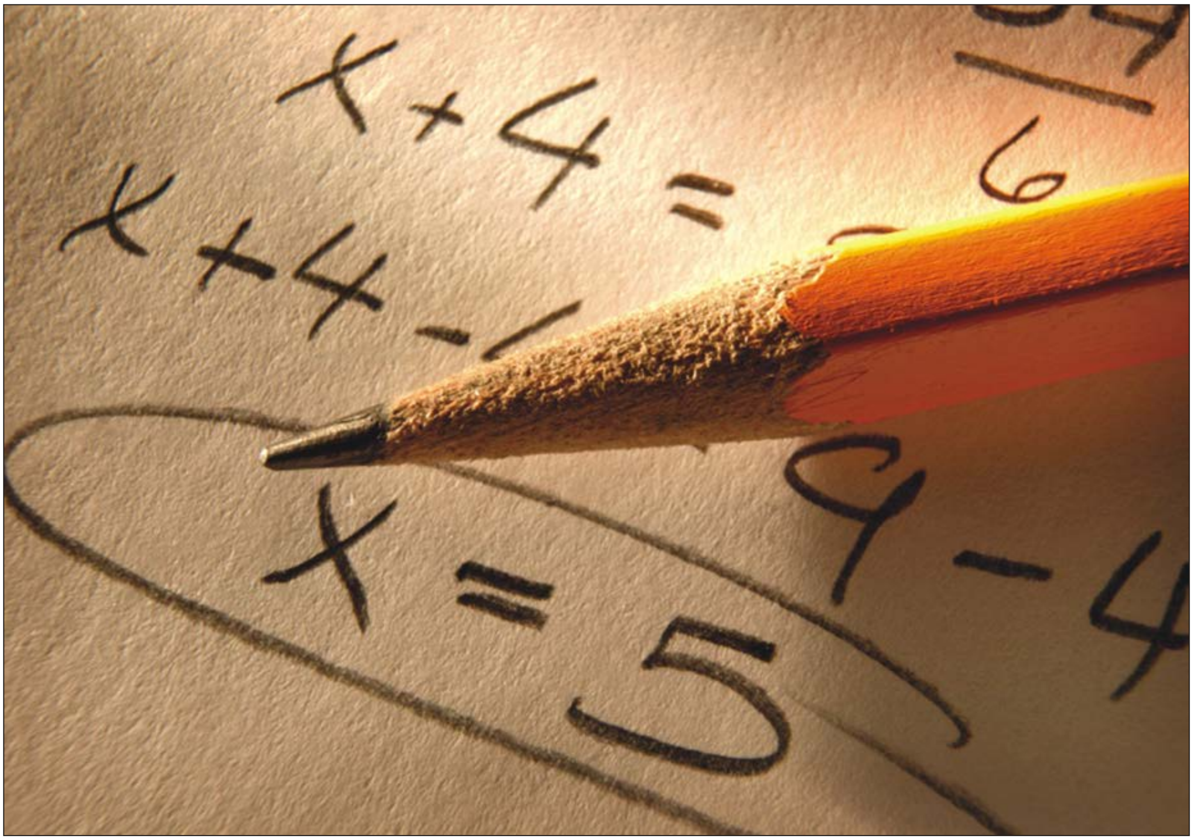

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Characterization of Lattice Measurable Functions on Product Lattices

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ABSTRACT

This study introduce and advance the characteristics of S-lattice measurable function and T-lattice measurable function. It has been proved that the integrations of these lattice measurable functions are made equal. Also it establishes the result that the two iterated integrals of lattice measurable functions are finite and equal. Finally, it confirms that for the lattice σ-finiteness, the Lebesgue lattice measure cannot be omitted and the condition that f is a lattice measurable with respect to the lattice σ-algebra.

Key words: Lattice σ-algebra, measure, lattice measure, σ-finite measure, product lattice measurable functions and lattice σ-finiteness

INTRODUCTION

The concept of lattice measure was initiated by Gabor (1964). For the extension of this theme literature took some time. Afterward, the concepts of lattice sigma algebra and lattice measure on a lattice sigma algebra launched by Tanaka (2009). Recently the concepts of lattice measurable set, lattice measure space and lattice σ-finite measure were established by Kumar *et al.* (2011a, b).

The perception of measurable Borel lattices was introduced and studied by Kumar *et al.* (2011a). Further Radon-Nikodym theorem for signed lattice measure was expanded by Kumar *et al.* (2011a). A class of super lattice measurable sets was introduced by Pramada *et al.* (2011). Lebesgue decomposition and its uniqueness of a signed lattice measure were studied successfully by Kumar *et al.* (2012). A class of positive lattice measurable sets and positive lattice measurable functions was obtained by Pramada *et al.* (2012a). A characterization of complex integrable lattice functions and μ-free lattices was made by Pramada *et al.* (2012c). Further recently a characterization of boolean valued star and mega lattice functions was obtained by Pramada *et al.* (2012b).

This manuscript is aimed to the study of concept of product lattice measurable functions and their various characterizations. In particular these functions are observed by defined over topological spaces. Also it has been investigated the characteristics of S-lattice measurable function and T-lattice measurable functions. The concept of iterated integral of a product lattice measurable function has been defined in order to identify that the two iterated integrals of a product lattice measurable function are finite and equal. It is also aimed to get a condition that product lattice measurable function is lattice measurable is obtained and the condition that product lattice measurable function is lattice measurable cannot be dropped.

PRELIMINARIES

This section briefly reviews the well-known facts of Birkhoff (1967) lattice theory.

The system (L, $\wedge$, $\vee$), where L is a non empty set, $\wedge$ and $\vee$ are two binary operations on L, is called a lattice if $\wedge$ and $\vee$ satisfies, for any elements x, y, z, in L:

- (L1) commutative law: $x\wedge y = y\wedge x$ and $x\vee y = y\vee x$
- (L2) associative law: $x\wedge(y\wedge z) = (x\wedge y)\wedge z$ and $x\vee(y\vee z) = (x\vee y)\vee z$
- (L3) absorption law: $x\vee(y\wedge x) = x$ and $x\wedge(y\vee x) = x$. Hereafter, the lattice (L, $\wedge$, $\vee$) will often be written as L for simplicity. A lattice (L, $\wedge$, $\vee$) is called distributive if, for any x, y, z, in L
- (L4) distributive law holds: $x\vee(y\wedge z) = (x\vee y) \wedge(x\vee z)$ and $x\wedge(y\vee z) = (x\wedge y)\vee(x\wedge z)$

A lattice L is called complete if, for any subset A of L, L contains the supremum $\vee$ A and the infimum $\wedge$ A. If L is complete, then L itself includes the maximum and minimum elements which are often denoted by 1 and 0 or I and O, respectively.

A distributive lattice is called a Boolean lattice if for any element x in L, there exists a unique complement x^c such that:

$x\vee x^c = 1$ (L5) the law of excluded middle
$x\wedge x^c = 0$ (L6) the law of non-contradiction

Let L be a lattice and c: L→L be an operator. Then c is called a lattice complement in L if the following conditions are satisfied.

(L5) and (L6):	$\forall x\in L$, $x\vee x^c = 1$ and $x\wedge x^c = 0$
(L7) the law of contrapositive:	$\forall x, y\in L$, $x<y$ implies $x^c>y^c$
(L8) the law of double negation:	$\forall x\in L (x^c)^c = x$

Definition 1: If a lattice L satisfies the following conditions, then it is called a lattice s-algebra:

(1) for all $h\in L$, $h^c\in L$
(2) if $h_n \in L$ for n = 1, 2, 3 ....., then:

$$\bigvee_{n=1}^{\infty} h_n \in L$$

We denote σ(L) = ß, as the lattice σ-algebra generated by L.

Example 1 (Halmos, 1974): 1: {Φ, X}(that is the empty set together with entire set) is a lattice σ-algebra. 2: P (X) power set of any nonempty set X is a lattice σ-algebra.

Example 2: Let X = $\Re$ and L = {measurable subsets of $\Re$} with usual ordering (≤).

Here, L is a lattice and $\sigma(L) = ß$ is a lattice σ-algebra generated by L, where, $\Re$ is a extended real number system.

Example 3: Let X be any non-empty set, L = {All topologies on X}. Here, L is a complete lattice but not a σ-algebra.

Example 4 (Halmos, 1974): Let $X = \Re$ and $L = \{E \subset \Re / E$ is finite or E^c is finite$\}$.

Here, L is lattice algebra but not lattice σ-algebra.

Definition 2: The entire set X together with a lattice σ-algebra ß is said to be lattice measurable space, it is denoted by the ordered pair (X, ß).

Example 5: Let $X = \Re$ and L = {All Lebesgue measurable sub sets of $\Re$}. Then it can be verified that ($\Re$, ß) is a lattice measurable space.

Definition 3: If the mapping μ: ß$\rightarrow R \cup \{\infty\}$ satisfies the following properties, then μ is called a lattice measure on the lattice σ-algebra $\sigma(L)$.

(1) $\mu(\varphi) = \mu(0) = 0$
(2) For all h, g$\in$ß, such that $\mu(h)$, $\mu(g) \geq 0$ and $h \leq g \Rightarrow \mu(h) \leq \mu(g)$
(3) For all h, g$\in$ß, $\mu(h \vee g) + \mu(h \wedge g) = \mu(h) + \mu(g)$
(4) If $h_n \subset$ ß, $n \in N$ such that $h_1 \leq h_2 \leq \dots \leq h_n \leq \dots$, then:

$$\mu\left(\overset{\infty}{\underset{n=1}{\vee}} h_n\right) = \lim \mu(h_n)$$

Note 1: Let μ_1 and μ_2 be lattice measures defined on the same lattice σ-algebra ß. If one of them is finite, then the set function $\mu(E) = \mu_1(E) - \mu_2(E)$, E$\in$ß is well defined and is countably additive on ß.

Example 6 (Royden, 1981): Let X be any set and ß = P(X) be the class of all sub sets of X. Define for any A$\in$ß, $\mu(A) = +\infty$ if A is infinite = $|A|$ if A is finite, where $|A|$ is the number of elements in A.

Then μ is a countable additive set function defined on ß and hence μ is a lattice measure on ß.

Definition 4: A set A is said to be lattice measurable set or lattice measurable if A belongs to ß.

Example 7 (Kumar *et al.*, 2011a): The interval (a, ∞) is a lattice measurable under usual ordering.

Example 8 (Kumar *et al.*, 2011a, b): The closed interval $[0, 1] < \Re$ is lattice measurable under usual ordering.

Let $X = \Re$, L= {Lebesgue measurable subsets of $\Re$} with usual ordering ($\leq$) clearly $\sigma(L)$ is a lattice σ-algebra generated by L. Here, [0, 1] is a member of $\sigma(L)$. Hence, it is a Lattice measurable set.

Example 9 (Kumar *et al*., 2011a, b): Every Borel lattice is a lattice measurable.

Definition 5: The lattice measurable space (X, ß) together with a lattice measure μ is called a lattice measure space and it is denoted by (X, **β**, μ).

Example 10: Suppose $\Re$ is a set of real numbers μ is the lattice Lebesgue measure on $\Re$ and ß is the family of all Lebesgue measurable subsets of real numbers. Then ($\Re$, ß, μ) is a lattice measure space.

Example 11: Let $\Re$ be the set of real numbers, ß be the class of all Borel lattices and μ be a lattice Lebesgue measure on $\Re$. Then ($\Re$, ß, μ) is a lattice measure space.

Definition 6: Let (X, ß, μ) be a lattice measure space. If μ(X) is finite, then μ is called lattice finite measure.

Example 12: The lattice Lebesgue measure on the closed interval [0, 1] is a lattice finite measure.

Example 13: When a coin is tossed, either head or tail comes when the coin falls. Let us assume that these are the only possibilities. Let X = {H, T}, H for head and T for tail. Let ß = {φ, {H}, {T}, X}. Define the mapping P: ß→[0, 1] by P (φ) = 0 P ({H}) = P ({T}) = ½, P (X) = 1. Then P is a lattice finite measure on the lattice measurable space (X, ß).

Definition 7: If μ is a lattice finite measure, then (X, ß, μ) is called a lattice finite measure space.

Example 14: Let ß be the class of all Lebesgue measurable sets of [0, 1] and μ be a lattice Lebesgue measure on [0, 1]. Then ([0, 1], ß, μ) is a lattice finite measure space.

Definition 8: Let (X, ß, μ) be a lattice measure space. If there exists a sequence of lattices measurable sets $\{x_n\}$ such that (i) $X = \overset{\infty}{\underset{n=1}{\vee}} x_n$ and (ii) $\mu(x_n)$ is finite, then μ is called a lattice σ-finite measure.

Example 15: The lattice Lebesgue measure on ($\Re$, μ) is a lattice σ-finite measure since:

$$\Re = \overset{\infty}{\underset{n=1}{\vee}} (-n,\ n)$$

and μ(-n,n) = 2n is finite for every n.

Definition 9: If μ be a lattice σ-finite measure, then (X, ß, μ) is called lattice σ-finite measure space.

Example 16: Let ß be the class of all Lebesgue measurable sets on:

$$\Re = \overset{\infty}{\underset{n=1}{\vee}} (-n,\ n)$$

and μ be a lattice measure on $\Re$. Then ($\Re$, ß, μ) is a lattice σ-finite measure space.

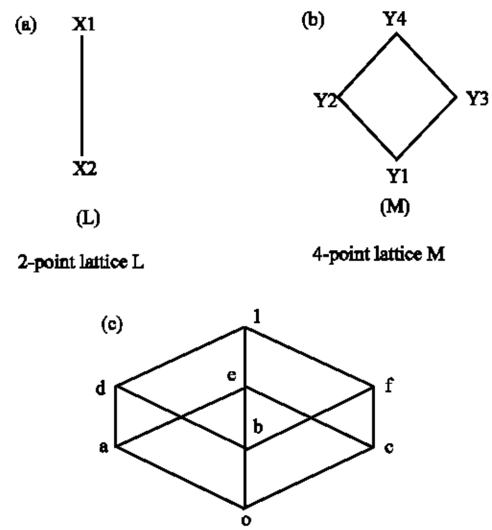

Fig. 1(a-c): (a) 2 point lattice L, (b) 4 point lattice M and (c) L×M; the Cartesian product of lattices L and M

Definition 10 (Gabor, 1964): Let X and Y be two lattices. Then their Cartesian product denoted by X×Y is defined as X×Y = {(x, y)/x∈X, y∈Y}. It is called product lattice.

Example 17: Let L and M be two lattices shown in the Fig. 1.

Consider L×M in Fig. 1, where, 1 = (x_2, y_4), d = (x_2, y_2), e = (x_1, y_4), f = (x_2, y_3), a = (x_1, y_2), b = (x_2, y_1), c = (x_1, y_3) and O = (x_1, y_1).

Definition 11: The lattice measure m defined on S×T is called the product of the lattice measures μ and λ and is denoted by μ×λ.

Example 18: If μ is a lattice measure on R, then m = μ×μ is a product lattice measure on R×R.

Definition 12: If A<X and B<Y, then A×B<X×Y. Any lattice of the form A×B is called super lattice in X×Y.

Example 19: If A⊂B and C⊂D, then (A×C)⊂(B×D).

Let (x, y) be any element of A×C. Then by definition of product lattice we have x∈A, y∈C.

But it is given that A⊂B and C⊂D.

Therefore x∈B and y∈D.

That is (x, y) is an element of B×D. Hence, (A×C)⊂(B×D) is a super lattice in B×D.

Remark 1: Counting measure: Let X be a non-empty set. Let σ (L) = P (X).

Define μ: σ (L)→[0, ∞] by |E| = number of lattice measurable sets in E, if E is finite, ∞if E is infinite. Then μ is a lattice measure on P (X) called the lattice counting measure on X.

Definition 13 (Pramada *et al*., 2011): Let f be a complex lattice measurable function on X. Then |f| is a lattice measurable function from X→[0, ∞]. If:

$$\int_X |f|\, d\mu < \infty$$

then we say that f is a complex integrable lattice function with respect to μ. The set of all complex integrable lattice measurable functions with respect to μ on X is denoted by L^1.

Definition 14 (Pramada *et al*., 2011): Let f = u+iv where u and v are real lattice measurable functions on X. Let f∈L^1. Then we define:

$$\int_E f\, d\mu = \int_E u^+\, d\mu - \int_E u^-\, d\mu + \int_E v^+\, d\mu - \int_E v^-\, d\mu$$

for every lattice measurable set E, where, $u^+ = \max\{u, 0\}$, $u^- = -\min\{u, 0\}$ and $v^+ = \max\{v, 0\}$, $v^- = -\min\{v, 0\}$.

Definition 15 (Kumar *et al*., 2011a): If E is a lattice measurable set and then the characteristic function $\chi_E(x)$ is defined as if $\chi_E(x) = 1$, if x∈E = 0, if x∉E.

Remark 2: Let (X, S) (Y, T) be lattice measurable spaces.
Then S is a lattice σ-algebra in X and T is a lattice σ-algebra in Y.

Definition 16: If A∈S and B∈T, then the lattice of the form A×B is called super lattice measurable set where S, T are lattice σ-algebras on X and Y, respectively.

Example 19: Every member of S×T is a super lattice measurable set.

Definition 17: Let E<X×Y where x∈X, y∈Y. We define x-section lattice of E by $E_x = \{y/ (x, y)\in E\}$ and y-section lattice of $E_y = \{x/(x, y)\in E\}$.

Note 2: $E_x<Y$ and $E_y<X$.

Definition 18 (Kumar *et al*., 2011a): Let f be an extended real valued measurable function on the lattice of real numbers such that {x∈L/f(x)>α} is lattice for each α∈L. Then f is lattice measurable function.

Definition 19 (Kumar *et al*., 2011a): A function s on a lattice measurable space X whose range consists of only finitely many points in [0, ∞] is called a simple lattice measurable function.

Theorem 1 (Pramada *et al*., 2011): If E∈S×T, then E_x∈T and E_y∈S for every x∈X and y∈Y.

Theorem 2 (Rudin, 1987) and (Pramada *et al*., 2011): Let f: X→[0,∞] be a lattice measurable function. Then there exists simple lattice measurable functions s_n on X such that:

i: $0 \le s_1 \le s_2 \le \ldots\ldots \le f$

ii: $s_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$

Theorem 3 (Rudin, 1987) and (Pramada *et al.*, 2011): Let $\{f_n\}$ be a sequence of lattice measurable functions on X such that $0 \le f_1(x) \le f_2(x) \ldots\ldots \le \infty$ for every $x \in X$ and $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$. Then f is lattice measurable and:

$$\int_X f_n \, d\mu \to \int_X f \, d\mu \text{ as } n \to \infty.$$

Note 3: Let E = [a, b]. Then:

$$\int_E d\mu = \int_a^b d\mu = [\mu]_a^b = b - a = \mu(b-a) = \mu(E)$$

CHARACTERIZATION OF LATTICE MEASURABLE FUNCTIONS ON PRODUCT LATTICES

Definition 20: A lattice measurable function f: $X \times Y \to Z$ where z is a topological space. For each $x \in X$, we define f_x: $Y \to Z$ by $f_x(y) = f(x, y)$. Then f_x is called Y-lattice measurable function. For each $y \in Y$, we define $f_y : X \to Z$ by $f_y(x) = f(x, y)$. Then f_y is called X-lattice measurable function.

Theorem 4: Let f be an $(S \times T)$ lattice measurable function on $X \times Y$, The:

(1): For each $x \in X$, f_x is a T-lattice measurable function

(2): For each $y \in Y$, f_y is a S-lattice measurable function

Proof: Let V be an open set in Z. Let $Q = \{(x, y) \in X \times Y: f(x, y) \in V\}$.

Since f is $S \times T$ lattice measurable, $Q \in S \times T$.

$$Q_x = \{y: (x, y) \in Q\} = \{y: f(x,y) \in V\} = \{y: f_x(y) \in V\}$$

By theorem 1, $Q_x \in T$. Therefore f_x is a T-lattice measurable function.

A similar argument shows that f_y is an S-lattice measurable function.

Theorem 5: Let (X, S, μ) and (Y, T, λ) be a lattice σ-finite measure spaces. Let f be an $(S \times T)$-lattice measurable function on product lattice $X \times Y$. Then the following conditions are hold good:

1: If $0 \le f \le \infty$ and if:

$$\Phi(x) = \int_Y f_x \, d\lambda, \quad \Psi(y) = \int_X f_y \, d\mu, \ (x \in X, y \in Y)$$

then Φ is S-lattice measurable. Ψ is T-lattice measurable and:

$$\int_X \Phi d\mu = \int_{X \times Y} f\, d(\mu \times \lambda) = \int_Y \Psi d\lambda$$

2: If f is complex and if:

$$\Phi^*(x) = \int_Y |f|_x \, d\lambda$$

and

$$\int_X \Phi^* d\mu < \infty$$

then $f \in L^1(\mu \times \lambda)$.

3: If $f \in L^1(\mu \times \lambda)$ then $f_x \in L^1(\lambda)$ for almost all $x \in X$, $f_y \in L^1(\mu)$ for almost all $y \in Y$; the functions Φ and Ψ defined by:

$$\Phi(x) = \int_Y f_x d\lambda, \quad \Psi(y) = \int_X f_y d\mu$$

almost every where, are in $L^1(\mu)$ and $L^1(\lambda)$, respectively and:

$$\int_X \Phi \, d\mu = \int_Y \Psi d\lambda = \int_{X \times Y} f\, d(\mu \times \lambda)$$

Proof: By theorem 4, we get f_x is a T-lattice measurable function for each $x \in X$ and f_y is an S-lattice measurable function for each $y \in Y$. Hence the definitions of Φ and Ψ make sense.

Part (a): Let $Q \in S \times T$. Let $f = \chi_Q$

Then:

$$\Phi(x) = \int_Y (\chi_Q)_x \, d\lambda = \int_Y \chi_{Q_x} d\lambda = \lambda(Q_x)$$

Similarly:

$$\Psi(y) = \int_X (\chi_Q)_y \, d\mu = \int_X \chi_{Q_Y} d\mu = \mu\left(\chi_{Q_Y}\right)$$

Therefore, by theorem 4:

$$\int_X \Phi \, d\mu = \int_Y \Psi d\lambda = (\mu \times \lambda)(Q) = \int_{X \times Y} \chi_Q \, d(\mu \times \lambda)$$

Hence, we get (a) for characteristic functions.

Let f be a non-negative (S x T) simple lattice measurable function.
Then:

$$f = \sum_{i=1}^{n} \alpha_i \chi_{Q_i} \quad \alpha_i \geq 0 \quad f_x = \sum_{i=1}^{n} \alpha_i \chi_{Q_{iX}}, \; f_y = \sum_{i=1}^{n} \alpha_i \chi_{Q_{iy}}$$

$$\varphi(x) = \int_Y f_x d\lambda = \int_Y \sum_{i=1}^{n} \alpha_i \chi_{Q_{iX}} d\lambda = \sum_{i=1}^{n} \alpha_i \int_Y \chi_{Q_{iX}} d\lambda = \sum_{i=1}^{n} \alpha_i \quad \lambda(Q_{iX})$$

Let:

$$\Phi_i(x) = \lambda(Q_{iX}) \quad 1 \leq i \leq n$$

Similarly:

$$\Psi(y) = \sum_{i=1}^{n} \alpha_i \mu(Q_{iy}) \qquad 1 \leq i \leq n$$

Now:

$$\int_x \Phi_i \; d\mu = \int_y \psi_i \; d\lambda = \int_{X x Y} \chi_{Q_i} \; d(\mu x \lambda)$$

Therefore:

$$\sum_{i=1}^{n} \alpha_i \int_X \Phi_i \; d\mu \quad = \sum_{i=1}^{n} \alpha_i \int_Y \Psi_i \; d\lambda \quad = \sum_{i=1}^{n} \alpha_i \int_{X x Y} \chi_{Q_i} \; d(\mu x \lambda)$$

That is:

$$\int_X \sum_{i=1}^{n} \alpha_i \Phi_i \; d\mu \quad = \int_Y \sum_{i=1}^{n} \alpha_i \Psi_i \; d\lambda \quad = \int_{X x Y} \sum_{i=1}^{n} \alpha_i \chi_{Q_i} \; d(\mu x \lambda)$$

That is:

$$\int_X \Phi \; d\mu \quad = \int_Y \Psi \; d\lambda \; = \int_{X x Y} f \, d(\mu x \lambda)$$

Hence, (a) holds for all non-negative (S×T)-simple lattice measurable functions S.

Let f be any (S x T)-lattice measurable function. Then by theorem 2, there exist (S x T)-simple lattice measurable functions s_n on X x Y such that $0 \leq s_1 \leq s_2 \leq \ldots\ldots . \leq f$ and s_n (x, y) $\rightarrow$f(x, y) as $n \rightarrow \infty$ for every (x, y) ϵX x Y. Let Φ_n be associated with s_n in the same way as Φ is associated to f. we have:

$$\int_{x} \Phi_n \, d\mu = \int_{X \times Y} s_n \, d(\mu \times \lambda) \tag{1}$$

Now:

$$\Phi_n(x) = \int_{Y} s_{n_x} \, d\lambda \text{ and } s_{n_x} \to f_x$$

$$0 \leq (s_1)_x \leq \ldots \ldots .. \leq f_{x.}$$

Therefore, if we apply theorem 3, on (Y, T, λ) then this shows that:

$$\int_{Y} s_{n_x} \, d\lambda \to \int_{Y} f_x \, d\lambda$$

That is, $\Phi_n(x)$ increase to $\Phi(x)$ for every $x \in X$ as $n \to \infty$.
Again applying theorem 3, to the integrals of (1) we get:

$$\int_{x} f \, d\mu = \int_{X \times Y} f \, d(\mu \times \lambda)$$

By interchanging the role of x and y we get:

$$\int_{y} \Psi \, d\lambda = \int_{X \times Y} f \, d(\mu \times \lambda)$$

Therefore:

$$\int_{x} \Phi \, d\mu = \int_{y} \Psi \, d\lambda = \int_{X \times Y} f \, d(\mu \times \lambda)$$

This completes proof of (a).

Part (b): Let f be complex. Then $0 \leq |f| \leq \infty$.
Let:

$$\Phi^*(x) = \int_{Y} |f|_x \, d\lambda$$

Given:

$$\int_{x} \varphi^* \, d\mu < \infty$$

Then by using (a) for $|f|$, we get:

$$\int_{X \times Y} |f| \, d(\mu \times \lambda) = \int_{X} \Phi^{*} \, d\mu < \infty$$

therefore, $f \in L^1(\mu \times \lambda)$.

Proof of (c): First we prove for real $f \in L^1 (\mu \times \lambda)$

Let f be in $L^1 (\mu \times \lambda)$ and let f be real. Then $0 \le f^+ \le \infty$ and $0 \le f^- \le \infty$.

Let Φ_1 and Φ_2 correspond to f^+ and f^-, respectively as Φ corresponds to f.

Now $f \in L^1 (\mu \times \lambda)$ and $f^+ \le |f|$

Since, (a) holds for f^+, we get that:

$$\int_{X} \Phi_1 \, d\mu = \int_{X \times Y} f^+ \, d(\mu \times \lambda) \le \int_{X \times Y} |f| \, d(\mu \times \lambda) < \infty \text{ (since } f \in L^1 (\mu \times \lambda))$$

Therefore, $\Phi_1 \in L^1 (\mu)$.

Similarly, $\Phi_2 \in L^1 (\mu)$.

Now $f_x = (f^+)_x - (f^-)_x$. Also $\Phi_1(x) = \int_{Y} f^+_x \, d\lambda$ shows that $f^+_x \in L^1(\lambda)$ for every x for which both $\Phi_1(x)$ and $\Phi_2(x) < \infty$.

Similarly $(f^-)_x \in L^1(\lambda)$.

Therefore $f_x \in L^1 (\lambda)$ for every x for which both $\Phi_1 (x)$ and $\Phi_2(x)$ are $< \infty$.

Since, Φ_1, Φ_2, $\in L^1(\mu)$ we get that $\Phi_1(x)$ and $\Phi_1(x)$ are $< \infty$, almost every where.

Hence, $f_x \in L^1$ for almost all $x \in X$.

For such x, we have $\Phi (x) = \Phi_1 (x) - \Phi_2(x)$

Hence, $\Phi \in L^1(\mu)$ using (a) we get:

$$\int_{X} \Phi_1 \, d\mu = \int_{X \times Y} f^+ \, d(\mu \times \lambda)$$

$$\int_{X} \Phi_2 \, d\mu = \int_{x \times y} f^- \, d(\mu \times \lambda)$$

Therefore:

$$\int_{X} (\varphi_1 - \varphi_2) \, d\mu = \int_{X \times Y} (f^+ - f^-) \, d(\mu \times \lambda)$$

That is:

$$\int_{X} \Phi \, d\mu = \int_{X \times Y} f \, d(\mu \times \lambda)$$

Similarly, we can prove that:

$$\int_{Y} \psi \, d\lambda = \int_{X \times Y} f \, d(\mu \times \lambda)$$

by using f_y in place of f_x and Ψ in the place of ϕ.

Suppose f is complex and $f \in L^1(\mu \times \lambda)$.

Let $f = u+iv$. Then $u, v \in L^1(\mu \times \lambda)$ and u, v are real.

Then applying what we proved above to u, v we get:

$$\int_X \Phi_u \, d\mu = \int_{X \times Y} u \; d(\mu \times \lambda) = \int_Y \Psi_u \, d\lambda$$

$$\int_X \Phi_v \, d\mu = \int_{X \times Y} v \; d(\mu \times \lambda) = \int_Y \Psi_v \, d\lambda$$

where, Φ_u, Φ_v corresponds to u, v as Φ corresponds to f.

Thus:

$$\int_X (\Phi_u + i\Phi_v) \, d\mu = \int_{X \times Y} (u + iv) \; d(\mu \times \lambda) = \int_Y (\Psi_u + i\Psi_v) \, d\lambda$$

That is:

$$\int_X \Phi \, d\mu = \int_{X \times Y} f \; d(\mu \times \lambda) = \int_Y \Psi \, d\mu$$

This proves (c).

Hence the theorem.

Note 4:

$$\int_X \Phi \, d\mu = \int_{X \times Y} f \; d(\mu \times \lambda) = \int_Y \Psi \, d\lambda$$

can be written as:

$$\int_X d\mu(x) \int_Y f(x, y) \; d\lambda(y) = \int_{X \times Y} f \; d(\mu \times \lambda) = \int_Y d\lambda(x) \int_X f(x, y) \; d\mu(x)$$

The integrals at the ends are the so called iterated integrals of f.

The middle integral is often referred to as a double integral.

Result 1: The two iterated integrals are finite and equal.

Proof: From part (b) and part (c) we get the following useful result.

Let f is (S×T)-lattice measurable and let:

$$\int_X \varphi^* \, d\mu = \int_X d\mu(x) \int_Y |f(x,y)| \; d\lambda(y) < \infty$$

Then $\Phi \in L^1(\mu)$. $\Psi \in L^1(\lambda)$ that is:

$$\int_x \Phi \, d\mu < \infty, \int_y \psi \, d\lambda < \infty \text{ that implies, } \int_x \Phi \, d\mu = \int_y \Psi \, d\lambda$$

Therefore, the two iterated integrals are finite and equal.

Note 5: The order of integration may be reversed for (S×T)-lattice measurable functions f whenever $f \geq 0$ or when ever one of the iterated integrals of $|f|$ is finite.

Result 2: For lattice σ-finiteness μ can not be omitted.

Proof: Let X = [0, 1] = Y, μ = Lebesgue measure on [0,1], λ = lattice counting measure on Y.

Let f(x, y) = 1 if x = y, f(x, y) = 0 if x≠y:

$$\int_y f(x,y) d\mu(x) = 0$$

since, for a given y, f (x, y) = 1 when x = y and 0 at all other x. also Lebesgue lattice measure of a single point is 0.

That is:

$$\int_X f(x,y) \, d\mu\,(x) = \int_{\{y\}} d\mu = 0$$

$\int_y f(x,y) \, d\lambda\,(x) = \int_{\{x\}} d\lambda = 1$ (since λ is the lattice counting measure)

Hence:

$$\int_y d\lambda(y) \int_x f(x,y) \; d\mu(x) = 0$$

So:

$$\int_x d\lambda\,(y) \int_y f(x,y) \; d\lambda\,(y) = \int_0^1 d\mu(x) = \mu[0,1] = 1$$

Hence:

$$\int_x d\lambda(y) \int_y f(x,y) \; d\lambda(y) \neq \int_y d\lambda(y) \int_x f(x,y) \; d\mu(x)$$

To show that f in (S×T)-lattice measurable.

(where, S is the class of all Lebesgue lattice measurable sets in [0, 1] and T consists of all subsets of [0, 1]).

Since f (x, y) = 1 if x = y, f (x, y) = 0 if x $\neq$ y, we see that f = χ_D where, D is the diagonal of the unit square.

Given a positive integer n:

$$\text{Let}I_j = \left[\frac{j-I}{n}, \frac{j}{n}\right] \quad i \leq j \leq n$$

Let $Q_n = (I_1 \times I_1) \vee (I_2 \times I_2) \vee \ldots\ldots\ldots\ldots \vee (I_n \times I_n)$
Where n = 1,
$I_1 = [0, 1]$, $Q_1 = I_1 \times I_1$ is the unit square.

When:

n = 2
$I_1 = [0, 1/2]$, $I_2 = [1/2, 1]$
$Q_2 = [0, 1/2] \times [0, 1/2] \vee [1/2, 1] \times [1/2, 1]$
$Q_3 = [0, 1/3] \times [0, 1/3] \vee [1/3, 2/3] \times [1/3, 2/3] \vee [2/3, 1] \times [2/3, 1]$ etc.

Thus, Q_n is finite union of super lattice measurable sets and $D = \Lambda Q_n$.
Hence, $D \epsilon S \times T$.
Therefore, $f = \chi_D$ is S×T-lattice measurable.
Since λ is the lattice counting measure, if:

$$y = \overset{\infty}{\underset{n=1}{\vee}} Y_n$$

a disjoint union such that λ (Yn) < ∞ for all n, then every Y_n is a finite set.
Hence, Y is countable, a contradiction since Y = [0, 1]
Thus λ is not lattice σ-finite.
Thus the lattice σ-finiteness of λ, so μ can not be omitted.

Result 3: The condition that f is lattice measurable with respect to S x T can not be dropped.

Proof: Consider X = Y = [0, 1], $\mu = \lambda$ = Lebesgue lattice measure on [0, 1],
S = T = class of all Lebesgue lattice measurable sets in [0, 1].

Let us assume the following consequence of continuum hypothesis: there exists a one-to-one map θ from [0, 1] onto a well- ordered set W such that θ (x) has at most countably many predecessors in W for each $x \epsilon [0, 1]$.
Let $Q = \{(x, y) \in X \times Y: \theta(x) \text{ precedes } \theta(y) \text{ in } W\}$.
For each $x \epsilon [0, 1]$ $Q_x = \{y: (x, y) \epsilon Q\}$.
$Y \epsilon Q_x$ if and only if $(x, y) \epsilon Q$ if and only if $\theta(x)$ precedes $\theta(y)$ in W.

Since $\theta(x)$ has at most countably many precedessors in W, there will be only countably many $y^{'s}$ in [0, 1] such that $\theta(y)$ processes $\theta(x)$.

Hence, all but countably many $y^{'s}$ in [0, 1] are such that $\theta(x)$ precedes $\theta(y)$ that is, Q_x contains all but countably many points of [0, 1].

For each $y\epsilon[0, 1]$ $Q_y = \{x: (x, y)\epsilon Q\}$.

That is, $x\epsilon Q_y$ if and only if $(x, y)\epsilon Q$ if and only if $\theta(x)$ precedes $\theta(y)$.

But $\theta(y)$ has at most countably many predecessors in W.

Hence, Q_y contains at most countably many points of [0,1].

Let $f = \chi_Q$.

Since Q_x and Q_y are Borel lattice measurable, we get that f_x and f_y are Borel lattice measurable and:

$$\Phi(x) = \int_y f_x \, d\lambda = \int_0^1 f(x, y)\, dy = 1$$

$$\psi(x) = \int_y f_y \, d\mu = \int_0^1 f(x, y)\, d\lambda = 0 \text{ for all } x \text{ and } y$$

since for any fixed x, f (x, y) = χ_{Q_y} and Q_x contains all but countably many points:

$$\int_0^1 \chi_{Qx} \, dx = 1$$

All since χ_{Qy} contains at most countably many number of points:

$$\int_0^1 \chi_{Qy} \, dx = 0$$

Hence:

$$\int_0^1 dx \int_0^1 f(x, y)\, dy = 1 \neq 0 \int_0^1 dy \int_0^1 f(x, y)\, dx$$

In this result. f is not lattice measurable w.r.t. lattice σ-algebra S x T.

Hence, the condition that f is lattice measurable with respect to S x T can not be dropped.

CONCLUSION

This manuscript illustrate the concept of product lattice measurable functions and their various characterizations. In particular these functions were defined over topological spaces. Also it has been introduced and advanced the characteristics of S- lattice measurable function and T-lattice measurable function. The concept of iterated integral of a product lattice measurable function has been defined and proved that the two iterated integrals of a product lattice measurable function are finite and equal. The condition that product lattice measurable function is lattice measurable is obtained and it has been derived that the condition that product lattice measurable function is lattice measurable cannot be dropped.

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