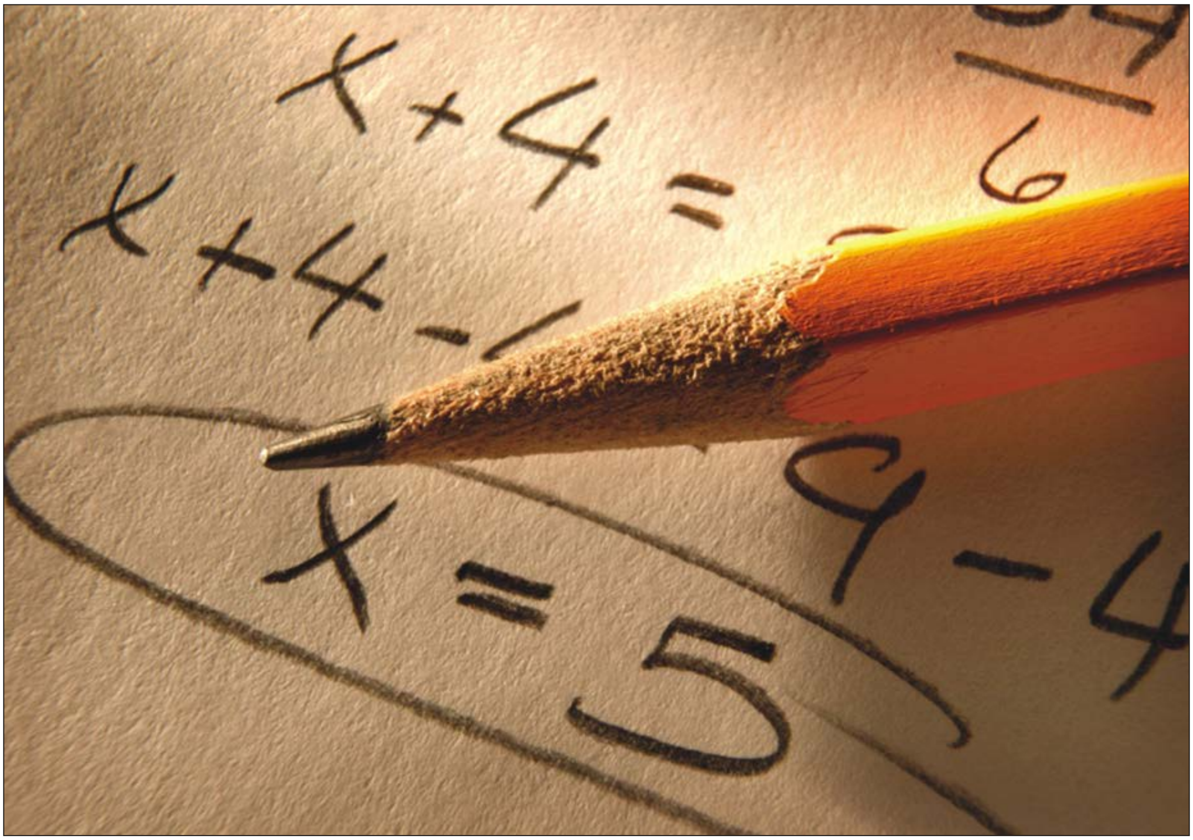

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Characterizations of Lie Lattice Sigma Algebras in Formal and Conformal Systems

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ABSTRACT

This study is an exploration on lie lattice σ-algebra, indiscrete lattice σ-algebra, formal system and co-formal systems. It has acknowledged some characterizations of formal and co-formal systems. Finally, it corroborates that the lie lattice σ-algebra generated by formal system contained in the lie lattice σ-algebra is generated by co-formal system.

Key words: Lattice, σ-algebra, formal system, conformal system

INTRODUCTION

The notion of outer measure connected to lattice measure was introduced by Hus (2000). Subsequent to that Khare and Singh (2005) contributed to the concept of weakly tight functions and their decomposition. Later on Khurana (2008) developed the idea of lattice valued Borel measures. Hann decomposition in signed fuzzy measure version was established by Tanaka (2008) and further Tanaka (2009) derived a Hann decomposition for signed lattice measure and built-up the concept of σ-algebra. Recently the structure of gamma lattice was through by Kumar *et al.* (2011a). Most recently Radon-Nikodym theorem and its uniqueness of signed lattice measure was established by Kumar *et al.* (2011b). Jordan decomposition and its uniqueness of signed lattice measure were developed by Kumar *et al.* (2011c).

The class of positive lattice measurable sets and positive lattice measurable functions were exposed by Pramada *et al.* (2011). Further the class of super lattice measurable sets was successfully studied by Pramada *et al.* (2011). Complex integrable lattice functions and ì-free lattices were recognized by Pramada *et al.* (2012b,c). Further Pramada *et al.* (2012a) initiated the Boolean valued star and mega lattice functions. Putcha and Malladi (2010) formulated a mathematical model on litter, detritus and predators in mangrove estuarine ecosystem and solved system by extending the Adomian's decomposition method. Deekshitulu *et al.* (2011) established some fundamental inequalities and comparison results of fractional difference equation of Volterra type. Anand *et al.* (2011) found multiple symmetric positive solutions for a system of higher order two-point boundary-value problems on time scales by determining growth conditions and applying a fixed point theorem in cones under suitable conditions. Putcha (2012) constructed the approximate analytical solutions of two species and three species ecological systems using homotopy ananlysis and homotopy perturbation methods.

A class of measurable Borel lattices was established by Kumar *et al.* (2011d). The concepts Boolean valued measurable functions, function lattice, σ-lattice and lattice measurable space were contributed by Kumar *et al.* (2011e).

This study established a general agenda for the study of characterization of formal and conformal systems. Further, it has been noticed that measures of theoretical concepts were generalized in terms of σ-algebra. Some elementary characteristics of lie lattice σ-algebra has been proved and finally confirmed that the lie lattice σ-algebra generated by formal system contained in the lie lattice σ-algebra generated by conformal system.

PRELIMINARIES

In this manuscript it has been considered that the union and intersection of set theory as the binary operations $\wedge$ and $\vee$. Further, it was briefly reviewed the well-known facts described by Birkhoff (1967), proposed an extension lattice and investigated its properties.

The system (L, $\wedge$, $\vee$) where L is a non empty set together with binary operations $\wedge$, $\vee$ called a lattice if it satisfies, for any elements x, y, z, in L:

- **The commutative law (L1):** $x\wedge y = y\wedge x$ and $x\vee y = y\vee x$
- **The associative law (L2):** $x\wedge (y\wedge z) = (x\wedge y)\wedge z$ and $x\vee (y\vee z) = (x\vee y)\vee z$
- **The absorption law (L3):** $x\vee (y\wedge x) = x$ and $x\wedge (y\vee x) = x$
- Hereafter, the lattice (L, $\wedge$, $\vee$) will often be written as L for simplicity

A mapping h from a lattice L to another lattice L^1 is called a lattice-homomorphism, if it satisfies:

- $h(x\wedge y) = h(x)\wedge h(y)$ and $h(x\vee y) = h(x)\vee h(y)$, for all $x, y\in L$

If h is a bijection, that is, h is one-to-one and onto, it is called a lattice isomorphism and in this case, L^1 is said to be lattice-isomorphic to L.

A lattice (L, $\wedge$, $\vee$) is called distributive if, for any x, y, z, in L.

- **The distributive law holds (L4):** $x\vee (y\wedge z) = (x\vee y)\wedge (y\vee z)$ and $x\wedge (y\vee z) = (x\wedge y)\vee (y\wedge z)$

A lattice L is called complete if, for any subset A of L, L contains the supremum $\vee A$ and the infimum $\wedge A$. If L is complete, then L itself includes the maximum and minimum elements which are often denoted by 1 and 0 or I and O, respectively.

A distributive lattice is called a Boolean lattice if for any element x in L, there exists a unique complement x^c such that:

- **The law of excluded middle (L5):** $x\vee x^c = 1$
- **The law of non-contradiction (L6):** $x\wedge x^c = 0$

Let L be a lattice and c: L→L be an operator. Then c is called a lattice complement in L if the following conditions are satisfied:

- **L5 and L6:** for all $x\in L$, $x\vee x^c = 1$ and $x\wedge x^c = 0$
- **The law of contrapositive (L7):** for all $x, y\in L$, $x<y$ implies $x^c>y^c$

- **The law of double negation (L8):** for all $x \in L$, $(x^c)^c = x$

Throughout this study, it has been considered the lattices as complete lattices which obey L1-L8 except for L6 the law of non-contradiction.

LIE LATTICE SIGMA ALGEBRAS

Unless otherwise stated, X is the entire set and F is a lattice of any subsets of X.

Definition 1: If a lattice F satisfies the following conditions, then it is called a lattice σ-algebra:

- For all $h \in F$, $h^c \in F$
- If $h_n \in F$ for n = 1, 2, 3 ....., then $\bigvee_{n=1}^{\infty} h_n \in F$.

Denote σ (F) is a lattice σ-algebra generated by F.

Example 1: (i) {ϕ, X} is a lattice σ-algebra and (ii) P(X) power set is a lattice σ-algebra.

Example 2: Let $X = \Re$, F = {measurable subsets of $\Re$} with usual ordering (≤). Here F is a lattice, σ (F) is a lattice σ-algebra generated by F. Where $\Re$ is an extended real number system.

Example 3: Let X be any non-empty set and F = {all topologies on X}. Here F is a complete lattice but not σ-algebra.

Example 4: Let $X = \Re$ and F = {$E<\Re/E$ is finite or E^c is finite}. Here F is lattice algebra but not lattice σ-algebra.

Definition 2: The lattice σ-algebra F of all sub sets of X lies between {ϕ, X}<F<P(X) is called a lie lattice σ-algebra.

Example 5: A partition of X is a collection of disjoint subsets of X whose union is all of X. For simplicity, consider a partition consisting of a finite number of sets $A_1, A_2, \dots\dots$. Thus:

$$A_i \wedge A_j = \phi \text{ and } A_1 \vee A_2 \dots . A_n = X$$

Then the collection F of all unions of the sets A_j forms a lie lattice σ-algebra.

Theorem 1: If F be a lie lattice σ-algebra of subsets of X, then the following conditions hold good:

- $X \in F$
- If $A_1, A_2 \dots\dots . A_n \in F$, then $A_1 \vee A_2 \vee \dots\dots .. A_n \in F$
- If $A_1, A_2 \dots\dots . A_n \in F$, $A_1 \wedge A_2 \wedge \dots\dots . A_n \in F$
- If $A_1, A_2 \dots\dots$ is a countable collections of sets in F then $\bigwedge_{n=1}^{\infty} A_n \in F$
- If A, $B \in F$ then $A-B \in F$

Proof:

- Since $\phi \in F$ and $X = \phi^c$ it follows that $X \in F$
- $A_1 \vee A_2 \dots . A_n = A_1 \vee A_2 \dots . A_n \vee \phi \vee \phi \vee \dots\dots \in F$ (definition of 2)
- Since $A_1 \wedge A_2 \dots . A_n = (A_1^c \wedge A_2^c \dots\dots A_n^c)^c$ which is in F because each $A_i^c \in F$ and F is closed under finite unions, from (2) it follows that $A_1 \wedge A_2 \wedge \dots\dots . A_n \in F$
- $\overset{\infty}{\underset{n=1}{\wedge}} A_n$ can be expressed as $\left[\overset{\infty}{\underset{n=1}{\wedge}} A_n^c\right]^c$ and is in F. Since F is closed under complementation and countable unions
- Since A, $B^c \in F$ it follows that $A\text{-}B = A \wedge B^c$ is in F

Definition 3: Let B a non-empty collection of subsets of a set X. The smallest lattice σ-algebra containing all the sets of B is denoted by σ(B) and is called the indiscrete lattice σ-algebra generated by the collection B.

Note 1: Any lattice σ-algebra containing the sets of B must contain all the sets of σ(B). In the entire discussion it is assumed that the symbol <represents the set operation proper subset (⊂).

Observation 1: From the definition 2 of lie lattice σ-algebra it follows that if G is any non-empty collection of lie lattice σ-algebras of subsets of X, then the meet $\wedge G$ is indiscrete lie lattice σ-algebra of subsets of X. That is $\wedge G = \{A < X \mid A \in F \text{ for every } F \in G\}$ consists of all sets A which belong to each lie lattice σ-algebra F of G.

Note 2: Given a collection B of subsets of X, let G_B be the collection of all lattice σ-algebras including containing all the sets of B. Note that $P(X) \in G_B$ and so G_B is non empty. Then $\wedge G_B$ is a lie lattice σ-algebra, contains all the sets of B and is minimal among such lie lattice σ-algebras. Minimally means if F is a lie lattice σ-algebra such that $B < F$ then $\wedge G_B < F$ thus $\wedge G_B$ is the lie lattice σ-algebra. This lie lattice σ-algebra is a indiscrete lie lattice σ-algebra.

Definition 4: Let X be a set, A collection P of subsets of X is called a formal system. If it is closed under finite intersections that is if A_1, $A_2 \dots\dots A_n$ are a finite number of sets in P, then their intersection $A_1 \wedge A_2 \dots\dots A_n$ is also in P.

Definition 5: A collection L of subsets of X is called a conformal system if:

- L contains the empty set ϕ
- L is closed under complementation. That is if $A \in L$ then $A^c \in L$
- L is closed under countable disjoint union. That is if $A_1, A_2, \dots\dots \in L$ and $A_i \wedge A_j = \phi$ for every $i \neq j$, then $\overset{\infty}{\underset{n=1}{\vee}} A_n \in L$

Result 1: Every conformal system is closed under proper differences, that is, if $A, B \in L$, where L is a conformal system and $A < B$ then the difference B-A is also in L.

Proof: Since B-A can be expressed as $A \vee B^c$ when ever $A < B$ and L being a conformal system it follows that $B\text{-}A \in L$. The same thing can also be realized from the fact that $B\text{-}A = B \wedge A^c = (B^c \vee A)^c$.

Result 2: A family which is both formal and conformal system is a lie lattice σ-algebra.

Proof: Let S be a collection of subsets of X which is both formal system and conformal system. To prove that S is a lie lattice σ-algebra it is sufficient to show that S is closed under countable union (not just disjoint countable unions).

Let A_1, A_2 .... ϵS. By rewriting $\overset{\infty}{\underset{n=1}{\vee}} A_n$ as a countable union of disjoint sets $\overset{\infty}{\underset{n=1}{\vee}} B_n$, where $B_1 = A_1$ and $B_n = A_n\text{-}(A_1 \vee A_2 \vee \ldots\ldots A_{n-1}) = A_n \wedge A_1^c \wedge A_2^c \wedge \ldots\ldots \wedge A_n^c$, for n = 1.

Thus, B_n consists of all elements of A_n which do not appear in all A_i, 1 = i = n-1. From the construction of B_i's (i=1,2,3,...), it follows that they are mutually disjoint. Since S is conformal and formal it follows that S is closed under complementation and finite intersection, respectively.

Result 3: Suppose L^1 is a conformal system of X. For any set Aϵ L^1, let S_A be the set of all B<X for which $A \wedge B \epsilon L^1$. Then S_A is a conformal system.

Proof: The set S_A contains the null set ϕ since $A \wedge \phi = \phi$ and is in L^1. It is also clear that S_A is closed under countable disjoint unions.

Let $B \epsilon S_A$ and observe that $A \wedge B^c = A\text{-}B = A\text{-}(A \wedge B)$ and is in L^1.

Therefore S_A is closed under complementation.

Result 4: The intersection l(P) of all conformal systems containing P is formal.

Proof: Let Aϵl(P) and let S_A be the set of all sets B<X for which $A \wedge B$ is in l(P).

- From result 3 it follows that S_A is a conformal system and P is in S_A
- Thus $P<S_A$. Therefore, $l(P)<S_A$
- Similarly $P<S_B$ whenever Bϵl(P)
- Result 3, infers that S_B is a conformal system
- Therefore, $l(P) <S_B$. Thus, l(P) is a formal system

Theorem 3: The lie lattice σ-algebra generated by a formal system P and a conformal system generated by L is contained in L.

Proof: Let P is a formal system and L is a conformal system, with P<L:

- The line of attack is to establish the existence of a lie lattice σ-algebra between P and L
- This will imply that σ(P) is the smallest lie lattice σ-algebra containing P and is contained in L
- From result 4 it follows that l(P) is also a formal system
- Now result 2 infers that l(P) is a lie lattice σ-algebra
- From result 4 and 2 it follows, respectively that l(P) is a formal system and is a lie lattice σ-algebra
- From the definition of l(P), P<l(P) <L and L is just one conformal system containing P
- Thus it was shown the existence of lie lattice σ-algebra l(P) lying between P and L
- Therefore, P<σ(P) <l(P) <L, where σ(P) is the intersection of all lie lattice σ-algebras which contain P

CONCLUSION

This study illustrates the notions of lie lattice σ-algebra, indiscrete lattice σ-algebra, formal system and conformal system. Also it establishes some characterizations of formal and conformal systems. Finally it confirms that, the lie lattice σ-algebra generated by formal system is contained in the lie lattice σ-algebra generated by conformal system.

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REFERENCES

Anand, P.V.S., P. Murali and K.R. Prasad, 2011. Multiple symmetric positive solutions for the system of higher order boundary value problems on time scales. Electron. J. Differ. Equat. USA., 2011: 1-12.

Birkhoff, G.D., 1967. Lattice Theorey. 3rd Edn., American Mathematical Society, Colloguim Publications, Rhode Island, New Delhi.

Deekshitulu, G.V.S.R., J.J. Mohan and P.V.S. Anand, 2011. Fractional difference inequalities of volterra type. Int. J. Pure Applied Math. Bul., 70: 137-149.

Hus, P.S., 2000. Characterization of outer measures associated with lattice measures. Int. J. Math. Math. Sci., 24: 237-249.

Khare, M. and B. Singh, 2005. Weakly tight functions and their decomposition. Int. J. Math. Math. Sci., 18: 2991-2998.

Khurana, S.S., 2008. Lattice-valued borel measures. III. Arch. Math., 44: 307-316.

Kumar D.V.S.R.A., J.V. Rao and E.S.R.R. Kumar, 2011a. Charecterization of class of measurable borel lattices. Int. J. Contemp. Math. Sci., 6: 439-446.

Kumar, D.V.S.R.A., J. Venkateswara Rao and E.S.R.R. Kumar, 2011b. Jordan decomposition and its uniqueness of signed lattice measure. Int. J. Contemp. Math. Sci., 6: 431-438.

Kumar, D.V.S.R.A., J. Venkateswara Rao and J. Pramada, 2011c. Construction of gamma lattice. Int. J. Applied Math., 205: 314-323.

Kumar, D.V.S.R.A., J. Venkateswara Rao and J. Pramada, 2011d. Radon-nikodym theorem and its uniqueness of signed lattice measure. Int. J. Math. Comput., 12: 19-27.

Kumar, D.V.S.R.A., J.V. Rao and J. Pramada, 2011e. Lattice boolean valued measurable functions. Int. J. Applied Math. Stat., Vol. 23.

Pramada, J., J. Venkateswara Rao and D.V.S.R.A. Kumar, 2011. Characterization of class of super lattice measurable sets. J. Applied Sci., 11: 3525-3529.

Pramada, J., J. Venkateswara Rao and D.V.S.R.A. Kumar, 2012a. Characterization of boolean valued star and mega lattice functions. Asian J. Algebra, 5: 1-10.

Pramada, J., J. Venkateswara Rao and D.V.S.R.A. Kumar, 2012b. Characterization of class of positive lattice measurable sets and positive lattice measurable functions. Asian J. Applied Sci., 5 : 43-51.

Pramada, J., J. Venkateswara Rao and D.V.S.R.A. Kumar, 2012c. Characterization of complex integrable lattice functions and ì-free lattices. Asian J. Math. Stat., 5: 1-20.

Putcha, V.S. and R. Malladi, 2010. A mathematical model on detritus in mangrove estuarine eco system. Int. J. Pure Applied Math., Bulg., 63: 169-182.

Putcha, V.S., 2012. Two Species and Three Species Ecological Modeling-Homotopy Analysis. In: Diversity of Ecosystems, Ali, M. (Ed.). InTech Publisher, Croatia, ISBN-13: 978-953-51-0572-5, pp: 221-250.

Tanaka, J., 2008. Hahn decomposition theorem of signed fuzzy measure. Adv. Infuzzy Sets Syst., 30: 315-323.

Tanaka, J., 2009. Hahn decomposition theorem of signed lattice measure. arXiv:0906.0147v1, Cornell University Library. http://arxiv.org/abs/0906.0147