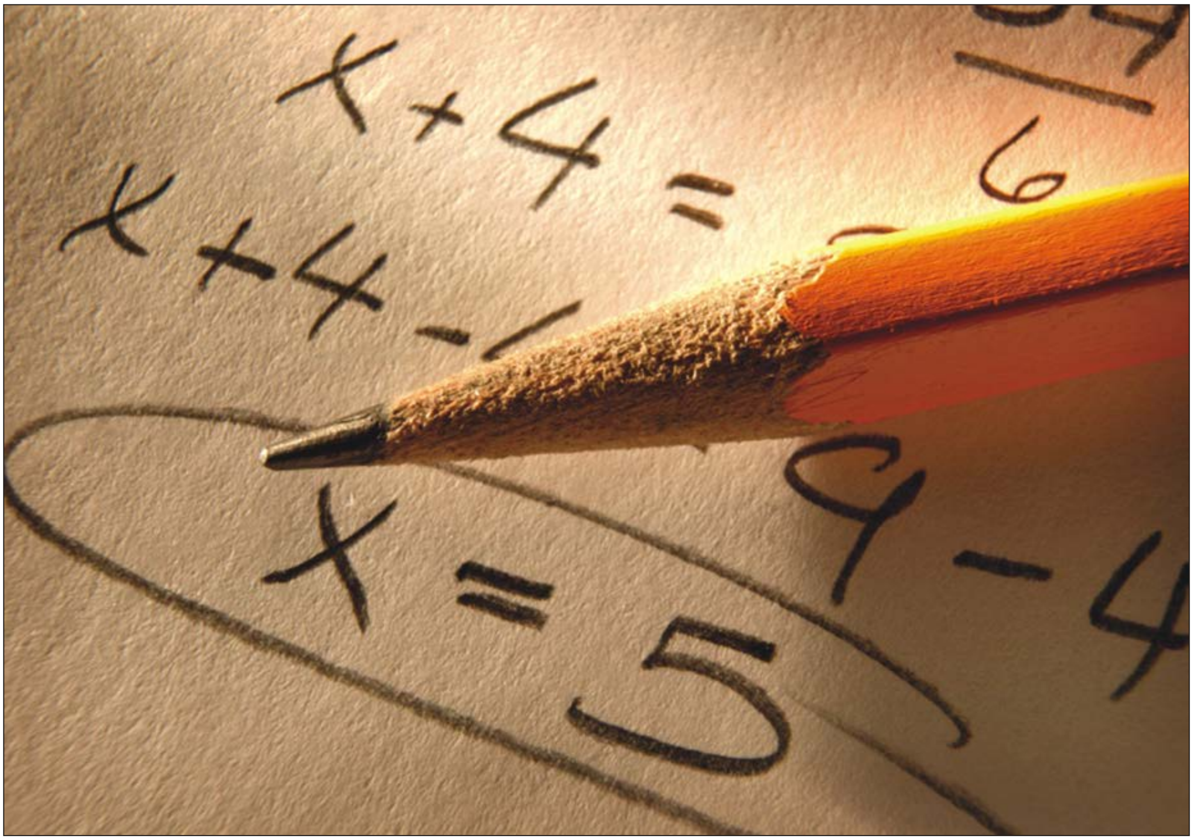

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An Algebraic-Analytic Approach of Diophantine Equations

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Abstract: Our purpose in this research is to show how much Fermat equation is rich in analytic applications. Effectively, this equation allows to build amazing sequences, series and numbers. The question of the elementary proof of the theorem remains of course, we will see it in this communication. We will make also an allusion to the very known Fermat numbers (x^{2^i}). We will see how this problem of the proof is actual and how it can be solved using Fermat sequences and series.

Key words: Diophantine equations, fermat, beal, algebraic-analytic approach

INTRODUCTION

We show that Fermat equation allows to build rational sequences and series. After the formulation of those sequences and series, we calculate their limits. We generalize the sequences and series and their limits to Beal equation and to a generalized Diophantine equation. We define also complex sequences. Of course, all the development is available for other Diophantine equations, we show an example, but there are many others, like Pilai, Catalan and Smarandache equations.

THE SEQUENCES

Fermat equation is
$U^n = X^n+Y^n$
$GCD(X,Y) = 1$

We will consider in this study two equivalent equations. Effectively, let us pose
$u = U^{2n}$
$x = U^nX^n$
$y = U^nY^n$
$z = X^nY^n$
After a little calculus

$$u = U^{2n} = U^n(X^n + Y^n) = x + y \tag{1}$$

And

$$\frac{1}{z} = \frac{1}{X^nY^n} = \frac{U^{2n}}{U^nX^nU^nY^n} = \frac{u}{xy} = \frac{x+y}{xy} = \frac{1}{x} + \frac{1}{y} \tag{2}$$

We deduce that if U, X, Y, are integers verifying Fermat equation, then u, x, y, z as defined verify simultaneously the new Fermat Eq. 1 and 2 which follow:

Lemma 1

$$u = x+y \tag{3}$$

$$\frac{1}{z} = \frac{1}{x} + \frac{1}{y} \tag{4}$$

Let us build the sequences. If we pose

$x_1 = x$

$y_1 = y$

And $\forall x_1, y_1$ integers $\exists z_1$ verifying

$$\frac{1}{z_1} = \frac{1}{x_1} + \frac{1}{y_1}$$

And

$$z_1 = \frac{xy}{x+y} = z$$

Then

$$(x_1 + y_1)z_1 = x_1 y_1$$

And

$$x_1(y_1 - z_1) = x_1 y_1$$

We pose

$$y_2 = y_1 - z_1 = \frac{z_1 y_1}{x_1}$$

Also

$$y_1(x_1 - z_1) = \frac{z_1 x_1}{y_1}$$

We pose

$$x_2 = x_1 - z_1 = \frac{z_1 x_1}{y_1}$$

And

$$x_2 y_2 = z_1^{\,2}$$

Which means that

$$x_1 = x_2 + z_1 = x_2 + \sqrt{x_2 y_2}$$

$$y_1 = y_2 + z_1 = y_2 + \sqrt{x_2 y_2}$$

$$u_1 = u = (x_1 + y_1) = (\sqrt{x_2} + \sqrt{y_2})^2 > x_2 + y_2 > 0$$

and

$$x_1 = \sqrt{x_2}(\sqrt{x_2} + \sqrt{y_2}) > x_2 > 0$$

$$y_1 = \sqrt{y_2}(\sqrt{x_2} + \sqrt{y_2}) > y_2 > 0$$

$$z_1 = \frac{x_1 y_1}{x_1 + y_1} = \sqrt{x_2 y_2} > z_2 = \frac{x_2 y_2}{x_2 + y_2} > 0$$

Because $\forall x_2, y_2, \exists z_2$ verifying

$$\frac{1}{z_2} = \frac{1}{x_2} + \frac{1}{y_2}$$

The process is available until infinity. For i

$$u_i = x_i + y_i = (\sqrt{x_{i+1}} + \sqrt{y_{i+1}})^2 > x_{i+1} + y_{i+1} > 0$$

$$x_i = \sqrt{x_{i+1}}(\sqrt{x_{i+1}} + \sqrt{y_{i+1}}) > x_{i+1} > 0$$

$$y_i = \sqrt{y_{i+1}}(\sqrt{x_{i+1}} + \sqrt{y_{i+1}}) > y_{i+1} > 0$$

$$z_i = \frac{x_i y_i}{x_i + y_i} = \sqrt{x_{i+1} y_{i+1}} > z_{i+1} = \frac{x_{i+1} y_{i+1}}{x_{i+1} + y_{i+1}} > 0$$

And of course

$$\frac{1}{z_{i+1}} = \frac{1}{x_{i+1}} + \frac{1}{y_{i+1}}$$

We have built the sequences.

Lemma 2

x_i, y_i have an expression

$$x_i = x^{2^{i-1}} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})^{-1} \quad (H)$$

$$y_i = y^{2^{i-1}} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})^{-1} \quad (H')$$

Proof of Lemma 2

By traditional induction, for i = 2

$$x = \sqrt{x_2}(\sqrt{x_2} + \sqrt{y_2}) = \sqrt{x_2}(x + y)^{\frac{1}{2}}$$

$$x_2 = \frac{x^2}{x + y}$$

Also

$$y = \sqrt{y_2}(\sqrt{x_2} + \sqrt{y_2}) = \sqrt{y_2}(x + y)^{\frac{1}{2}}$$

$$y_2 = \frac{y^2}{x + y}$$

We suppose (H) and (H') true for i, then

$$x_i = \sqrt{x_{i+1}}(\sqrt{x_{i+1}} + \sqrt{y_{i+1}}) = \sqrt{x_{i+1}}(x_i + y_i)^{\frac{1}{2}}$$

$$x_{i+1} = \frac{x_i^{\,2}}{x_i + y_i} = x^{2^i} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})^{-2} (x^{2^{i-1}} + y^{2^{i-1}})^{-1} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j}) = x^{2^i} \prod_{j=0}^{j=i-1} (x^{2^j} + y^{2^j})^{-1}$$

Also

$$y_i = \sqrt{y_{i+1}}(\sqrt{x_{i+1}} + \sqrt{y_{i+1}}) = \sqrt{y_{i+1}}(x_i + y_i)^{\frac{1}{2}}$$

$$y_{i+1} = \frac{y_i^{\,2}}{x_i + y_i} = y^{2^i} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})^{-2} (x^{2^{i-1}} + y^{2^{i-1}})^{-1} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j}) = y^{2^i} \prod_{j=0}^{j=i-1} (x^{2^j} + y^{2^j})^{-1}$$

It is proved, but $\forall x, y$

$$\prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j}) = \frac{x^{2^{i-1}} - y^{2^{i-1}}}{x - y}$$

Then, for $x \neq y$

$$x_i = \frac{x^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x - y)$$

$$y_i = \frac{y^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x - y)$$

Lemma 4

Fermat equation has the constant

$$x_i - y_i = x - y$$

THE SERIES

As we saw

$\sqrt{x_i y_i} = y_{i-1} - y_i = x_{i-1} - x_i$

It implies the following sum

$$\sum_{j=2}^{j=i+1} (\sqrt{x_j y_j}) = x - x_2 + x_2 - x_3 + x_3 - x_4 + ... + x_i - x_{i+1} = x - x_{i+1}$$

then

$$\sum_{j=2}^{j=\infty} (\sqrt{x_j y_j}) = \lim_{i\to\infty}(x - x_{i+1})$$

And the limits, if x>y

$$\lim_{i\to\infty}(y_i) = \lim_{i\to\infty}(\frac{y^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x-y)) = 0$$

And

$$\lim_{i\to\infty}(x_i) = \lim_{i\to\infty}(\frac{x^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x-y)) = x - y$$

If x<y

$$\lim_{i\to\infty}(x_i) = \lim_{i\to\infty}(\frac{x^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x-y)) = 0$$

$$\lim_{i\to\infty}(y_i) = \lim_{i\to\infty}(\frac{y^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x-y)) = y - x$$

Let us study series. If x>y then $\sum_{j=2}^{j=\infty} (\sqrt{x_j y_j}) = \lim_{i\to\infty}(x - x_{i+1}) = x - (x - y) = y$

And if x<y then $\sum_{j=2}^{j=\infty} (\sqrt{x_j y_j}) = \lim_{i\to\infty}(x - x_{i+1}) = x$

The Applications of the Sequences and the Series

We will consider firstly that x>y

$$\sum_{j=2}^{j=i} ((-1)^j \sqrt{x_j y_j}) = x - x_2 - x_2 + x_3 + ... + (-1)^i (x_{i-1} - x_i)$$

$$= x - 2x_2 + 2x_3 - ... + 2(-1)^i x_{i-1} + (-1)^{i+1} x_i$$

$$= 2\sum_{j=1}^{j=i} ((-1)^{j+1} x_j) - x - (-1)^{i+1} x_i$$

Also

$$\sum_{j=2}^{j=i} ((-1)^j \sqrt{x_j y_j}) = y - y_2 - y_2 + y_3 + ... + (-1)^i (y_{i-1} - y_i)$$

$$= y - 2y_2 + 2y_3 - ... + 2(-1)^i y_{i-1} + (-1)^{i+1} y_i$$

$$= 2\sum_{j=1}^{j=i} ((-1)^{j+1} y_j) - y - (-1)^{i+1} y_i$$

Then $2\sum_{j=1}^{j=i} ((-1)^{j+1} x_j) = \sum_{j=2}^{j=i} ((-1)^j \sqrt{x_j y_j}) + x + (-1)^{i+1} x_i$

And $2\sum_{j=1}^{j=i} ((-1)^{j+1} y_j) = \sum_{j=2}^{j=i} ((-1)^j \sqrt{x_j y_j}) + y + (-1)^{i+1} y_i$

We will study now the convergence of the series. As $\sum_{j=2}^{j=\infty} ((-1)^j \sqrt{x_j y_j})$ is convergent and $\lim_{i\to\infty}(y_i) = \lim_{i\to\infty}(\frac{y^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x-y)) = 0$ and

$\sum_{j=2}^{j=\infty}((-1)^j\sqrt{x_j y_j}) = 2\sum_{j=1}^{j=\infty}((-1)^{j+1}x_j) - x - \lim_{i\to\infty}((-1)^{i+1}x_i)$

$= 2\sum_{j=1}^{j=\infty}((-1)^{j+1}y_j) - y - \lim_{i\to\infty}((-1)^{i+1}y_i) = 2\sum_{j=1}^{j=\infty}((-1)^{j+1}y_j) - y$ Is convergent. It implies that $2\sum_{j=1}^{j=\infty}((-1)^{j+1}y_j)$ is convergent. Then $\lim_{k\to\infty}(2\sum_{j=1}^{j=2k+1}((-1)^{j+1}y_j)) = \lim_{k\to\infty}(2\sum_{j=1}^{j=2k}((-1)^{j+1}y_j))$

It means one thing: $\lim_{i\to\infty}((-1)^{i+1}x_i) = 0$ and x−y = 0, then $\lim_{i\to\infty}(x_i) = x - y = 0$. It is confirmed by the fact the limit of the general term of the series (here x-y) is equal to zero, because $\sum_{j=1}^{j=\infty}((-1)^{j+1}x_j)$ is convergent. And $x - y = U^n(X^n - Y^n) = 0$ and X = Y = 0, because GCD(X,Y)=1 (The reasoning is the same for x<y)

Our question is now: Why are there solutions for n = 2? The answer is in the equations. Effectively, there are trivially an infinity of solutions for n = 1. But the sequences for n = 1 are as it follows;

$$x_i = \frac{x^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x-y) = U\frac{X^{2^{i-1}}}{X^{2^{i-1}} - Y^{2^{i-1}}}(X-Y) = \frac{X^{2.2^{i-2}}}{X^{2.2^{i-2}} - Y^{2.2^{i-2}}}(X^2 - Y^2)$$

$$y_i = \frac{y^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x-y) = U\frac{Y^{2^{i-1}}}{X^{2^{i-1}} - Y^{2^{i-1}}}(X-Y) = \frac{Y^{2.2^{i-2}}}{X^{2.2^{i-2}} - Y^{2.2^{i-2}}}(X^2 - Y^2)$$

these are the expressions of the x_{i-1} and y_{i-1} of the exponent n = 2. And for n = 2, the sequences are

$$y_i = \frac{y^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x-y) = U^2\frac{Y^{2^{i-1}}}{X^{2^{i-1}} - Y^{2^{i-1}}}(X^2-Y^2) = \frac{Y^{4.2^{i-3}}}{X^{4.2^{i-3}} - Y^{4.2^{i-3}}}(X^4 - Y^4)$$

$$x_i = \frac{x^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x-y) = U^2\frac{X^{2^{i-1}}}{X^{2^{i-1}} - Y^{2^{i-1}}}(X^2-Y^2) = \frac{X^{4.2^{i-3}}}{X^{4.2^{i-3}} - Y^{4.2^{i-3}}}(X^4 - Y^4)$$

these are the expressions of the sequences and they do not guarantee the existence of the series for i = 2. So, the case n = 2 is the only exception.

Other Applications of the Sequences and the Series

Let the Beal equation $U^c = X^a + Y^b$; GCD (X,Y) = 1. If we pose

$u = U^{2c} = U^c(X^a + Y^b)$

$x = U^c X^a$

$y = U^c Y^b$

$z = X^a Y^b$

Then

$$u = x + y \quad (5)$$

And

$$\frac{1}{z} = \frac{1}{X^a Y^b} = \frac{U^{2c}}{U^c X^a U^c Y^b} = \frac{u}{xy} = \frac{x+y}{xy} = \frac{1}{x} + \frac{1}{y} \quad (6)$$

Equation 5 and 6 are the new Fermat equations, they imply after the same reasoning and formulas than for Fermat equation $x - y = U^c(X^a - Y^b) = 0$

Which means

X = Y = 0

Because GCD (X,Y) = 1

Then Beal equation has not solutions, with the same calculus and reasoning than for Fermat equation, for c>2 and a>2 and b>2.

Now, let the general following equation

$Y^n = X_1^{n_1} + X_2^{n_2} + ... + X_i^{n_i}$

$GCD(X_k) = 1$

We pose

$u = Y^{2n}$

$x = Y^n(Y^n - X_k^{n_k})$

$y = Y^n X_k^{n_k}$

$z = X_k^{n_k}(Y^n - X_k^{n_k})$

With k = 1,2,…,i

Then

$$u = x+y \tag{7}$$

$$\frac{1}{z} = \frac{1}{X_k^{n_k}(Y^n - X_k^{n_k})} = \frac{Y^{2n}}{Y^n X_k^{n_k} Y^n (Y^n - X_k^{n_k})} = \frac{u}{xy} = \frac{x+y}{xy} = \frac{1}{x} + \frac{1}{y} \tag{8}$$

Equation 7 and 8 are the new Fermat equation, generalized equation has no solution for n>i (i-1) and n_k>i (i−1) other than

$Y^n = 2X_k^{n_k} = 2X_m^{n_m}; m \neq k$

Then

$X = Y = 0$

Because GCD (X_k) = 1

The new Fermat equations must be used with precaution, for example for the following equation

$$kU^n = X^n + Y^n$$

There are solutions for k = 7 and there are not for k = 2. We must pose judiciously

$u = U^{2n}$

$x = U^n X^n$

$y = U^n Y^n$

$z = X^n Y^n$

And the new equations are

$ku = kU^{2n} = kU^n(X^n + Y^n) = k(x + y)$

$\frac{1}{z} = \frac{U^{2n}}{U^n X^n U^n Y^n} = \frac{u}{k(xy)} = \frac{ku}{k^2(xy)} = \frac{k(x+y)}{k^2(xy)} = \frac{1}{kx} + \frac{1}{ky}$

Which are not new Fermat equations and have not the same solutions. It is false to pose

$u = k^2 U^{2n}$

$x = kU^n X^n$

$y = kU^n Y^n$

$z = X^n Y^n$

Conclusion

The new Fermat equations allow to build sequences and series which allows to test the impossibility of the resolution of an equation. If they are a consequence of some Diophantine equations, they remain an intellectual building. They must be used with precaution, but they are very efficient.

Generalization

Now, we will generalize the results. Let the following equation

$Y^n = X_1^{n_1} + X_2^{n_2} + ... + X_i^{n_i}$ (E)

We will prove that this equation has not solution for

$n > i(i-1), n_j > i(i-1), \forall j \in \{1, 2, ..., i\}$

When $n \le i\ (i-1)$, $n_k \le i\ (i-1)$, there are solutions, for example

$i = 2$ has $3^2+4^2 = 5^2$

$i = 3$ has $3^2+4^2+5^2 = 6^3$

$95800^4+217519^4+414560^4 = 422481^4$

$i = 4$ has $27^5+84^5+110^5+133^5 = 144^5$

It seems to have solutions only for i+1, but we will prove that it is for i (i-1)

We will suppose that X_k are coprime, let

$x_k = Y^{(i-1)n} X_k^{n_k}, \forall k \in \{1, 2, ..., i\}$

$u = Y^{in}$

$v = X_1^{n_1} X_2^{n_2} ... X_i^{n_i}$

Lemma 5

$$x_1 + x_2 + ... + x_i = Y^{(i-1)n}(X_1^{n_1} + X_2^{n_2} + ... + X_i^{n_i}) = Y^{in} = u \tag{9}$$

$$\frac{1}{v} = \frac{1}{X_1^{n_1} X_2^{n_2} ... X_i^{n_i}} = \frac{Y^{i(i-1)n}}{Y^{(i-1)n} X_1^{n_1} Y^{(i-1)n} X_2^{n_2} ... Y^{(i-1)n} X_i^{n_i}} = \frac{u^{i-1}}{x_1 x_2 ... x_i} \tag{10}$$

We will define the sequences

$x_{k,0} = x_k$

$u_0 = u$

$v_0 = v$

$x_{k,1} = x_k^{\,i}(x_1 + x_2 + ... + x_i)^{-(i-1)}, \forall k \in \{1, 2, ..., i\}$

Which implies

$u = x_1 + x_2 + ... + x_i = (x_{1,1}^{\frac{1}{i}} + x_{2,1}^{\frac{1}{i}} + ... + x_{i,1}^{\frac{1}{i}})^i > u_1 > 0$

$x_{k,0} = x_k = x_{k,1}^{\frac{1}{i}}(x_1 + x_2 + ... + x_i)^{\frac{i-1}{i}} = x_{k,1}^{\frac{1}{i}}(x_{1,1}^{\frac{1}{i}} + x_{2,1}^{\frac{1}{i}} + ... + x_{i,1}^{\frac{1}{i}})^{i-1} > x_{k,1} > 0$

$v = \frac{x_{1,0} x_{2,0} ... x_{i,0}}{u^{i-1}} = x_{1,1}^{\frac{1}{i}} x_{2,1}^{\frac{1}{i}} ... x_{i,1}^{\frac{1}{i}} > v_1 = \frac{x_{1,1} x_{2,1} ... x_{i,1}}{u_1^{\,i-1}} > 0$

The reasoning is available until infinity. Then

$x_{k,j} = x_{k,j+1}^{\frac{1}{i}}(x_{1,j+1}^{\frac{1}{i}} + x_{2,j+1}^{\frac{1}{i}} + ... + x_{i,j+1}^{\frac{1}{i}})^{i-1} > x_{k,j+1} > 0$

$u_j = x_{1,j} + x_{2,j} + ... + x_{i,j} = (x_{1,j+1}^{\frac{1}{i}} + x_{2,j+1}^{\frac{1}{i}} + ... + x_{i,j+1}^{\frac{1}{i}})^i > u_{j+1} > 0$

$v_j = \frac{x_{1,j} x_{2,j} ... x_{i,j}}{u_j^{\,i-1}} = x_{1,j+1}^{\frac{1}{i}} x_{2,j+1}^{\frac{1}{i}} ... x_{i,j+1}^{\frac{1}{i}} > v_{j+1} = \frac{x_{1,j+1} x_{2,j+1} ... x_{i,j+1}}{u_{j+1}^{\,i-1}} > 0$

Lemma 6

(P) is the following expression

$x_{k,j} = x_k^{\,i^j}(\prod_{l=0}^{l=j-1} x_1^{\,i^l} + x_2^{\,i^l} + ... + x_i^{\,i^l})^{-(i-1)}$

Proof of Lemma 6

By traditional induction, it is verified for j=1, we suppose that (P) is true for j, so

$$x_{k,j+1}^{\frac{1}{i}} = x_{k,j}(x_{1,j+1}^{\frac{1}{i}} + x_{2,j+1}^{\frac{1}{i}} + \ldots + x_{i,j+1}^{\frac{1}{i}})^{-(i-1)}$$

$$x_{k,j+1} = x_{k,j}^{\ i}(x_{1,j+1}^{\frac{1}{i}} + x_{2,j+1}^{\frac{1}{i}} + \ldots + x_{i,j+1}^{\frac{1}{i}})^{-i(i-1)} = x_{k,j}^{\ i}(x_{1,j}^{\frac{1}{i}} + x_{2,j}^{\frac{1}{i}} + \ldots + x_{i,j}^{\frac{1}{i}})^{-(i-1)}$$

$$= x_k^{\ i^{j+1}}(\prod_{1=0}^{1=j-1} x_1^{\ i^1} + x_2^{\ i^1} + \ldots + x_i^{\ i^1})^{-i(i-1)}(x_1^{\ i^j} + x_2^{\ i^j} + \ldots + x_i^{\ i^j})^{-(i-1)} \prod_{1=0}^{1=j-1} x_1^{\ i^1} + x_2^{\ i^1} + \ldots + x_i^{\ i^1})^{(i-1)^2}$$

$$= x_k^{\ i^{j+1}}(\prod_{1=0}^{1=j} x_1^{\ i^1} + x_2^{\ i^1} + \ldots + x_i^{\ i^1})^{-(i-1)}$$

And it is true for j+1.

Lemma 7

The equation (E) conducts to an impossibility, effectively, if we pose

$$u = Y^{2n}$$

$$x = Y^n X_k^{\ n_k}$$

$$y = Y^n(Y^n - X_k^{\ n_k})$$

$$z = X_k^{\ n_k}(Y^n - X_k^{\ n_k})$$

u, x, y and z verify the lemma 1

$$u = x + y$$

$$\frac{1}{z} = \frac{1}{x} + \frac{1}{y}$$

Which conducts, we saw it, to

$$x = y$$

$$U = X_k = 0, \forall k \in \{1, 2, \ldots, i\}$$

Because they are coprime. Now, the question is: why are there solutions for

$$n \le i(i-1), n_k \le i(i-1)?$$

Let us pose

$$n = i(i-1), n_k = i(i-1), \forall k \in \{1, 2, \ldots, i\}$$

The expression (P) becomes

$$x_{k,j} = x_k^{\ i^j}(\prod_{1=0}^{1=j-1} x_1^{\ i^1} + x_2^{\ i^1} + \ldots + x_i^{\ i^1})^{-(i-1)}$$

$$= Y^{i(i-1)i^j} X_k^{\ i(i-1)i^j}((\prod_{1=0}^{1=j-1} Y^{i(i-1)i^1} X_1^{\ i(i-1)i^1} + Y^{i(i-1)i^1} X_2^{\ i(i-1)i^1} + \ldots + Y^{i(i-1)i^1} X_i^{\ i(i-1)i^1}))^{-(i-1)}$$

$$= Y^{(i-1)i^{j+1}} X_k^{\ (i-1)i^{j+1}}((\prod_{1=0}^{1=j-1} Y^{i(i-1)i^1} X_1^{\ i(i-1)i^1} + Y^{i(i-1)i^1} X_2^{\ i(i-1)i^1} + \ldots + Y^{i(i-1)i^1} X_i^{\ i(i-1)i^1}))^{-(i-1)}$$

$$= Y^{(i-1)i^{j+1}} X_k^{\ (i-1)i^{j+1}}((\prod_{1=0}^{1=j-1} Y^{(i-1)i^{1+1}} X_1^{\ (i-1)i^{1+1}} + Y^{(i-1)i^{1+1}} X_2^{\ (i-1)i^{1+1}} + \ldots + Y^{(i-1)i^{1+1}} X_i^{\ (i-1)i^{1+1}}))^{-(i-1)}$$

$$= Y^{(i-1)i^{j+1}} X_k^{\ (i-1)i^{j+1}}((\prod_{1=1}^{1=j} Y^{(i-1)i^1} X_1^{\ (i-1)i^1} + Y^{(i-1)i^1} X_2^{\ (i-1)i^1} + \ldots + Y^{(i-1)i^1} X_i^{\ (i-1)i^1}))^{-(i-1)}$$

It is the expression for the exponent (i-1). If there are solutions for the exponent (i-1), there will be solutions for the exponent i(i-1). It is not true for i, because of the exponent –(i-1) in the expression (P).

Conclusion

The sequences and the series as we defined them have several applications in several diophantine equations, we saw Fermat and Beal, we saw the generalized equation (E), but there are many others like Pilai, Smarandache, Catalan… They are truly very amazing !

THE ALGEBRAIC APPROACH

Now, let Fermat equation

$U^n = X^n + Y^n = X^n + i(-iY^n)$

$i^2 = -1$

We pose

$x' = U^n X^n$

$y' = -iU^n Y^n$

$$u' = U^{2n} = U^n(X^n + i(-iY^n)) = x + iy \tag{11}$$

$$z' = X^n(-iY^n) = \frac{x'y'}{u'}$$

$$\frac{1}{z'} = \frac{1}{x'} + \frac{i}{y'} \tag{12}$$

We will build sequences

$x'_1 = x'$

$y'_1 = y$

$u'_1 = u'$

$z'_1 = z'$

And

$(y'_1 - z'_1)x'_1 = y'_2 x_1{}' = iy'_1 z'_1$

$(x'_1 - iz'_1)y'_1 = x'_1 z'_1 = x'_2 y'_1$

$iz'^2_1 = x'_2 y'_2$

$y'_1 = y'_2 + z'_1 = y'_2 + \sqrt{\frac{x'_2 y'_2}{i}}$

$x'_1 = x'_2 + iz'_1 = x'_2 + i\sqrt{\frac{x'_2 y'_2}{i}}$

And

$\frac{1}{z'_2} = \frac{1}{x'_2} + \frac{i}{y'_2}$

The process is available until infinity, for j

$y'_j = y'_{j+1} + z'_j = y'_{j+1} + \sqrt{\frac{x'_{j+1} y'_{j+1}}{i}}$

$x'_j = x'_{j+1} + iz'_j = x'_{j+1} + i\sqrt{\frac{x'_{j+1} y'_{j+1}}{i}}$

$x'_j + iy'_j = (\sqrt{x'_{j+1}} + \sqrt{iy'_{j+1}})^2$

And

$\frac{1}{z'_{j+1}} = \frac{1}{x'_{j+1}} + \frac{i}{y'_{j+1}}$

The expressions are

$x'_j = x'^{2^{j-1}} \prod_{m=0}^{m=j-2} (x'^{2^m} + (iy')^{2^m})^{-1}$

$y'_j = i^{2^{j-1}-1} y'^{2^{j-1}} \prod_{m=0}^{m=j-2} (x'^{2^m} + (iy')^{2^m})^{-1}$

We prove it by induction, as we did for rational sequences

So

$x'_j = x_j$

$y'_j = i^{-1} y_j$

y_j is solution of

$iy'^2_j - iy'_{j+1}\, y'_j - y'_{j+1}\, x'_j = 0$

$$y'_j = \frac{iy'_{j+1} + \sqrt{-y'^2_{j+1} + 4iy'_{j+1}\, x'_j}}{2i}$$

$$\sqrt{-y'^2_{j+1} + 4iy'_{j+1}\, x'_j} = u + iv = y'_{j+1}\left(\sqrt{\frac{-1 + \sqrt{1 + 16\frac{x'^2_j}{y'^2_{j+1}}}}{2}} + i\sqrt{\frac{1 + \sqrt{1 + 16\frac{x'^2_j}{y'^2_{j+1}}}}{2}}\right)$$

And

$x'_j = x_j$

$y'_j = i^{-1} y_j$

It means that

$$\sqrt{\frac{-1 + \sqrt{1 + 16\frac{x'^2_j}{y'^2_{j+1}}}}{2}} = 0 \Rightarrow x' = 0$$

Also for x'_j

So the only solution is

$x = y = 0$

CONCLUSION

It appeared since the beginning, before the change of the data, that the equation contains a symmetry between x and y. Effectively, we found $u = x+y$. We broke the symmetry by changing the equation in two equations $u = x+y$ and $\frac{1}{z} = \frac{1}{x} + \frac{1}{y}$. We have solved the equation and found a method of resolution of Beal equation The conclusion is that Fermat equation (E) conducts always to an impossibility. It is also the case of Beal equation and generalized Fermat equation.

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