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Groups Satisfying Some Normalizer Conditions

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Abstract: The aim of the study is to characterize all finite groups that satisfy the normalizer conditions stated in this manuscript. Group and character theoretic methods are used in the study and it is proved that such groups are not simple. Specifically, the following result is established. Let a finite group $G$ have a maximal subgroup $H$ satisfying: (I) $H = XP<T>$, where $P = <x, y; x^3 = y^3 = [x, y] = 1>$ and $t^2 = (zt)^2 = 1$ for all $z$ in $H$. (II) $X = R\cdot K\cdot T$, where $R$ has odd order and $y$ acts fixed-point-free on $X$, $K$ and $T$ are 2-groups, $xy$ centralizes $K$ and acts fixed-point-free on $T$; $x$ centralizes $T$ and acts fixed-point-free on $K$, $T\cdot 1$, $K\cdot 1$. (III) $H$ is the only maximal subgroup of $G$ containing $XP$ and $|\Omega(Z(K\cdot T))|\geq 4$. Then $G$ is not a simple group.

Keywords: Maximal subgroup, normalizer conditions, fixed-point-free, centralize, simple group

INTRODUCTION

The problem originated from the study of alternating group $A_7$ on 7 letters, by Liggonah (1977). It is the generalization of the proposition in the proof of the main result established by Liggonah (1977).

The notations used are standard, as used and defined in Gorenstein (1968). Conditions (i) through (iii) are conditions on the subgroup $H$ of $G$, giving the structure of $H$. These are the normalizer conditions referred to in the theorem.

The proof of the theorem is by contradiction. We suppose that $G$ is simple and satisfies the conditions of the theorem and aim at arriving at a contradiction.

The method used in the proof involves studying the fusion of involutions in $H$. By using character and group theoretic methods.

We divide the proof through a series of lemmas. From now on, $G$ is assumed to be simple and satisfies conditions (i) through (iii).

PROOF OF THE THEOREM

Lemma 1

For each class $L$ of involutions of $H$, either there is an element $1 \in L$ such that $l^{-1}yl = y^{-1}$ or $L \subseteq \text{Cl}_y(H)$.

Proof

Conditions (i) to (iii) imply that $N_G(X) = H = N_G(K\cdot T)$. Since $K\cdot T$ admits $<y><t>$ and $y$ acts fixed-point-free, $K\cdot T$ is unique in $(K\cdot T)<t>$ and of class $\leq 2$ by Stephen and Tyrer (1973a, b). Hence $(K\cdot T)<t>$ is a Sylow 2-subgroup of its own normalizer, so it is a Sylow 2-subgroup of $G$. Consider $H_1 := (R\cdot K\cdot T)<y><t>$. The principal 3-block $B_3(H)$ of $H_1$ has the form:
\[
\begin{array}{ccc}
e & y & L \\
1 & 1 & 1 \\
d & 1 & \delta \\
d+1 & -1 & \delta + 1
\end{array}
\]

If \(L^{-1}yL = y^{-1}\) is false for all \(l \in L\), then \(\#(L^{-1} = sy) = 0\) for all \(3\)-elements \(s\) commuting with \(y\), by Higman (1968), where \(\#(L^{-1} = sy)\) is the number of conjugates of \(L\) with product equal to \(sy\). Applying Higman (1968) results, we have

\[
\#(L^{-1} = sy) = |H|/[C_n(y)] \Sigma \chi(f^n) \chi(y) = 0
\]

where, summation is over all characters \(\chi\) in \(B_n(H)\). Then this implies that

\[
\Sigma \chi(y)\chi(f^n) = 0 \text{ giving } 1 + \delta^2/d - (\delta + 1)^2/(d + 1) = 0.
\]

That is, \((\delta - d)^2 = 0\), giving \(\delta = d\). Then \(L\) lies in the kernel of every character in the principal \(3\)-block of \(H\). By Brauer (1964a, b), \(L\) lies in \(O_2^n(H) = O_2^n(H)\) and \(O_2^n(H) = R \lt K \lt T\) and hence \(L \subset O_2^n(H)\) as required.

Hence we have that a class \(L\) of involutions in \(H\), either there is an element \(l \in L\) such that \(L^{-1}yL = y^{-1}\) or \(L \subset O_2^n(H)\).

**Corollary**

The involutions of \(H\) not in \(X = R \lt K \lt T\) are all conjugate (even in \(H\)).

**Proof**

The extended centralizer of \(y\) in \(H\) is \(C_{\delta}(y) = C \langle y \rangle \langle x \rangle\), so all involutions of \(H\), not in \(X\) must be in \(\langle y \rangle \langle x \rangle = D_n\) by Lemma 1 (otherwise they will lie in \(O_2^n(H) = X\)). Using the fact that all involutions of \(D_n\) are conjugate and \(C_{\delta}(y) = P \langle x \rangle\), all involutions of \(H\) not in \(O_2^n(H) = O_2^n(H)\) are conjugate to \(t\) required.

We have seen in the proof of Lemma 1 that \(K \lt T\) is weakly closed in \((K \lt T) \lt T\), a Sylow 2-subgroup of \(G\) and of class \(\leq 2\) and any Involution of \(H\) is conjugate in \(H\) to \(t\) or lies in \(K \lt T\). Since \(O_2^n(Z(K \lt T))\) is characteristic in \(K \lt T\), elements of \(O_2^n(Z(K \lt T))\) are conjugate in \(G\) if they are conjugate in \(N_G(O_2^n(Z(K \lt T)))\).

By maximality of \(H\) and the supposition that \(G\) is simple, we must have that \(N_G(O_2^n(Z(K \lt T)))/H\). In particular, if \(k\) is an element of \(O_2^n(Z(K \lt T))\), its conjugates in \(O_2^n(Z(K \lt T))\) are \(k, k^2, k^3\). Since \(O_2^n(Z(K \lt T))\) is a group of order \(4\), we can always pick \(k\) as not to be conjugate to \(t\) in \(H\). From now on, we assume that such a \(k\) has been picked.

**Lemma 2**

The only conjugates of \(k\) in \(G\) lying in \((K \lt T) \langle t \rangle\) are \(k, k^2, k^3\).

**Proof**

By the corollary and the underlying assumption, any further conjugates \(k^4\) lies in \(K \lt T\), whence \(k, k^2, k^3, k^4\) generate an abelian group. Let \(A\) be a subgroup of \(K \lt T\) chosen such that:

- \(A\) contains the greatest possible number of conjugates of \(k\)
- \(A\) is as large as possible
We first show that the conjugates of \( k \) lying in \( A \) are already conjugate in \( N_G(A) \). Indeed, let \( k \in A \). Then \( k^t \in A^t \) and so \( A^t \leq C_2(k) \). Since \((K \times T)_{\langle t \rangle} \) is a Sylow 2-subgroup of \( G \) and of \( C_2(k) \), we can assume \( A^t \leq (K \times T)_{\langle t \rangle} \). An element of \((K \times T)_{\langle t \rangle} \) not in \( K \times T \) transforms \( k^t \) to \( k^{t^2} \) since it must involve \( t \), so that an abelian subgroup of \((K \times T)_{\langle t \rangle} \) not in \( K \times T \) cannot contain \( k^t \) or \( k^{t^2} \).

Furthermore, the conjugates of \( k \) in \( A^t \) lie in \( K \times T \). Thus, if \( A^t \) is not contained in \( K \times T \), then \((A^t)_{\langle t \rangle} \cap (K \times T)_{\langle t \rangle} \) is an abelian subgroup of \( K \times T \) containing more conjugates of \( k \) than \( A \), contrary to the choice of \( A \). Hence \( A^t \leq K \times T \).

By choice of \( A \), each \( A \) and \( A^t \) are maximal abelian subgroups of \( K \times T \) and since \( K \times T \) is of class at most 2, \((K \times T)^- = Z(K \times T) - A \) and because \([K \times T, A] = (K \times T)^- = A \) and applying theorem 2.1 in Gorenstein (1968), page 18, gives \( A, A^t \) are normal in \( K \times T \). By weak closure of \( K \times T \) in \((K \times T)_{\langle t \rangle} \), it means that \( A \) and \( A^t \) are conjugate in \( N_G(K \times T) \) which is \( H \). Since \( y \) acts fixed-point-free, \( A, A^t \) is abelian by Stephen and Taylor, so it implies \( A = A^t \) by maximality of \( A \). Thus the conjugates of \( A \) in \( H \) are \( A \) and \( A^t \). Replacing \( g \) by \( t g \) if necessary, we can assume \( A = A^t \), which implies \( g \in N_G(A) \). Hence conjugates of \( k \) lying in \( A \) are already conjugate in \( N_G(A) \). A is normalized by \( P \) so \( N_G(A) = X P \) and hence \( N_G(A) = H \) or \( N_G(A) = X P \). This implies the only conjugates of \( k \) in \( A \) are \( k^t \) and \( k^{t^2} \) as required.

**Lemma 3**

The element \( t \) is not conjugate to any element of \( K \times T \) in \( C_2(k) \).

**Proof**

By Lemma 2, the conjugates of \( k \) are \( k, k^t, k^{t^2} \) and \( k^t k^{t^2} = 1 \) because \( y \) fixes \( k k^t k^{t^2} \) and \( y \) acts fixed-point-free. This implies the only conjugates of \( k \) in \( C_2(k) \) are \( k^t \). By Glauberg and Glauberg (1966), it implies that \( k^t \leq Z(C_2(k)) \). This means that for some normal subgroup \( M \) of \( C_2(k) \) of odd order, \( M \) is normal in \( C_2(k) \). By Frattini argument, this gives \( C_2(k) = M N_G(k^t) \). Since \( k, k^t \leq N_G(Z(K \times T)) \) and because of the fact that the conjugates of \( k \) in \( H \) lying in \( (K \times T)_{\langle t \rangle} \) are \( k, k^t, k^{t^2} \), then \( k, k^t \) is normal. Hence by maximality of \( H \), we have \( N_G(k, k^t) = H \). Hence \( C_2(k) = M X < t > \) or \( C_2(k) = MX < t > \). Since \( K \times T \) is not abelian, we have \( t \) is not conjugate to any element of \( K \times T \) from the structure of \( C_2(k) \) as required.

We could stop here, at this stage, by quoting Goldstone’s results, presented by Thompson (1968), concerning strongly closed abelian 2-subgroups because we have shown that \( k, k^t \) is a strongly closed abelian 2-subgroup of \( C_2(k) \). But we can also finish more concisely.

**Lemma 4**

The element \( t \) is conjugate in \( C_2(k) \) to some element of \( K \times T \).

**Proof**

By Thompson’s Transfer Theorem given by Thompson (1968), \( t \) is conjugate in \( G \) to some element of \( K \times T \), say \( t^* \in K \times T \). Then both \( k^t \) and \( k, k^t \) centralize \( t^* \), so we can choose \( h \) in \( C_2(t^*) \) so that \( k^h \) lies in the same Sylow 2-subgroup of \( C_2(t^*) \) as \( k, k^t \). By Lemma 3, a Sylow 2-subgroup of \( G \) contains only three conjugates of \( k \), so this implies \( k^i = k^{i+2} \) for some \( i = 1, 2, 3 \). Since \( h \in C_2(t^*) \), \( t^{h^{i-1}} \) lies in \( K \times T \) and \( g h t^{e^{i-1}} \in C_2(k) \), so \( t \) is conjugate in \( C_2(k) \) to some element of \( K \times T \), proving the lemma.

The contradiction between Lemma 3 and Lemma 4 completes the proof of the theorem. That is, \( G \) is not simple.
REFERENCES