Characterization of Boolean Valued Star and Mega Lattice Functions

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ABSTRACT

This paper is a study on Boolean valued star and mega lattice functions. It establishes a positive lattice measure, complex lattice measure, star lattice function and founds that every star lattice function is a positive lattice measure on \( \sigma(L) \) and every star lattice function is a \( \sigma \)-additive on \( X \). Also, it classifies mega lattice function, mega lattice and demonstrates that in a mega lattice the limit of the integral is equal to the integral of the limit and every mega lattice is a \( \sigma \)-additive. Finally, it confirms that every mega lattice preserves Fatou’s lemma.

Key words: Lattice, measure, \( \sigma \)-algebra, lattice simple function and measurable functions

INTRODUCTION

The concept of function lattice was distinguished by Rudin (1987). Anger and Portemnier (1992) introduced the concept of lattice measurable function. Tanaka (2009) has established the notion of Hahn Decomposition Theorem of Signed Lattice. Recently, Anil Kumar et al. (2011) made a Characterization of Class of Measurable Borel Lattices and introduced the concept of lattice \( \sigma \)-Algebra \( \sigma(L) \). Also, Kumar et al. (2011) introduced the notion of lattice boolean valued measurable functions and made contribution on the concepts of characteristic function of a lattice and \( \delta \)-lattice function.

In this study, we establish a general framework for the study of characterization of Boolean valued star lattice functions and mega lattice functions. Here, some concepts in measure theory can be generalized by means of lattice \( \sigma \)-Algebra \( \sigma(L) \).

We prove that every star lattice function is a positive lattice measure on \( \sigma(L) \) and the star lattice function is a \( \sigma \)-additive on \( X \). Also we establish that in a mega lattice the limit of the integral is equal to the integral of the limit and every mega lattice is a \( \sigma \)-additive. And finally we conclude that the mega lattice preserves Fatou’s lemma and strict inequality occurs in Fatou’s lemma.

PRELIMINARIES

Here, we shall briefly review the well-known facts about lattice theory specified by Birkhoff (1967).

\((L, \wedge, \vee)\) is called a lattice if it is enclosed under operations \( \wedge \) and \( \vee \) and satisfies, for any elements \( x, y, z \), in \( L \):

- (L1) commutative law: \( x \wedge y = y \wedge x \) and \( x \vee y = y \vee x \)
- (L2) associative law: \( x \wedge (y \wedge z) = (x \wedge y) \wedge z \) and \( x \vee (y \vee z) = (x \vee y) \vee z \)
• (L3) absorption law: $x \vee (y \wedge x) = x$ and $x \wedge (y \vee x) = x$. Here, after, the lattice $(L, \wedge, \vee)$ will often be written as $L$ for simplicity. A lattice $(L, \wedge, \vee)$ is called distributive if, for any $x, y, z$ in $L$

• (L4) distributive law holds: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

A lattice $L$ is called complete if, for any subset $A$ of $L$, $L$ contains the supremum $\vee A$ and the infimum $\wedge A$. If $L$ is complete, then $L$ itself includes the maximum and minimum elements which are often denoted by $1$ and $0$ or $1$ and $0$, respectively.

A distributive lattice is called a Boolean lattice if for any element $x$ in $L$, there exists a unique complement $x^c$ such that:

• (L5) the law of excluded middle: $x \vee x^c = 1$
• (L6) the law of non-contradiction: $x \wedge x^c = 0$

Let $L$ be a lattice and $\varepsilon: L \rightarrow L$ be an operator. Then $\varepsilon$ is called a lattice complement in $L$ if the following conditions are satisfied:

• (L5) and (L6): $\forall x \in L, x \vee x^c = 1$ and $x \wedge x^c = 0$
• (L7) the law of contrapositive: $\forall x, y \in L, x < y$ implies $x^c > y^c$
• (L8) the law of double negation: $\forall x \in L, (x^c)^c = x$

Throughout this study, we consider lattices as complete lattices which obey (L1)-(L8) except for (L6) the law of non-contradiction. Unless otherwise stated, $X$ is the entire set and $L$ is a lattice of any subsets of $X$.

**Definition 1:** If a lattice $L$ satisfies the following conditions, then it is called a lattice $\sigma$-Algebra:

• $\forall h \in L, h' \in L$
• if $h_n \in L$ for $n = 1, 2, 3, ..., then \bigvee_{n=1}^{\infty} h_n \in L$

We denote $\sigma (L)$, as the lattice $\sigma$-Algebra generated by $L$ and ordered pair $(X, \sigma (L))$ is said to be lattice measurable space.

**Note 1:** By Definition 1, it is clear that $\sigma (L)$ is closed under finite unions and finite intersections.

**Definition 2:** Let $\sigma (L)$ be a lattice $\sigma$-algebra of subset sets of a set $X$. A function:

$\mu: \sigma (L) \rightarrow [0, \infty]$ is called a positive lattice measure defined on $\sigma (L)$ if:

• $\mu (\emptyset) = 0$
• $\mu \left( \bigvee_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu (A_i)$

**Definition 3:** A function lattice is a collection $L'$ of extended real valued functions defined on a lattice $L$ with respect to usual partial ordering on functions. That is if $f, g \in L'$ then $f \vee g \in L'$, $f \wedge g \in L'$.

**Definition 4:** If $f$ and $g$ are extended real valued lattice measurable functions defined on $L'$, then $f \vee g, f \wedge g$ are defined by $(f \vee g)(x) = \sup \{f(x), g(x)\}$ and $(f \wedge g) (x) = \inf \{f (x), g (x)\}$ for any $x \in L$. 

Definition 5: A complex positive lattice measure is a complex-valued countably additive lattice function defined on a lattice $\sigma$-algebra $\sigma (L)$.

Definition 6: A function $s$ on a lattice measurable space $X$ whose range consists of only finitely many points in $[0, \infty]$ is called a simple lattice function.

Note 2: Kumar et al. (2011): Every simple lattice function is lattice measurable.

Definition 7: Let $\sigma (L)$ be a lattice $\sigma$-algebra on $X$. Let $\mu$ be a positive lattice measure on $\sigma (L)$. Let $s$ be a simple lattice function on $X$ of the form:

$$ s = \sum_{i=1}^{n} a_i \chi_{A_i} $$

where, $a_1, a_2, \ldots, a_n$ are the distinct values of $s$ and $A_i = \{ x \in X | s(x) = a_i \} \ 1 = i = n$. Let $E \in \sigma (L)$ then we define, a star lattice function:

$$ \varphi(E) = \int_{E} s \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i \cap E) $$

Definition 8: Let $f: X \rightarrow [0, \infty]$ be lattice measurable function. Let $E \in \sigma (L)$. Then $\int_{E} f \, d\mu$ is defined as $\sup \{ \int_{E} s \, d\mu \}$ where the supremum being taken over all simple lattice functions $s$ such that $0 = s = f$. We now call $\int_{E} f \, d\mu$ is a mega lattice function over $E$ with respect to the lattice measure $\mu$ and $X$ is called mega lattice.

Definition 9: An extended real valued function $f$ defined on a lattice measurable set $E$ is said to be lattice measurable function if the set $f^{-1} = \{ x \in E | f(x) > \alpha \}$ is a lattice measurable for every real number $\alpha$.

Definition 10: Kumar et al. (2011): Countable intersection of lattice measurable functions is called a $\delta$-lattice function.

Note 3: Kumar et al. (2011): Every $\delta$-lattice function is lattice measurable.

Definition 11: Let $A$ be a lattice. The characteristic function $\chi_{A}$ of a lattice $A$ is defined by $\chi_{A} (x) = 1$ if $x \in A$, $0$ if $x \notin A$.

Definition 12: If $\mu$ is a positive lattice measure on $\sigma (L)$ then the numbers of $\sigma (L)$ are called positive lattice measurable sets or simply positive lattice measurable.

Theorem 1: Rudin (1987): If $f_{n}: X \rightarrow [-\infty, \infty]$ is measurable. For $n = 1, 2, 3, \ldots$ and $h = \lim f_{n}$ then $h$ is measurable.

Theorem 2: Rudin (1987): Let $f: X \rightarrow [0, \infty]$ be measurable. Then there exists simple lattice measurable functions $S_{n}$ on $X$ such that:
• 0 ≤ S_1 ≤ S_2 ≤ ... ≤ f
• S_α (x) → f (x) as n → ∞ for every x ∈ X

**Theorem 3: Rudin (1987):** Outer measure of an interval is its length.
That is for an interval I, μ(I) = 1(I).

**Result 1: Rudin (1987):** Let E = [a, b]. Now:
\[
\int_{E} dμ = \int_{a}^{b} dμ = [μ]^b_a = b - a = l([a, b]) = l(E) = μ(E)
\]

**CHARACTERIZATIONS OF STAR AND MEGA LATTICE FUNCTIONS**

**Theorem 4:** Every star lattice function is a positive lattice measure on σ (L).

**Proof:** Let φ be a star lattice function. We prove φ is a positive lattice measure on σ (L).

Let’s be a simple lattice function and:
\[
S = \sum_{n=1}^{N} a_n x_{n,n^*}
\]

Let:
\[
E ∈ σ(L) \text{ then } φ(E) = \sum_{n=1}^{N} a_n μ(A_n ∧ E) ≥ 0
\]
(by definition 7). Thus φ: σ (L) → [0, ∞].

To show φ is countably additive. Let \( \{E_n\}_{n=1}^{\infty} \) be a disjoint countable collection of lattice measurable sets of σ (L). Let \( E = \bigvee_{n=1}^{\infty} E_n \). Then:
\[
A_i ∧ E = A_i ∧ (\bigvee_{n=1}^{\infty} E_n) = \bigvee_{n=1}^{\infty} (A_i ∧ E_n)
\]

Therefore:
\[
μ(A_i ∧ E) = μ(\bigvee_{n=1}^{\infty} (A_i ∧ E_n)) = \sum_{n=1}^{\infty} μ(A_i ∧ E_n).
\]

Therefore:
\[
φ(E) = \sum_{n=1}^{\infty} a_n μ(A_i ∧ E) = \sum_{n=1}^{\infty} a_n \sum_{n=1}^{\infty} μ(A_i ∧ E_n) = \sum_{n=1}^{\infty} \left( \int_{E} dμ \right) = \sum_{n=1}^{\infty} φ(E_n).
\]

Hence, φ is countably additive. Also:
\[
φ(ϕ) = \int_{E} dμ = \sum_{n=1}^{\infty} a_n μ(A_i ∧ ϕ) = \sum_{n=1}^{\infty} a_n μ(ϕ) = 0.
\]
Hence, $\varphi$ is not identically $\infty$. Thus, $\varphi$ is a positive lattice measure on $\varnothing$ (L).

**Theorem 5:** Every star lattice function is a $\varnothing$-additive on $X$.

**Proof:** Let's and $t$ be simple lattice functions given by:

$$S = \sum_{i=1}^{n} a_i \chi_{A_i}, \text{ and } T = \sum_{j=1}^{m} \beta_j \chi_{B_j}.$$  

Let, $E_{ij} = A_i \wedge B_j, 1 \leq i \leq n, 1 \leq j \leq m$. Therefore, for any $x \in E_{ij}, s(x) = a_i, t(x) = \beta_j$. Hence, $(s+t)(x) = s(x) + t(x) = a_i + \beta_j$. Therefore:

$$\int_{E_{ij}} (s+t) d\mu = (a_i + \beta_j) \mu(E_{ij})$$  

(1)

Since, $s+t$ is a simple lattice function with values $a_i + \beta_j$ in $E_{ij}, 1 \leq i \leq n, 1 \leq j \leq m$. Note that:

$$\bigvee_{i=1}^{n} A_i \wedge \bigvee_{j=1}^{m} B_j = X$$

$$\int_{E_{ij}} s d\mu + \int_{E_{ij}} t d\mu = \sum_{i=1}^{n} a_i \mu(A_i \wedge E_{ij}) + \sum_{j=1}^{m} \beta_j \mu(B_j \wedge E_{ij}) = \mu(\bigcup_{i=1}^{n} A_i \wedge \bigcup_{j=1}^{m} B_j \wedge E_{ij}) = \mu(X \wedge E_{ij})$$

Hence:

$$\int_{E_{ij}} (s+t) d\mu = \int_{E_{ij}} s d\mu + \int_{E_{ij}} t d\mu$$  

(2)

Since, $E_{ik} \wedge A_i$ and $E_{kj} \wedge B_j$ if $k \neq i, E_{ik} \wedge A_i = \emptyset$, if $1 \neq j, E_{kj} \wedge B_j = \emptyset$.

Now, $X$ is a disjoint union of lattice measurable sets $E_{ij} (1 \leq i \leq n, 1 \leq j \leq m)$. Therefore:

$$f_{ij}(x) = \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij}(E_{ij})$$

Since, $\varphi$ is positive lattice measure on $\varnothing$ (L). Therefore:

$$\int_{X} (s+t) d\mu = \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{E_{ij}} (s+t) d\mu = \sum_{i=1}^{n} \sum_{j=1}^{m} \left( \int_{E_{ij}} s d\mu + \int_{E_{ij}} t d\mu \right) = \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{E_{ij}} s d\mu + \int_{E_{ij}} t d\mu = \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij}(E_{ij}) + \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij}(E_{ij}) = \sum_{i=1}^{n} f_i(X) + \sum_{j=1}^{m} f_j(X) = \int_{X} s d\mu + \int_{X} t d\mu.$$

Therefore, every star lattice function is a $\varnothing$-additive on $X$.  

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Lemma 1: Let \( s = f = c \) be any simple lattice measurable function such that \( 0 = s = f \) and \( c \) be any constant, such that \( 0 \leq c \leq 1 \). Define \( E_n = \{ x \in A_n | x = c.s \} \) for \( n = 1, 2, 3, ... \) then each \( E_n \) is lattice measurable set and \( X = \bigvee E_n \).

Proof: Claim (1): To show that each \( E_n \) is a lattice measurable set.

Let:

\[
S = \sum_{i=1}^{n} a_i \chi_{A_i}
\]

for all \( x \in A_i \), \( 1 \leq i \leq n \) implies \( c.s \) \( (x) = c \alpha_i \), for all \( x \in A_i \), \( 1 \leq i \leq n \).

Then:

\[
E_n = \bigwedge_{i=1}^{n} f_i^{-1}[\alpha_i, \infty]
\]

Since, each \( f_i^{-1}[\alpha_i, \infty] \) is a lattice measurable implies \( E_n \) is a lattice measurable set (by definition 10 and by note 3).

Claim 2: To show that \( E_n < E_{n+1} \).

Let, \( E_n < E_n < \ldots \), \( x \in E_n \). Then \( f_n(x) = c.s \) \( (x) \). But \( f_{n+1}(x) \geq f_n(x) \). Therefore, \( f_{n+1}(x) = c.s \) \( (x) \) implies \( x \in E_{n+1} \). Therefore, \( E_n < E_{n+1} \).

Claim 3: To show that \( X = \bigvee E_n \). Let \( x \in X \). If \( f(x) = 0 \) then \( s(x) = 0 \) (Since, \( s = f \)). As \( f_1(x) \geq 0 \) we get \( x \in E_1 \) (In fact \( x \in E_n \) for all \( n \)). If \( f(x) > 0 \) then \( c.s(x) < f(x) \) (Since, \( s \leq f \) and \( 0 \leq c \leq 1 \)).

Suppose \( f_n(x) < c.s(x) \) for all \( n \), then \( \lim f_n(x) = c.s(x) \) that is \( f(x) = c.s(x) \) a contradiction to the fact that \( c.s(x) < f(x) \). Hence, for some \( n \), \( f_n(x) = c.s(x) \), that is \( x \in E_n \), for all \( n \). Thus \( X = \bigvee E_n \).

Theorem 6: In a mega lattice the limit of the integral is equal to the integral of the limit.

Proof: Let \( \{ f_n \} \) be a sequence of lattice measurable functions on \( X \) such that \( 0 \leq f_1(x) \leq f_2(x) \ldots \leq f_n(x) \) for every \( x \in X \) and \( f_n(x) \rightarrow f(x) \) as \( n \rightarrow \infty \) for every \( x \in X \).

Since, \( f_n \leq f_{n+1} \):

\[
\int_X f_n \, d\mu \leq \int_X f_{n+1} \, d\mu
\]

for all \( n \). Hence, \( \{ f_n \} \) is an increasing sequence of non-negative real numbers there exists an \( \alpha \in [0, \infty] \) such that:

\[
\int_X f_n \, d\mu \rightarrow \alpha \text{ as } n \rightarrow \infty \quad (3)
\]

Since, \( \{ f_n \} \) is a sequence of lattice measurable functions and since, \( f_n \rightarrow f \), \( f \) is a lattice measurable function (by Theorem 1).
Since, \( f_n \leq f \), we have \( \int_X f_n \, d\mu \leq \int_X f \, d\mu \) for all \( n \). So from Eq. 1 we get:

\[
a \leq \int_X f \, d\mu
\]  

(4)

By Lemma 1 we have \( E_n < X \) and c.s \( \leq f_n \) on \( E_n \) we get:

\[
\int_X f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq \int_{E_n} c.s \, d\mu = c \int_{E_n} s \, d\mu (n=1, 2, 3, \ldots)
\]  

(5)

Let \( n \to \infty \). As \( E_1 < E_2 < \ldots \) and \( X = \bigcup_{i=1}^{\infty} E_i \), \( E_i \in \sigma(L) \). We get \( \varphi(E) \to \varphi(E) \) where, \( \varphi \) is a positive lattice measure on \( \sigma(L) \) given by \( \varphi(E_n) = \int_{E_n} s \, d\mu \) that is \( \int_X s \, d\mu \to \int_X s \, d\mu \) (by theorem 2, 4 and 5). Therefore, from Eq. 3 we have lim:

\[
\int_X f_n \, d\mu \geq \int_X s \, d\mu \quad \text{that is} \quad a \leq \int_X s \, d\mu
\]  

(6)

Since, Eq. 4 holds for every c, \( 0 < c < 1 \) we get \( a \geq \int_X s \, d\mu \). This is true for every simple lattice measurable function \( s \) satisfying \( 0 \leq s \leq f \).

Hence:

\[
a \geq \sup_{s \in \mathcal{S}} \int_X s \, d\mu = \int_X f \, d\mu
\]  

(7)

From Eq. 2 and 5 we get \( a = \int_X f \, d\mu \). Hence, \( \lim \int_X f_n \, d\mu = \int_X f \, d\mu \). Implies \( \lim \int_X f_n \, d\mu = \int_X (\lim f_n) \, d\mu \) (Since, \( f_n \to f \) implies \( \lim f_n = f \)).

Therefore, the limit of the integral is equal to the integral of the limit.

**Theorem 7:** Every mega lattice is a \( \sigma \)-additive.

**Proof:** Let \( f_n : X \to [0, \infty] \) is a lattice measurable function for \( n = 1, 2, 3, \ldots \) and:

\[
f(x) = \sum_{n=1}^{\infty} f_n(x)
\]

for \( x \in X \) to prove:

\[
\int_X f \, d\mu = \int_X \sum_{n=1}^{\infty} f_n \, d\mu.
\]

First we prove:

\[
\int_X f_1 + f_2 + \ldots f_n \, d\mu = \int_X f_1 \, d\mu + \int_X f_2 \, d\mu + \ldots \int_X f_n \, d\mu
\]
by induction: Let \( n = 2 \). Let \( f_1, f_2 \) be lattice measurable functions then (by theorem 2) there exists monotonic increasing sequences \( \{S_1\} \) and \( \{S_2\} \) of simple lattice measurable non negative functions such that \( S_1 \rightarrow f_1 \) and \( S_2 \rightarrow f_2 \). Let \( S_1 = S_1 + S_2 \). By theorem 7 we have:

\[
\int \frac{(S_1 + S_2)}{x} \, d\mu = \int \frac{S_1}{x} \, d\mu + \int \frac{S_2}{x} \, d\mu.
\]

Therefore, \( \lim \)

\[
\int \frac{S_1}{x} \, d\mu = \lim \int \frac{S_1}{x} \, d\mu + \lim \int \frac{S_2}{x} \, d\mu.
\]

That is:

\[
\int \frac{(f_1 + f_2)}{x} \, d\mu = \int \frac{f_1}{x} \, d\mu + \int \frac{f_2}{x} \, d\mu.
\]

By Theorem 6.

Assuming the result for \( n, n \geq 2 \) we can easily get the result for \( n + 1 \). Thus by induction we get Eq. 1. Let \( g_{n} = f_1 + f_2 + \ldots + f_n \), as \( \sum_{i=0}^{n} f_n(x) = f(x) \) and \( f_n(x) \geq 0 \) we get \( \{g_{n}\} \) converges monotonically to \( f \).

Also by Eq. 1:

\[
\int \frac{g_n}{x} \, d\mu = \sum_{n=1}^{N} \int \frac{f_n}{x} \, d\mu.
\]

By Theorem 6:

\[
\int \frac{f}{x} \, d\mu = \lim \int \frac{g_n}{x} \, d\mu = \sum_{n=1}^{\infty} \int \frac{f_n}{x} \, d\mu.
\]

Hence, every mega lattice is a \( \sigma \)-additive.

**Corollary 1:** For \( a_{i} \geq 0 \) for \( i \) and \( j = 1, 2, 3, \ldots \) then:

\[
\sum_{i=1}^{\infty} \frac{a_i}{x} = \sum_{i=1}^{\infty} a_i.
\]

**Proof:** Let \( X \) be the set \( N \) of natural numbers. Let \( \mu \) be the lattice counting measure on \( N \). Let \( f_n: N \rightarrow [0, \infty] \) be defined by \( f_n(m) = a_{m} \) that is \( f_n(1) = a_1, f_n(2) = a_2, \ldots \).

Let:

\[
f(n) = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n + \ldots = \sum_{i=1}^{n} a_i.
\]

Now \( f_n, f \) are lattice measurable functions with respect to lattice counting measure. Now:
\begin{align*}
\int f \, d\mu &= \sum_{n=1}^{\infty} f(n) = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}, \\
\text{Now:} \\
\int f_n \, d\mu &= \sum_{m=1}^{\infty} f_n(m) = \sum_{m=1}^{\infty} a_{nm} = \sum_{m=1}^{\infty} a_{nm}, \\
\text{Therefore:} \\
\sum_{n=1}^{\infty} \int f_n \, d\mu &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mj}, \\
\text{But:} \\
\int f \, d\mu &= \sum_{n=1}^{\infty} \int f_n \, d\mu
\end{align*}

(from the theorem 7). Hence, we get:

\begin{align*}
\sum_{m=1}^{\infty} \sum_{j=1}^{\infty} a_{mj} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}.
\end{align*}

**Theorem 8:** Every mega lattice preserves Fatou's Lemma.

**Proof:** Let $X$ be a mega lattice such that $f_x: X \rightarrow [0, \infty]$ be a mega lattice function for each positive integer $n$ we prove $\int (\liminf f_n) \, d\mu \leq \liminf \int f_n \, d\mu$. Let $g_k(x) = \inf_{x \in X} (k=1,2,3,\ldots, \forall x \in X)$. Then $g_k$ is a lattice measurable function for each $k$. Also:

\begin{align*}
g_k \leq f_k \text{ so that } \int g_k \, d\mu \leq \int f_k \, d\mu (k=1,2,3,\ldots)
\end{align*}

Also:

\begin{align*}
0 \leq g_1 \leq g_2 \leq \ldots \text{ and } g_k(x) \rightarrow \lim \inf f_n(x) \text{ as } k \rightarrow \infty
\end{align*}

By theorem 6 we have lim:

\begin{align*}
\lim \int g_k \, d\mu &= \int (\lim \inf f_n) \, d\mu \text{ (from Eq. 2)}
\end{align*}

Using Eq. 1 we get:

\[ \lim \int_{X} f_{n} \, d\mu \leq \lim\inf \int_{X} f_{n} \, d\mu. \]

From Eq. 3-4 we get:

\[ \int \left( \lim\inf f_{n} \right) \, d\mu \leq \lim\inf \int_{X} f_{n} \, d\mu. \]

**Remark 2:** Strict inequality occurs in Fatou's lemma.

**Proof:** Let \( f_{n} : X \to [0, \infty] \), put \( f_{n}^{+} = \chi_{E} \) if \( n \) is odd and \( f_{n}^{-} = 1 \cdot \chi_{E} \) if \( n \) is even for some lattice measurable set \( E \subseteq X \). That is: if \( n \) is odd \( f_{n}(x) = 1 \) if \( x \in E \), 0 if \( x \notin E \) and if \( n \) is even \( f_{n}(x) = 0 \) if \( x \in E \), 1 if \( x \notin E \). Now \( \lim f_{n} = 0 \). Therefore:

\[ \int \left( \lim\inf f_{n} \right) \, d\mu = 0 \text{ implies } \int_{X} f_{n} \, d\mu = \int_{X} \chi_{E} \, d\mu \]

if \( n \) is odd = 1 and:

\[ \int_{X} f_{n} \, d\mu = \int_{X} \chi_{X \setminus E} \, d\mu = 1 \]

if \( n \) is even. Hence, \( \lim\inf \int_{X} f_{n} \, d\mu = 1 \). Therefore:

\[ \int \left( \lim\inf f_{n} \right) \, d\mu < \lim\inf \int_{X} f_{n} \, d\mu. \]

**REFERENCES**


