\sqrt{H_2b_2}F)^2 - 8R_1A_1A_2(A_1 + A_2)(M + N) > 0 \]  

Where:

\[ F = 2R_1a_1\sqrt{H_1b_1} \]
\[ G = 2R_2a_2\sqrt{H_2b_2} \]
\[ M = 2R_1^2a_1^2(H_1 + b_1) \]
\[ N = 2R_2^2a_2^2(H_2 + b_2) \]
\[ \alpha = (R_1\sqrt{H_1b_1}G + R_2\sqrt{H_2b_2}F)^2 \]
\[ \beta = 4R_1a_1(A_1 + A_2)\sqrt{H_1b_1}a_1G - R_1^2a_1^2G - R_2^2a_2^2F \]
\[ \gamma = (2R_1a_1\sqrt{H_1b_1}G + 2R_2a_2\sqrt{H_2b_2}F)^2 - 2(A_1 + A_2)(\sqrt{H_1}a_1 + \sqrt{H_2}a_2)FG \]

The resulting quadratic function Eq. 26 has two real roots which are:

\[ x_1 = \frac{-\beta + \sqrt{\Delta}}{2\alpha} \quad \text{and} \quad x_2 = \frac{-\beta - \sqrt{\Delta}}{2\alpha} \]

such that \( \Delta = \beta^2 - 4\alpha\gamma \). Clearly, if \( x_2 < A_2 < x_1 \) then \( K_{o*}K_{1} < 0 \) and if \( \left((A_2 > X_1 \text{ or } A_2 < X_2) \right) \) and \( A_2 > 0 \) then \( K_{o*}K_{1} < 0 \).

**Numerical example**

Consider the following data:

\[ A_1 = 10, \quad A_2 = 15, \quad R_1 = 1000, \quad R_2 = 300, \quad h_1 = 7, \quad h_2 = 4, \quad b_1 = 10, \quad b_2 = 6, \]

\[ X_1 = 58.3, \quad X_2 = -24.8 \]

\( A_2 = 4 \), using Eq. 11 and 14, the optimal values of \( K_{o*} \) and \( K_{1*} \) are \( K_{o*} = 510.93 \) and \( K_{1*} = 529.7 \), respectively. Therefore, the cost of traditional model is less than that of VMI model. Now suppose that the value of \( A_2 \) is changed to 15. Therefore we have:

\[ A_1 = 10, \quad A_2 = 15, \quad R_1 = 1000, \quad R_2 = 300, \quad h_1 = 7, \quad h_2 = 4, \quad b_1 = 10, \quad b_2 = 6 \]

\[ X_1 = 58.3, \quad X_2 = -24.8 \]
Using Eq. 11 and 14, the optimal values of $K_0$ and $K_1$ are $K_0^* = 722.65$ and $K_1^* = 622.1$, respectively. Therefore, the VMI model has lower cost than the traditional one. Table 3 shows the numerical results of example 3 for some different values of $A_x$.

**CONCLUSION**

We have presented a VMI model for one buyer and one supplier problem and three models for two buyers and one supplier problem. For the first model in which there are one buyer and one supplier, we assumed that the shortage is allowed. We have shown that even under the new assumption, the VMI model can outperform the traditional one. The other proposed models of this study are for the case of two buyers and one supplier. The proposed methods have been capable of determining when and how a VMI policy is adopted to reduce the inventory cost. The implementation of the methods has been demonstrated using some numerical examples. This study could be extended for problems with more than one supplier or two buyers.

**APPENDIX**

Here, we prove that the problem Eq. 4 is a pseudoconvex function. Let $K_i$ be a linear combination of five different components $f_i$ to $f_5$ where:

\[ f_i = \frac{A_i}{T}, \quad f_2 = \frac{(R, T - B)^2}{2R_i T}, \quad f_3 = \frac{b B^2}{2R_i T}, \quad f_4 = \frac{(R, T - B)^2}{2R_i T}, \quad f_5 = \frac{b B^2}{2R_i T} \]

By taking the derivatives of $f_i$ to $f_5$ with respect to $B$ and $T$, we can obtain a positive semi-definite Hessian matrix. As an example, for $f_2$ we have:

\[ \frac{\partial^2 f_2}{\partial B \partial T} = \begin{bmatrix} \frac{H_i}{R_i T} & -\frac{H_i B}{R_i T^2} \\ -\frac{H_i B}{R_i T^2} & \frac{H_i B^2}{R_i T^3} \end{bmatrix}, \quad \text{where} \quad \frac{H_i}{R_i T} > 0 \quad \text{and} \quad \frac{H_i B}{R_i T^2} = 0 \]

**REFERENCES**


