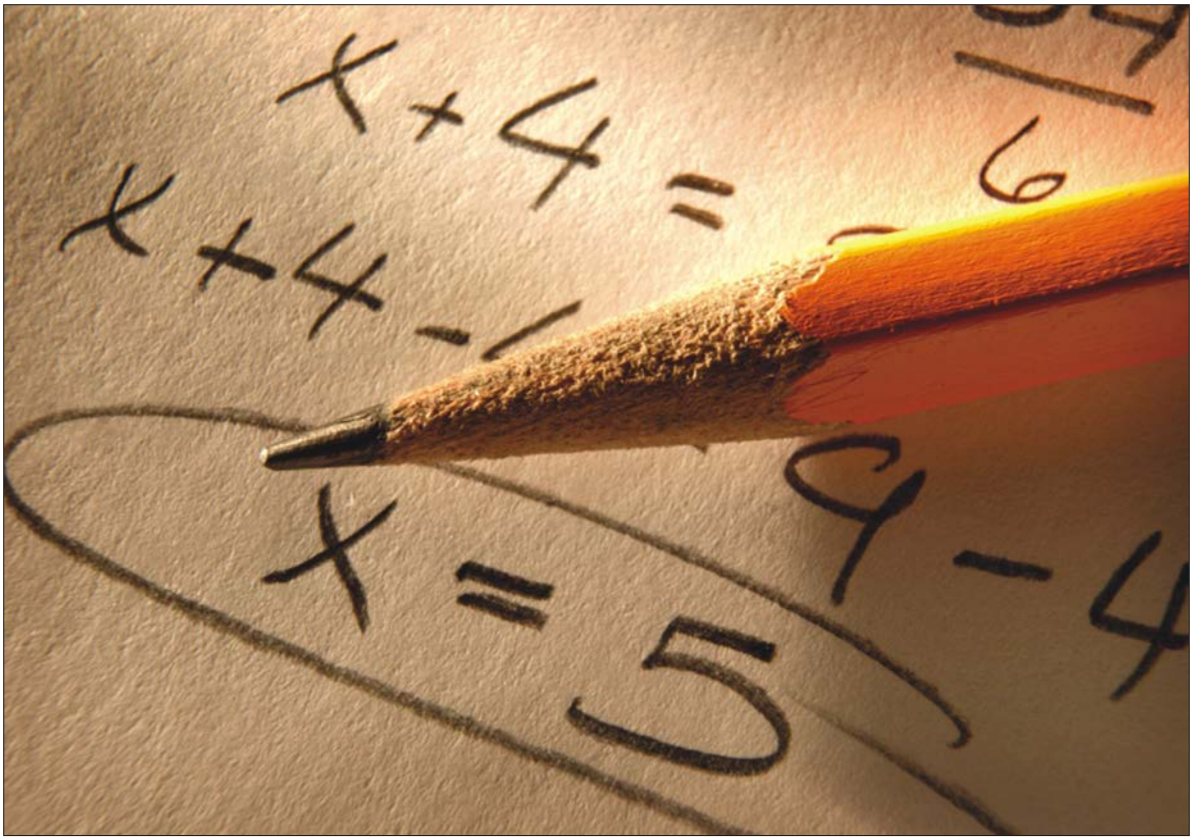

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Groups Satisfying Some Normalizer Conditions

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Abstract: The aim of the study is to characterize all finite groups that satisfy the normalizer conditions stated in this manuscript. Group and character theoretic methods are used in the study and it is proved that such groups are not simple. Specifically, the following result is established. Let a finite group G have a maximal subgroup H satisfying: (I)H = XP<t>, where P = <x, y: $x^3 = y^3 = [x, y] = 1$> and $t^2 = (zt)^2 = 1$ for all z in H. (II) X = R×K×T, where R has odd order and y acts fixed-point-free on X; K and T are 2-groups, xy centralizes K and acts fixed-pint-free on T; x centralizes T and acts fixed-point-free on K. T≠1, K≠1. (III) H is the ouly maximal subgroup of G containing XP and $|\Omega_1(Z(K\times T))|>4$. Then G is not a simple group.

Key words: Maximal subgroup, normalizer conditions, fixed-point-free, centralize, simple group

INTRODUCTION

The problem originated from the study of alternating group A_7 on 7 letters, by Liggonah (1977). It is the generalization of the proposition in the proof of the main result established by Liggonah (1977).

The notations used are standard, as used and defined in Gorenstein (1968). Conditions (i) through (iii) are conditions on the subgroup H of G, Giving the structure of H. These are the normalizer conditions referred to in the theorem.

The proof of the theorem is by contradiction. We suppose that G is simple and satisfies the conditions of the theorem and aim at arriving at a contradiction.

The method used in the proof involves studying the fusion of involutions in H, By using character and group theoretic methods.

We divide the proof through a series of lemmas. From now on, G is assumed to be simple and satisfies conditions (i) through (iii).

PROOF OF THE THEOREM

Lemma 1

For each class L of involutions of H, either there is an element l ε L such that $l^{-1}yl = y^{-1}$ or $L\subset O_{3}`(H)$.

Proof

Conditions (i) to (iii) imply that $N_G(X) = H = N_G(K\times T)$. Since K×T admits <y><t> and y acts fixed-point-free, K×T is unique in (K×T)<t> and of class ≤ 2 by Stephen and Tyrer (1973a, b). Hence (K×T)<t> is a Sylow 2-subgroup of its own normalizer, so it is a Sylow 2-subgroup of G. Consider H_1 =(R×K×T)<y><t>. The principal 3-block $B_0(H_1)$ of H_1 has the form:

e	y	L
1	1	1
d	1	δ
d+1	−1	δ+1

If $l^{-1}yl = y^{-1}$ is false for all $l \varepsilon L$, then $\#(l`l` = sy) = 0$ for all 3`-elements s commuting with y, by Higman (1968); where $\#(l`l` = sy)$ is the number of conjugates of l with product equal to sy. Applying Higman (1968) results, we

$$\#(l`l` = sy) = |H_1|/|C_{H_1}(y)|\Sigma\, \chi\, (l)^2\ \ \chi(y) = 0$$

where, summation is over all characters χ in $B_0(H_1)$. Then this implies that

$$\Sigma\, \chi(y)\chi(l)^2 = 0 \text{ giving } 1+\delta^2/d - (\delta+1)^2/(d+1) = 0.$$

That is, $(\delta-d)^2 = 0$, giving $\delta = d$. Then l lies in the kernel of every character in the principal 3-block of H_1. By Brauer (1964a, b), l lies in $O_3`(H_1) = O_3`(H)$ and $O_3`(H) = R\times K\times T$ and hence $L \subset O_3`(H)$ as required.

Hence we have that a class L of involutions in H, either there is an element $l \in L$ such that $l^{-1}yl = y^{-1}$ or $L \subset O_3`(H)$.

Corollary

The involutions of H not in $X = R\times K\times T$ are all conjugate (even in H).

Proof

The extended centralizer of y in H_1 is $C^*_{H_1}(y) = \langle y\rangle\langle t\rangle$, so all involutions of H_1 not in X must be in $\langle y\rangle\langle t\rangle \cong D_6$ by Lemma 1 (otherwise they will lie in $O_3`(H_1) = X$). Using the fact that all involutions of $D_6$ are conjugate and $C^*_H(y) = P\langle t\rangle$, all involutions of H not in $O_3`(H) = O_3`(H_1) = X$ are conjugate to t as required.

We have seen in the proof of Lemma 1 that $K\times T$ is weakly closed in $(K\times T)\langle t\rangle$, a Sylow 2-subgroup of G and of class ≤ 2 and any involution of H is conjugate in H to t or lies in $K\times T$. Since $\Omega_1(Z(K\times T))$ is characteristic in $K\times T$, elements of $\Omega_1(Z(K\times T))$ are conjugate in G only if they are conjugate in $N_G(\Omega_1(Z(K\times T)))$.

By maximality of H and the supposition that G is simple, we must have that $N_G(\Omega_1(Z(K\times T))) = H$. In particular, if k is an element of $\Omega_1(Z(K\times T))$, its conjugates in $\Omega_1(Z(K\times T))$ are k, k^y, k^{y2}. Since $|\Omega_1(Z(K\times T))|>4$, we can always pick k so as not to be conjugate to t in H. From now on, we assume that such a k has been picked.

Lemma 2

The only conjugates of k in G lying in $(KT)\langle t\rangle$ are k, k^y, k^{y2}.

Proof

By the corollary and the underlying assumption, any further conjugates k^g lies in $K\times T$, whence k^g, k, k^y, k^{y2} generate an abelian group. Let A be a subgroup of $K\times T$ chosen such that:

- A contains the greatest possible number of conjugates of k
- A is as large as possible

We first show that the conjugates of k lying in A are already conjugate in $N_G(A)$. Indeed, let $k \in A$. Then $k^g \in A^g$ and so $A^{g^{-1}} \leq C_G(k)$. Since $(K\times T)\langle t\rangle$ is a Sylow 2-subgroup of G and of $C_G(k)$, we can assume $A^{g^{-1}} \leq (K\times T)\langle t\rangle$. An element of $(K\times T)\langle t\rangle$ not in $K\times T$ transforms k^y to k^{y2} since it must involve t, so that an abelian subgroup of $(K\times T)\langle t\rangle$ not in $K\times T$ can not contain k^y or k^{y2}.

Furthermore, the conjugates of k in $A^{g^{-1}}$ lie in $K\times T$. Thus, if $A^{g^{-1}}$ is not contained in $K\times T$, then $(A^{g^{-1}}\ (K\times T))\ \langle k^y\rangle$ is an abelian subgroup of $K\times T$ containing more conjugates of k than A, contrary to the choice of A. Hence $A^{g^{-1}} \leq K\times T$.

By choice of A, each A and $A^{g^{-1}}$ are maximal abelian subgroups of $K\times T$ and since $K\times T$ is of class at most 2, $(K\times T)' = Z(K\times T) = A$ and because $[K\times T, A] = (K\times T)' = A$ and applying theorem 2.1 in Gorenstein (1968), page 18, gives A, $A^{g^{-1}}$ are normal in $K\times T$. By weak closure of $K\times T$ in $(K\times T)\langle t\rangle$, it means that A and $A^{g^{-1}}$ are conjugate in $N_G(K\times T)$ which is H. Since y acts fixed-point-free, $\langle A, A^{g^{-1}}\rangle$ is abelian by Stephen and Taylor, so it implies $A = A^{g^{-1}}$ by maximality of A. Thus the conjugates of A in H are A and A^t. Replacing g by tg if necessary, we can assume $A = A^g$, which implies $g\ \varepsilon N_G(A)$. Hence conjugates of k lying in A are already conjugate in $N_G(A)$. A is normalized by P so $N_G(A) = XP$ and hence $N_G(A) = H$ or $N_G(A) = XP$. This implies the only conjugates of k in A are k, k^y, k^{y2} as required.

Lemma 3

The element t is not conjugate to any element of $K\times T$ in $C_G(k)$.

Proof

By Lemma 2, the conjugates of k are k, k^y, k^{y2} and $kk^yk^{y2} = 1$ because y fixes kk^yk^{y2} and y acts fixed-point-free. This implies the only conjugates of $k^y\langle k\rangle$ in $C_G(k)/\langle k\rangle$ is $k^y\langle k\rangle$. By Glaubermann (1966), it implies that $k^y\langle k\rangle\ \varepsilon\ Z(C_G(k)/\langle k\rangle)$. This means that for some normal subgroup M of $C_G(k)$ of odd order, $M\langle k, k^y\rangle$ is normal in $C_G(k)$. By Frattini argument, this gives $C_G(k) = M\ N_{CG(k)}(\langle k, k^y\rangle)$. Since $\langle k, k^y\rangle \leq \Omega_1(Z(K\times T))$ and because of the fact that the conjugates of k in H lying in $(K\times T)\langle t\rangle$ are k, k^y, k^{y2}, then $\langle k, k^y\rangle = H$. Hence by maximality of H, we have $N_G(\langle k, k^y\rangle) = H$. Hence $C_G(k) = MX\langle t\rangle$ or $C_G(k) = MX\langle x\rangle\langle t\rangle$. Since $K\times T \leq X$, we have t is not conjugate to any element of $K\times T$ from the structure of $C_G(k)$ as required.

We could stop here, at this stage, by quoting Goldschmidt's results, presented by Thompson (1968), concerning strongly closed abelian 2-subgroups because we have shown that $\langle k, k^y\rangle$ is a strongly closed abelian 2-subgroup of $C_G(k)$. But we can also finish more concisely.

Lemma 4

The element t is conjugate in $C_G(k)$ to some element of $K\times T$.

Proof

By Thompson's Transfer Theorem given by Thompson (1968), t is conjugate in G to some element of $K\times T$, say $t^g \in K\times T$. then both k^g and $\langle k, k^y\rangle$ centralize t^g, so we can chose h in $C_G(t^g)$ so that k^{gh} lies in the same Sylow 2-subgroup of $C_G(t^g)$ as $\langle k, k^y\rangle$. By Lemma 3, a Sylow 2-subgroup of G contains ouly three conjugates of k, so this implies $k^{gh} = k^{y2}$ for some i = 1,2,3. Since $h \in C_G(t^g)$, $t^{ghy^{-1}}$ lies in $K\times T$ and $ghy^{-i} \in C_G(k)$, so t is conjugate in $C_G(k)$ to some element of $K\times T$, proving the lemma.

The contradiction between Lemma 3 and Lemma 4 completes the proof of the theorem. That is, G is not simple.

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