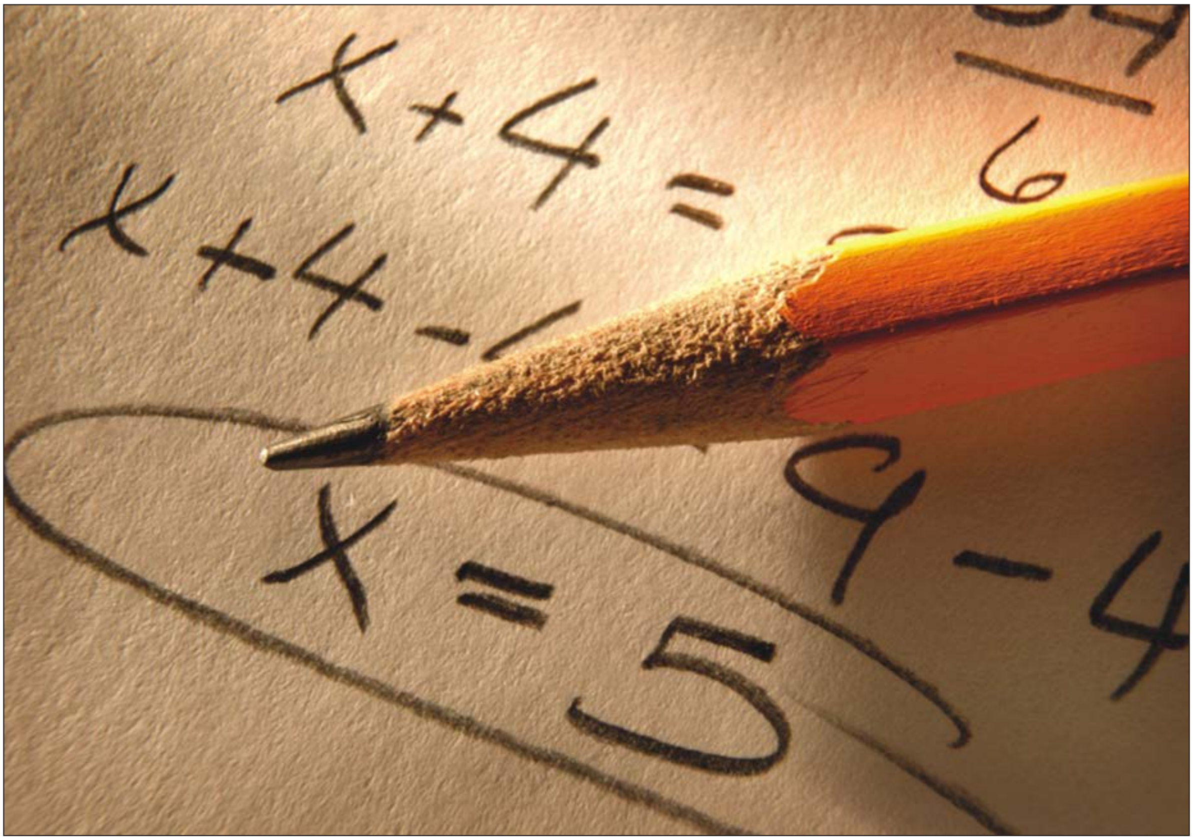

Asian Journal of Algebra

ISSN 1994-540X

Asian Journal of Algebra 1 (1): 1-9, 2008
ISSN 1994-540X

Retraction of Braid and Braid Group

[1]M. El-Ghoul and [2]M.M. Al-Shamiri
[1]Department of Mathematics, Faculty of Science,
Tanta University, Tanta, Egypt
[2]Department of Mathematics, Faculty of Science,
Menoufia University, Menoufia, Egypt

Abstract: In this study we introduce the retraction and conditional retraction of braids and braid groups, we show the retraction of braid group is not necessary a braid group also a retraction of a singular braid is not necessary a singular braid. We prove that a retraction of a braid is a braid and every retraction of a braid group is a monoid also we prove that a retraction is a braid invariant. The limit of all types of retraction is described.

Key words: Braid, braid group, retraction

INTRODUCTION

A braid theory introduced by E. Artin at 1925 and the concept of braid theory was found to have applications in other fields after the 1950s and this gave fresh impetus to the study of braids. More studies on braid theory are studied by many researches by this braid theory has gradually been prospected, refined and polished. In mathematics it is, now, recognized from one of the basic theories and is of benefit in such branches as topology and algebraic geometry. Also, it is of profound use in other areas of the sciences physics, statistical mechanics, chemistry and biology. The braid group was took an important role in this field. The iridescent hue of this concepts flowering into full bloom and activity occurred in 1984, when V. Jones put into action with inordinate success the original aim of Artin. i.e. the application of braids to knot theory. In (El-Ghoul *et al*., 2006, 2007) we introduced a new direction on knot theory called folding and retraction of knot. More studies of retraction in El-Ghoul (1985, 1995, 1998, 2002). In this study our intention introduce the concepts of retraction on the braid theory and braid group, continuations to the two articles above, we study the effect of retraction and conditional retraction on braid and braid group and singular braid.

DEFINITIONS

Here we will show some definitions and basic concepts which we will use it in the main results.

Definition 1

Let D be a unit cube, so D = {(x, y, z): $0 \le x, y, z \le 1$} on the top face of cube place n points, $a_1, a_2, ..., a_n$ and similarly, place n points on the bottom face $b_1, b_2, ..., b_n$ Now, join the points $a_1, a_2, ..., a_n$ with $b_1, b_2, ..., b_n$ by means of n arcs $d_1, d_2, ..., d_n$ (as smooth curves), this arcs are mutually disjoint and each d_1 connects some a_j to b_k ($J \ne K$ or $J = K$) not connect a_j to a_K or b_j to b_K. Each plane E_s, such that z = s, $0 \le s \le 1$ (parallel to xy-plane intersections each arc d_i at one and only one point.

A configuration of n arcs $d_1, d_2, ..., d_n$ with end points $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ is called n-braid or a braid with n strings Fig. 1 (Murasugi, 1996).

Corresponding Author: M.M. Al-Shamiri, Department of Mathematics, Faculty of Science, Menoufia University, Menoufia, Egypt

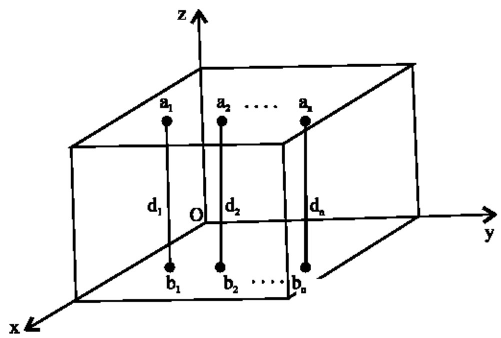

Fig. 1: Representation of braid in unit cube

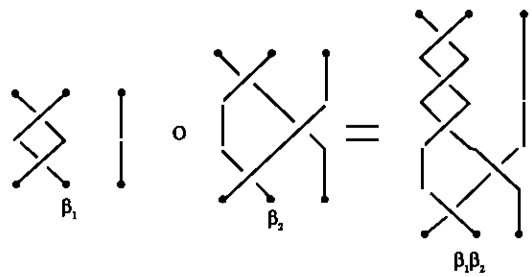

Fig. 2: Product of β_1 and β_2

Definition 2

Let B_n be a set of all n-braids and $\beta_1, \beta_2 \in B_n$, we may create a third n-braid from them which we shall call their product and denoted by $\beta_1\beta_2$ as follows:

Glue the bottom arcs of β_1 with the top arcs of β_2 (Murasugi and Kurpita, 1999) Fig. 2.

Remark 1

- For $\beta_1 \in B_n$ there is β_1^{-1} such that $\beta_1\beta_1^{-1} = e \in B_n$ Fig. 3.
- $\beta_1\beta_2 \neq \beta_2\beta_1$
- The product of braids is associative, i.e., $(\beta_1\beta_2)\beta_3 \approx \beta_1(\beta_2\beta_2)$

Theorem 1

The set of all n-braids, B_n forms a group. This group is usually called the n-braid group or Artin's n-braid group (Murasugi and Kurpita, 1999; Gemein, 2001).

Theorem 2

For any $n \geq 1$ the n-braid group B_n has the following presentation:

$$B_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} : \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{for } i = 1,2,\ldots,n-2$$
$$\text{and } \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for} |j - i| \geq 2 \rangle$$

Where, σ_i denotes the standard generator of the braid group (Kauffman, 1991; Murasugi and Kurpita, 1999; Gemein, 2001) Fig. 4.

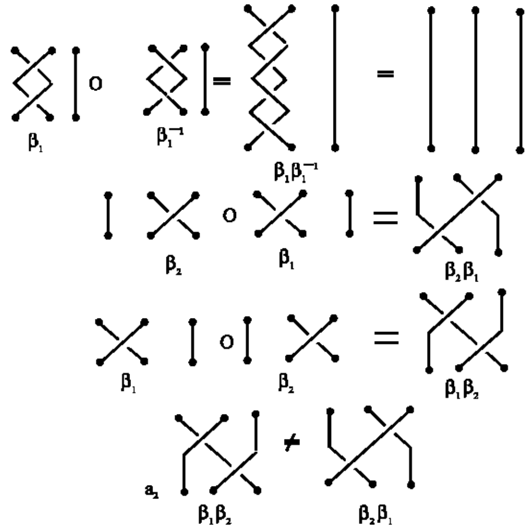

Fig. 3: Product of β_1 and β_1^{-1} and $\beta_1 \circ \beta_2 \neq \beta_2 \circ \beta_1$

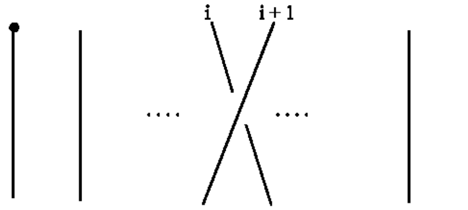

Fig. 4: Standard generator σ_i

Definition 3

An n-dimensional manifold is a Housdorff topological space M, such that every point of M has a neighborhood homeomorphic to open set $U \subset R^n$ (Munkers, 1975).

Definition 4

Let A be a subset of a topological space X. A continuous map $r: X \to A$ is said to be retraction if $r(a) = a$ for all $a \in A$ (Massay, 1967; Munkers, 1975).

Definition 5

A subset $A \subset X$ is deformations retract of X if there is a retraction $r: X \to A$ such that $i \circ r$ homotopic to the identity map. That is, there exists a continuous function $f: X \times [0, 1] \to X$ such that for $x \in X$, $f(x, 0) = x$ and $f(x, 1) = r(x)$ and for all $a \in A$ and all $t \in [0, 1]$, $f(a, t) = a$ (Massay, 1967).

Definition 6

Let β_1, β_2 be two n-braids in a cube D, we say β_1 ambient isotopic to β_2, denoted by $\beta_1 \approx \beta_2$, if there exists a homeomorphism $H: D\times[0,1]\to D\times[0,1]$, such that $H(x,t) = h_t((x),t)$, $t\in[0,1]$, $h_t: D\to D$ and $h_0(\beta_1) = \beta_1$, $h_t(\beta_2) = \beta_2$ (Murasugi and Kurpita, 1999).

Definition 7

Let β be a n-braid and suppose the ith string d_i of β joins a_i to $b_{j(i)}$ for i = 1, 2, 3, ….n. Define g: $B_n \rightarrow S_n$ (the set af all permutations of the set {1, 2, 3, 4, ….., n} as $g(\beta) = \begin{pmatrix} 1 & 2 & ..... & n \\ j(1) & j(2) & ..... & j(n) \end{pmatrix}$, the permutation for a given braid is called a braid permutation and denoted by π(β), if $g(\beta) = \begin{pmatrix} 1 & 2 & ..... & n \\ 1 & 2 & ..... & n \end{pmatrix}$ then β is called a pure n-braid (Murasugi and Kurpita, 1999).

Remark 2

If $\beta_1 \approx \beta_2 \Rightarrow \pi(\beta_1) = \pi(\beta_2)$.

Theorem 3

Let B_n be a n-braid group and S_n the symmetry group of n elements, then there exist a natural surjective homomorphism f from B_n onto S_n, takes any braid β to the permutation determined by β, f (β) = π (β).

The kernel of f is a pure n-braid group, denoted by P_n and B_n/P_n is isomorphic to S_n, the index of P_n in B_n is a finite, $[B_n:P_n] = n!$ (Murasugi and Kurpita, 1999).

THE MAIN RESULTS

Let $\beta = \sigma_i$ be a n-braid and r a retraction from β-{a} onto β', where a is a point on one arc of β, then we have two cases:

- β' is a trivial n-1-braid, if $a \in d_i$ or $a \in d_{i+1}$.
- $\beta' = \sigma_i$ as a n-1-braid if $a \in d_j$, $j \neq i, i + 1$ Fig. 5.

If the retraction r from β-{$a_1, a_2, ..., a_n$} onto β' by remove all points from top level or bottom level or together, then we have two cases:

- β' is a trivial n-braid.
- $\beta' = \sigma_i$ as a n-braid Fig. 6.

Theorem 4

A retract of any braid by remove a point or (points) from any arc or (arcs) is a braid.

Proof

Let β be a n-braid, β = {$d_1, d_2, ..., d_n$} where d_i is a string and *r* be a continuous map from β-{a} onto β' where a is a point on d_i, i = 1, 2, ..., n defined by $r(d_i) = \begin{cases} d_i & \text{if } a \notin d_i \\ 0 & \text{if } a \in d_i \end{cases}$, then $r(d_i) = d_i \ \forall d_i \in \beta'$ hence r is a retraction and β' is a n-1-braid.

Theorem 5

Let $\beta = \sigma_1\sigma_2.....\sigma_k$ be a n-braid and r: (β-a) → β' be a retraction then β' takes three cases:

- Trivial n-1-braid, if $a \in d_1$.
- $\beta' = \sigma_1\sigma_2....\sigma_{k-1}$ as a n-1-braid, if $a \in d_i$, $1 < i \leq k + 1$.
- $\beta' = \sigma_1\sigma_2....\sigma_k$ as a n-1-braid, if $a \in d_i$, $i > k + 1$.

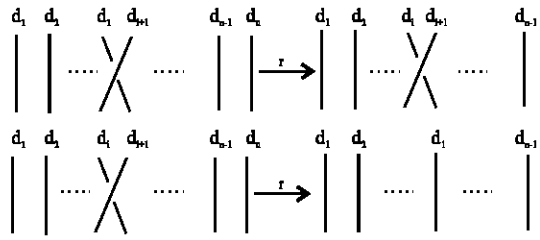

Fig. 5: Two cases of retraction of σ_i by remove one point

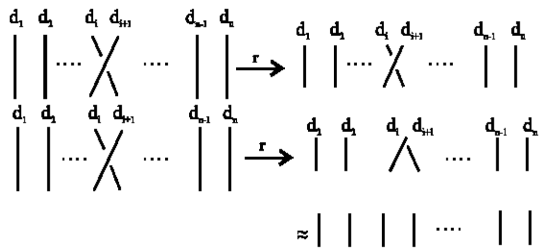

Fig. 6: Two cases of retraction of σ_i by remove more then one point

Proof

- Since d'_1 of σ_1 glue with d_1'' of σ_2, also d_1'' glue with d_1''' of σ_3...., d_1^{k-1} of σ_{k-1} glue with d_1^k of σ_k and since $d_1 = d_1'd_1''d_1'''....d_1^{k-1}d_1^k$ hence if we remove d_1, then all $\sigma_1, \sigma_2, ..., \sigma_k$ are finished.
- Since d_i, $1 < i \leq k+1$ is not glue with any arc of any σ, then the only σ which finished is σ_{i-1} and σ_i takes its place and represents σ_i in the retract.
- Since all d_i, $i > k+1$ are straight strings then remove it not finish any σ, hence the retract $\beta' = \sigma_1\sigma_2...\sigma_k$.

Corollary 1

The limit of retractions of i) in Theorem 5 is a 1-braid group, also the limit of the retraction of ii) is a 1-braid, but the limit of the retractions of iii) is $\sigma_1\sigma_2....\sigma_k$.

Lemma 1

Let β, β_1, β_2 be a three elements of a n-braid group B_n and r be a retraction from B_n-{a} onto $\bar{B}_n$, then not necessary

- $r(\beta_1 \circ \beta_2) = r(\beta_1) \circ r(\beta_2)$,
- $r(\beta^{-1}) = (r(\beta))^{-1}$, β^{-1} be the inverse element of β.

We can show the lemma by the following examples:

- Let $\beta_1 = \sigma_3\sigma_1$ and $\beta_2 = \sigma_1\sigma_2\sigma_3\sigma_1$ be a two elements of 4-braid group B_4, r be a retraction from B_4-{a} onto $\bar{B}_4$ $a \in d_1$, then $\beta_1 \circ \beta_2 = \sigma_3\sigma_1^2\sigma_2\sigma_3\sigma_1$, $r(\beta_1) = \sigma_2$ is a 3-braidas, $r(\beta_2) = \sigma_1$ is a 3-braid, $r(\beta_1 \circ \beta_2) = \sigma_2\sigma_1\sigma_2$ is a 3-braid and $r(\beta_1) \circ r(\beta_2) = \sigma_2\sigma_1$ as a 3-braid, hence $r(\beta_1 \circ \beta_2) \neq r(\beta_1) \circ r(\beta_2)$ Fig. 7.

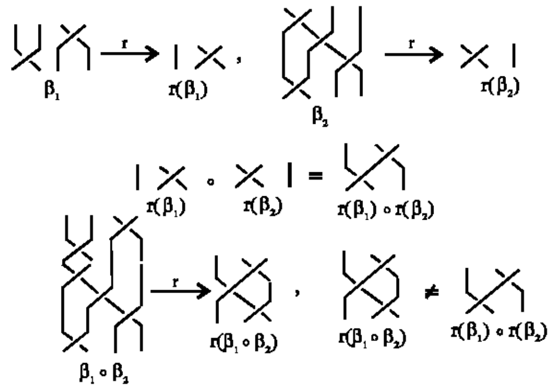

Fig. 7: $r(\beta_1 \circ \beta_2) \neq r(\beta_1) \circ r(\beta_2)$

- Let $\beta = \sigma_2\sigma_1\sigma_3$ be an element of 4-braid group B_4 and $\beta^{-1} = \sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1}$ be its inverse, r be a retraction from B_4-{a} onto $\overline{B}_4$ $a \in d_1$, then $r(\beta) = \sigma_2\sigma_3$ as a 3-braid, hence $(r(\beta))^{-1} = \sigma_3^{-1}\sigma_2^{-1}$ and $r(\beta^{-1}) = \sigma_1^{-1}$ as a 3-braid, hence $r(\beta^{-1}) \neq (r(\beta))^{-1}$ Fig. 8.

Theorem 6

The retraction of a n-braid group B_n is not necessary a braid group.

Proof

From the lemma above we can say easy there is not exist for all element in $r(B_n$-{a}) an inverse element.

Theorem 7

Let B_n be a n-braid group and $M = \{\sigma_1, \sigma_2, ..., \sigma_{n-1}\}$ be a set of its generators, r be a retraction from B_n-{a} onto $\overline{B}_n$, then i) if $a \in d_1$ or $a \in d_n$ then r(M) is a set of the generators of B_{n-1}, ii) if $a \in d_i$, $1 < i < n$, then $r(M) = \{\sigma_1, \sigma_2, ..., \sigma_{i-1}, \sigma_{i+1}, \sigma_{n-1}\}$

Proof

- If we remove an arc d_1 from all the generators, then σ_1 was vanished, also if we remove d_n, then σ_{n-1} was vanished and σ_2 becomes σ_1, etc.
- Proof ii) came directory from the proof i).

Theorem 8

Let $\beta = \sigma_2\sigma_4\sigma_6...\sigma_{2k-2}$ be a n-braid, then $\beta' = \sigma_1\sigma_3...\sigma_{2k-1}$ as a n-1-braid is a retract of β.

Proof

Let $\beta = \sigma_2\sigma_4\sigma_6...\sigma_{2k-2}$ be a n-braid, $\{d_1, d_2, ..., d_n\}$ be an arcs of β and r be a continuous map from β-{a} onto β' where a is a point on d_i.i =1, 2, ..., n defined, by $r(d_i) = \begin{cases} 0 & \text{if } i=1 \\ d_i & \text{if } 1<i\leq n \end{cases}$, then $r(d_i) = d_i \ \forall \ d_i \in \beta'$ hence r is a retraction and $\beta' = \sigma_1\sigma_3...\sigma_{2k-1}$ is a n-1-braid.

Theorem 9

Let $\beta = \sigma_1\sigma_3...\sigma_{2k-1}$ be a n-braid, then $\beta = \sigma_2\sigma_4\sigma_6...\sigma_{2k-2}$ as a n-1-braid is a retract of β.

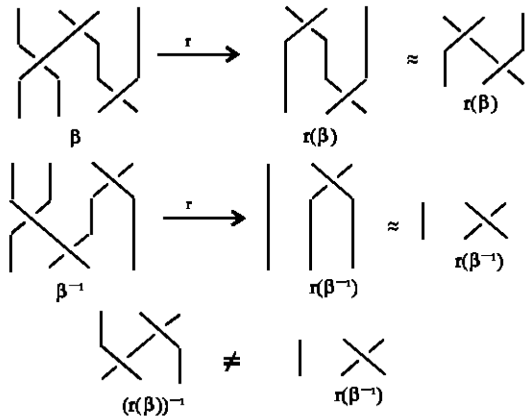

Fig. 8: $r(\beta^{-1}) \neq (r(\beta))^{-1}$

Proof

The proof is similar to the proof of Theorem 8.

Theorem 10

Let B_n be a pure n-braid group generated by $a_1, a_2, ..., a_{n-1}$, where, $a_i = (\sigma_{n-1}\sigma_{n-2}, ..., \sigma_{i+1})\sigma_i^2(\sigma_{i+1}^{-1}\sigma_{i+2}^{-1},.....,\sigma_{n-1}^{-1})$ and $r:(a_i\text{-}a) \to b_i$ be a retraction then

- b_i is a trivial n-1-braid if $a \in d_i$ or $a \in d_n$.
- $b_i = (\sigma_{n-1}\sigma_{n-2}.....\sigma_{j+1}\sigma_{j-1}....\sigma_{i+1})\sigma_i^2(\sigma_{i+1}^{-1}\sigma_{i+2}^{-1}.....\sigma_{j-1}^{-1}\sigma_{j+1}^{-1}.....\sigma_{n-2}^{-1}\sigma_{n-1}^{-1})$,
  if $a \in d_j$, $i < j < n$.
- $b_i = a_{i-1}$ in n-1-braid, if $a \in d_j$, $j < i$

Proof

The proof is clear.

Theorem 11

Let B_n be a n-braid group and r be a retraction from B_n-{a} onto $\overline{B}_n$, then $\overline{B}_n$ is a monoid.

Proof

Let B_n be a n-braid group and r be a retraction from B_n-{a} onto $\overline{B}_n$, then the all elements of $\overline{B}_n$ are n-1-braids, hence by definition the product of any two elements is also an element in $\overline{B}_n$, also the associative law is holds and we can see easy the identity element is exists, hence $\overline{B}_n$ is a monoid.

Theorem 12

A retract of a singular braid not necessary a singular braid.

Proof

We can show that by the following Example:

Example 1

Let $\beta = \sigma_1\sigma_3\sigma_2$ be a singular 4-braid and r: $(\beta\text{-}\{a\}) \to \overline{\beta}$ be a retraction and {a} be a singular point then $r(\beta\text{-}\{a\}) = \sigma_1$ which is not singular braid Fig. 9.

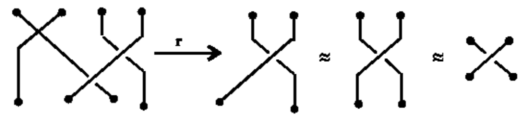

Fig. 9: Singular braid not necessary a singular braid

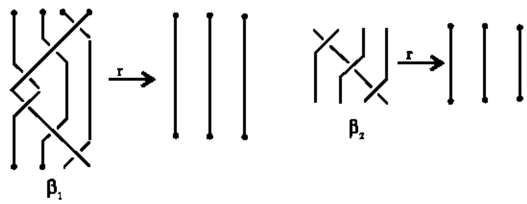

Fig. 10: $r(\beta_1) = r(\beta_2)$ but $\beta_1 \not\approx \beta_2$

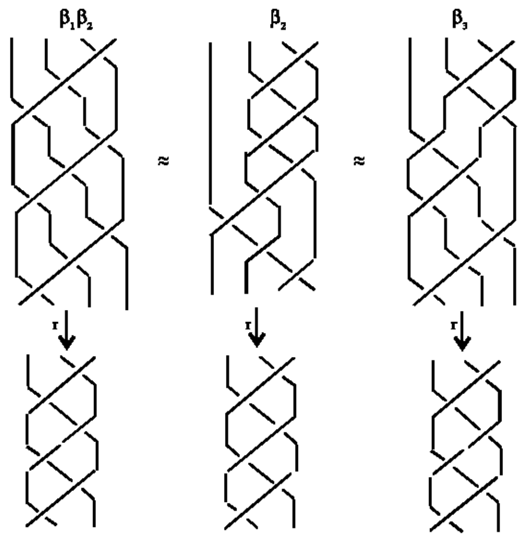

Fig. 11: β_1, β_2, β_3 are equivalents and has the same $r(\beta_{i}.\{a\})$, i = 1, 2, 3

Theorem 13

A retraction of any isotopic braids is an invariant.

Proof

Let β_1, β_2 be a two n-braids such that $\beta_1 \approx \beta_2$ and a Γ be a set of all n-braids and r be a retraction from Γ-{a} onto $\bar{\Gamma}$, we want to prove $r(\beta_1) = r(\beta_2)$, since $\beta_1 \approx \beta_2$ then $\pi(\beta_1) = \pi(\beta_2)$, this means the ends points of the arcs $d_1, d_2, ..., d_n$ of β_1 and β_2 have the same location, by elementary move Ω or Ω^{-1} on the arcs the ends points can not be changed, but the σ's which components of β_1 and β_2 are exchanged the places, hence by remove any arc from β_1 and β_2 gives the same result which means $r(\beta_1) = r(\beta_2)$.

Remark 1

If $r(\beta_1) = r(\beta_2)$, then not necessary $\beta_1 \approx \beta_2$.

By the following example we show that:

Example 2

Let $\beta_1 = \sigma_1\sigma_2\sigma_1^2\sigma_2^{-1}\sigma_3^{-1}$, $\beta_2 = \sigma_1\sigma_2\sigma_3$ be a two 4-braids, then $r(\beta_1) = r(\beta_2)$ but $\beta_1 \not\approx \beta_2$ Fig. 10.

Example 3

Let β_1, β_2, β_3 be a three 4-braids, easy we can get one from the another by some elementary moves, hence this three 4-braids are equivalents, if we apply the retraction on the three 4-braids, $r:(\beta_i\text{-}\{a\}) \rightarrow \overline{\beta}$, $a \in d_1$ we get $\overline{\beta} = (\sigma_2\sigma_1)^3$, (Fig. 11).

REFRANCES

El-Ghoul, M., 1985. Folding of manifolds. Ph.D Thesis, Univ. Tanta. Egypt.

El-Ghoul, M., 1995. The deformation retract of complex projective and its topological folding. J. Material Sci. Engl., 30: 4145-4148.

El-Ghoul, M., 1998. The deformation retraction and topological folding of manifold. Commun. Fac. Sic. Univ. Ankara Series Av., 37: 1-7.

El-Ghoul, M., 2002. Fractioul folding of manifold. Chaos, Soliton Fract., Engl., 12: 1019-1023.

El-Ghoul, M., A.I. Elrokh and M.M. Al-Shamiri, 2006a. Folding of trefoil knot andits graph. J. Math. Stat., 2: 368-372.

El-Ghoul, M., A.I. Elrokh and M.M. Al-Shamiri, 2006b. Folding of tubular trefoil knot. Int. J. Applied Math. (In Press).

Gemein, B., 2001. Representations of the singular braid monoid and group invariants of singular knots. Topol. Appl., 114: 117-140.

Kauffman, L.K., 1991. Knot and Physics, Singapore[u.a]: World Scientific.

Massay, W.S., 1967. Algebraic Topology. An Introduction, Harcout, Brace and World, New York.

Munkers, J.R., 1975. Topology. A First Course, Prentice-Hall Inc., Englewood Cliffs, New Jersey, USA.

Murasugi, K., 1996. Knot Theory and its Applications. Birkhaser, Boston, MA.

Murasugi, K. and B.I. Kurpita, 1999. A Study of Braids. Kluwer Academic Publication London.