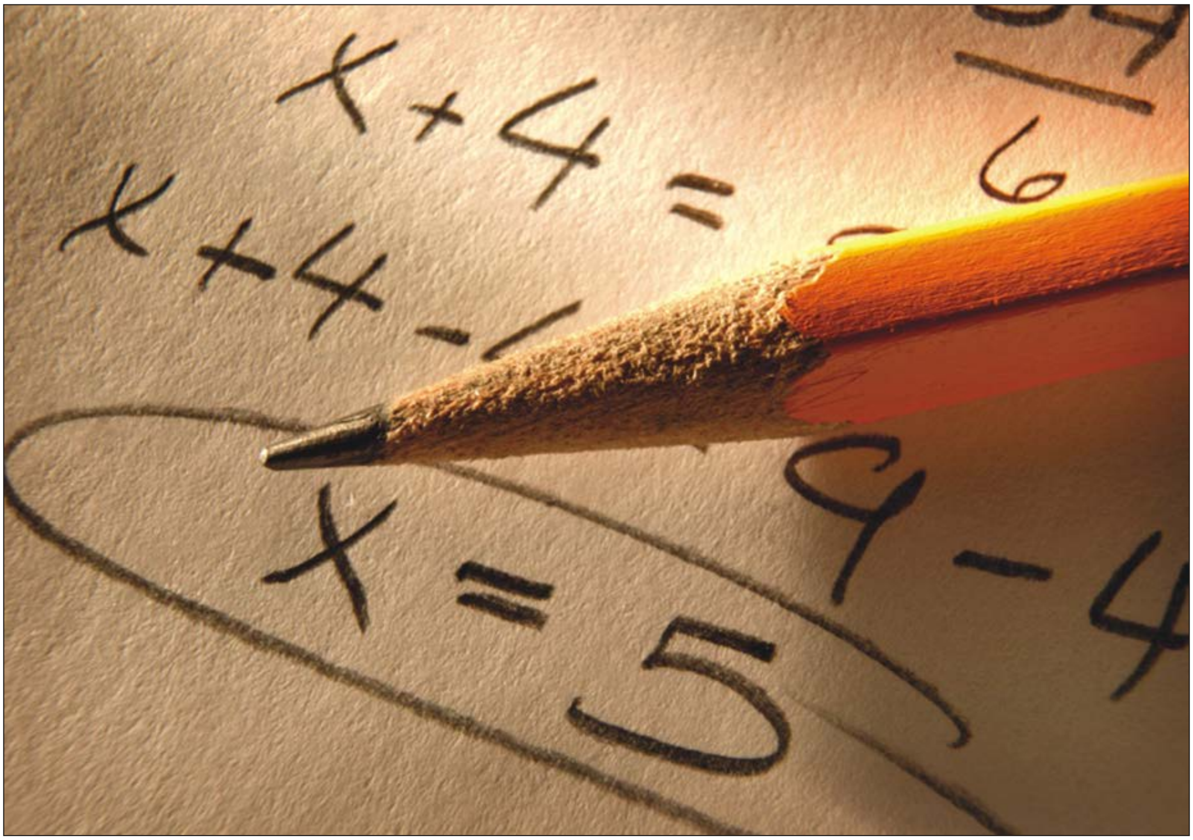

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Research Article
Flexible Lie-admissible Superalgebras of Vector Type

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Abstract

Background: First examples of simple nonassociative superalgebras were constructed by Shestakov in (1991 and 1992). Since then many researchers showed interest towards the study of superalgebras and superalgebras of vector type. **Materials and Methods:** Multiplication in M is uniquely defined by a fixed finite set of derivations and by elements of A. The types of derivations used in this article to obtain the results are the near derivation $\delta_{x,y}: a \mapsto (a, x, y)$, the derivation $D: a \mapsto \frac{1}{2}(x, a, x)$ and the derivation $D_{ij}: a \mapsto (x_i, a, x_j)$. **Results:** The flexible Lie-admissible superalgebra $F_{FLSA}[\phi; x]$ over a 2, 3-torsion free field Φ on one odd generator e is isomorphic to the twisted superalgebra B_0 ($\Phi[\Gamma]$, D, γ_0) with the free generator $\overline{1}$. In a 2, 3-torsion free flexible Lie-admissible superalgebras of vector type F, the even part A is differentiably simple, associative and commutative algebra and the odd part M is a finitely generated associative and commutative A-bimodule. **Conclusion:** A connection between the integral domains, the finitely generated projective modules over them, the derivations of an integral domain and the flexible Lie-admissible superalgebras of vector type has been established. If A is an integral domain and $M = Ax_1+...+Ax_n$ be a finitely generated projective A-module of rank 1, then F (A, Δ, Γ) is a flexible Lie-admissible superalgebra with even part A and odd part M provided that the mapping $A \mapsto (M \otimes_A M)^*$ is a nonzero derivation of A into the A-module $(M \otimes_A M)^*$, $\Delta = \{D_{ij} \mid i, j = 1,..., n\}$ is a set of derivations of A where $D_{ij}(a) = \bar{a}(x_i \otimes x_j)$.

Key words: Superalgebra, flexible Lie-admissible superalgebra, differentiably simple algebra, grassmann envelope, super algebra of vector type

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Data Availability: All relevant data are within the paper and its supporting information files.

INTRODUCTION

Okubo and Myung[1] gave the classification of simple flexible Lie-admissible algebra A such that A^- is reductive and the Levi-factor of A^- is simple. Later using the results of Okubo and Myung[1] and Benkart and Osborn[2] determined flexible Lie-admissible algebras A by module approach for which the radical A^- is abelian. Kamiya and Okubo[3] constructed some class of Lie superalgebras from non-super flexible Lie-admissible algebras. They concentrated especially on the case of any associative algebra giving rise to Lie superalgebras.

In Shestakov[4], the concept of a (-1, 1)-superalgebra of vector type was introduced and it was proved that a Jordan superalgebra of vector type may be obtained as the (super) symmetrized algebra $A^{(+)}$ of a (-1, 1)-superalgebra of vector type A. Furthermore, the (-1, 1)-superalgebras A_{VF}: = A(V, Φ, τ, λ) and the Jordan superalgebras J_{VF}: = J(V, Φ, τ, λ) of vector fields on a line associated with an additive homomorphism τ: V$\rightarrow\Phi$ of abelian groups and partial map λ: V$\rightarrow$V were introduced in Shestakov[4]. It was proved that if V $\neq$ 0 and the map τ is injective, then the superalgebras A_{VF} and J_{VF} are prime. These superalgebras were then used to construct prime degenerate (-1, 1) and Jordan algebras.

Simple (-1, 1)-superalgebras were obtained from the associative-commutative algebras which are differentiably simple with respect to a derivation. The 2, 3-torsion free simple (-1, 1)-superalgebras were described in Shestakov[5]. The (-1, 1)-superalgebras obtained in Shestakov[5] are also called the (-1, 1)-superalgebras of vector type. Zhelyabin[6] and Zhelyabin and Shestakov[7] described the product in M by some fixed finite sets of derivations and the elements of A for (-1, 1)-superalgebras.

In terms of the Lie algebra of derivations of an integral domain, some necessary and sufficient conditions were found by Zhelyabin[8], which allowed him to construct the Jordan superalgebras of vector type whose even part is the initial integral domain. Zhelyabin[9] studied analogous questions for the (-1, 1)-superalgebras. In particular, he gave some new examples of simple and prime (-1, 1)-superalgebras of vector type whose odd part is generated as a module by two elements in the case of simple superalgebras and by an arbitrary number of elements in the case of prime superalgebras. Some properties of the universal envelopings of the simple Jordan superalgebras of vector type were described as well. In particular, he proved that the even part of the universal enveloping contains two central orthogonal idempotents such that their sum is equal to unity.

Zhelyabin[9] described simple nonassociative 2, 3-torsion free (-1, 1)-superalgebras. If B has a positive characteristic, which is the only possibility in the finite dimensional case, then the even part A is local and B is isomorphic to B (Γ, D, γ). He posed an open question that whether M is always generated by one element as an A-bimodule. He also noticed that if there exists a simple (-1, 1)-superalgebra B which does not satisfy this condition and is therefore not isomorphic to B (Γ, D, γ) the attached superalgebra B^+ will infact exemplify a new simple Jordan superalgebra.

In the present article using the constraints of Shestakov[5] and Zhelyabin[9], a description of a simple 2, 3-torsion free flexible Lie-admissible superalgebra is obtained. It is proved that the flexible Lie-admissible superalgebra F_{FLSA} [ϕ; x] over a 2, 3-torsion free field Φ on one odd generator e is isomorphic to the twisted superalgebra B_0 (Φ [Γ], D, γ_0) with free generator $\overline{1}$. The even part A in such a superalgebra F is necessarily an associative and commutative algebra and the odd part M is finitely generated associative and commutative A-bimodule over A, which is a projective module of rank 1. Multiplication in M is defined by a fixed set of derivations and by elements of A. If the bimodule M is one-generated, i.e., M = Am for a certain m$\in$M, then F is isomorphic to a twisted superalgebra of vector type B (Γ, D, γ).

MATERIALS AND METHODS

A superalgebra $R = R_0 \oplus R_1$ is a Z_2-graded algebra. A typical example of a superalgebra is the associative Grassmann algebra $G = G_0 \oplus G_1$ on a countable set of generators with a natural Z_2-grading. If $\mathfrak{M}$ is an arbitrary homogeneous variety of algebras then $R = R_0 \oplus R_1$ is called an $\mathfrak{M}$-superalgebra if its Grassmann envelope $G(R) = G_0 \otimes R_0 \oplus G_1 \otimes R_1$ belongs to $\mathfrak{M}$. In particular $R = R_0 \oplus R_1$ is a flexible Lie-admissible superalgebra if and only if it satisfies the (super) identities:

$$(x, y, z)+(-1)^{|x||z|} (z, y, x) = 0 \quad (1)$$

$$(x, y, z)+(-1)^{|x|(|y|+|z|)} (y, z, x)+(-1)^{|z|(|x|+|y|)} (z, x, y) = 0 \quad (2)$$

where, x, y, z$\in R_0 \cup R_1$ and $|r| \in \{0, 1\}$ is the parity index of an homogeneous element r: $|r| = i$ for $r \in R_i$; $[x, y]_s = xy-(-1)^{|x||y|} yx$ is the supercommutator of homogeneous elements x, y.

Let flexible Lie-admissible superalgebra satisfy the following two identities:

$$x (yz) = (xz) y \quad (*)$$

$$[[x, y], z] = 0 \quad (**)$$

In a flexible Lie-admissible algebra A, the center defined by U (A) = $\{u\in A|[u, x] = 0$, for all $x\in A\}$ is the commutative center. Throughout this study A is an associative bimodule of U (A).

Any algebra satisfies Teichmuller and semi Jacobi identities:

$$f(w, x, y, z) = (wx, y, z)-(w, xy, z)+(w, x, yz) -w(x, y, z)-(w, x, y)z = 0 \tag{3}$$

$$[xy, z]-x[y, z]-[x, z]y = (x, y, z)-(x, z, y)+(z, x, y) \tag{4}$$

For $r, s\in A$ and $x, y\in M$, Eq. 1 and (*) yield:

$$(rs, x, y) = r(s, x, y)+(r, x, y)s \tag{5}$$

One can easily check that Eq. 1 and (*) gives rise to:

$$(x, y, yz) = (x, y, z)y \tag{6}$$

Anderson and Outcult[10] have derived an important identity for $r, s, x, y\in A\cup M$, $(r, [x, y], s) = 0$ for antiflexible rings. Applying the same argument for flexible Lie-admissible superalgebras, one can obtain:

$$(r, x\circ y, s) = 0 \tag{7}$$

One can easily check that flexible ring with (*) satisfies:

$$(z, x^2, y) = (z, x, xy+yx) \tag{8}$$

Let $u\in U(A)$, $a, x, y, z\in A$. The following identities are valid in flexible Lie-admissible algebras. From Eq. 4 and 5:

$$[ux, y] = u[x, y]+2(u, x, y) \tag{9}$$

From Eq. 5 and 1:

$$(x, y, zu) = (x, y, z)u+(x, y, u)z \tag{10}$$

From Eq. 5:

$$(u^2, x, y) = 2(u, x, y)u \tag{11}$$

From Eq. 6 and 8:

$$(u, a^2, x) = 2(u, a, x)a \tag{12}$$

RESULTS

Free flexible Lie-admissible superalgebra F_{FLSA} [ϕ; x]: Let $\Phi[\Gamma]$ be the algebra of polynomials on a countable set of variables $\Gamma = \{\gamma_0,..., \gamma_n,...\}$ and let D be the derivation of $\Phi[\Gamma]$ defined by the condition $D(\gamma_i) = \gamma_{i+1}$, i = 0, 1,$\cdots$ consider the twisted superalgebra of vector type $B(\Phi[\Gamma], D, \gamma_0)$. Let $\Phi_0[\Gamma]$ be a subalgebra of polynomials without constant terms, then the subspace $\Phi_0[\Gamma]\oplus\overline{\Phi[\Gamma]}$ is a subsuperalgebra of $B(\Phi[\Gamma], D, \gamma_0)$ and is denoted by $B_0(\Phi[\Gamma], D, \gamma_0)$. It can be checked that it is generated by the odd element $\overline{1}$.

Preliminary lemmas

Lemma 1: $e^2\in U$ (F), where, U (F) is the commutative center.

Proof: Identity Eq. 2 implies:

$$3(e, e, e) = 0, (e^2, e, e)-(e, e^2, e)+(e, e, e^2) = 0$$

that is:

$$(e, e, e) = [e^2, e] = 0 \tag{13}$$

$$2(e^2, e, e) = (e, e^2, e) \tag{14}$$

Identity Eq. 3 in the form of generators are:

$$(e^3, e, e) = -(e^2, e^2, e)+(e^2, e, e^2) = e^2(e, e, e)+(e^2, e, e)e,$$
$$(e^2, e^2, e)-(e, e^3, e)+(e, e, e^3) = e(e, e^2, e)+(e, e, e^2)e,$$
$$(e^2, e, e^2)-(e, e^2, e^2)+(e, e, e^3) = e(e, e, e^2)+(e, e, e)e^2$$

which from Eq. 5 implies:

$$(e^3, e, e) = (e^2, e^2, e)+(e^2, e, e)e \tag{15}$$

$$(e, e^3, e) = (e^2, e^2, e)-e(e, e^2, e) \tag{16}$$

$$(e^2, e^2, e) = (e^2, e, e)e+[e, (e, e, e^2)] \tag{17}$$

$$(e, e^2, e^2) = [e, (e, e, e^2)] \tag{18}$$

But the Jacobi identity Eq. 2 implies:

$$2(e^3, e, e)+(e, e^3, e) = 0 \tag{19}$$

Hence by Eq. 15 one can see that $2(e^2, e^2, e)+2(e^2, e, e)e+(e, e^3, e) = 0$ and eventually using Eq. 17, 16, 14, 5 and 18, one obtains:

$$(e^2, e^2, e) = 0 \quad (20)$$

In particular:

$$[e^3, e^2] = 0 = (e, e^2, e^2)$$

Let p(x) be a monomial of degree n>3 in x. And suppose that $[q(x), x^2] = 0$ for every monomial q(x) of degree less than n and flexible Lie-admissible superalgebra A satisfies the identity $[[r, s], D(A)] = 0$ where, D(A) is the associator ideal of A. Then in F, $[[x, x]_s, D(F)] = [2x^2, D(F)] = 0$ holds. Now, $[p(x), x^2] = [x^2(x^2 q(x)), x^2]$, where, deg. $q(x) \geq 0$. Replacing x^2 by r and q(x) by q, applying Eq. 4, induction assumption, (*) and Eq. 5 yields:

$$\begin{aligned}[p(x), x^2] &= [r(rq), r]\\ &= r[rq, r]+[r, r]rq+(r, rq, r)-(r, r, rq)+(r, r, rq)\\ &= (r, r, rq) = (r, r, q)r\end{aligned}$$

The above arguments show that (r, r, q) = [rq, r] = 0, which proves the lemma.

Lemma 2: Let F be a flexible Lie-admissible superalgebra. Then $[[F, F]_s, F]_s = 0$.

Proof: It is sufficient to show that $p(x) \in U(F)$ for every monomial p of even degree. This can be done by using induction on degree of p. One can check that the square of an ideal is an ideal in a flexible Lie-admissible algebra satisfying (*), it may be assumed that p = qx or p = xq, where, q is a monomial of odd degree. On the same lines again assume that q = tx or q = xt for some even $t \in F$. By lemma 1, $x^2 \in U(F)$. By theorem 13.10 in Zhevlakov *et al.*[11], $x^2 \in U(F)$ implies that:

$$y \circ z \in U(F) \quad \text{for any odd } y, z \in F \quad (21)$$

Applying this to q, we have $(q \circ x) \in U(F)$. So it suffices to prove that (tx)x and $x(xt) \in U(F)$. By induction $t \in U(F)$, thus tx^2, $x^2t \in U(F)$ and hence Eq. 8 gives:

$$(t, x, x) = \frac{1}{2}([tx, x]_s - t[x, x]_s) = \frac{1}{2}((tx)x + x(tx) - 2tx^2) \in U(F)$$

So, $(tx)x \in U(F)$ as $(tx)x = tx^2+(t, x, x)$.

Applying the same argument one obtains $x(xt) = x^2t-(x, x, t) = x^2t+(t, x, x) \in U(F)$.

Theorem 1: A 2, 3-torsion free flexible Lie-admissible superalgebra $F_{FLSA}[\phi; x]$ over a field Φ on one odd generator e is isomorphic to the superalgebra $B_0(\Phi[\Gamma], D, \gamma_0)$ with the free generator $\overline{1}$.

Proof: Let $A = F_0$, then $F_1 = M^1x$ and $F = A \oplus M^1x$, where by assumption A is an associative and commutative Φ-algebra and M^1x is a commutative and associative A-module generated by x. If $D: a \mapsto \frac{1}{2}(x, a, x)$ then D is a derivation of A. But $F = A_0+M_0^1x$ and since $x \in A_0+M_0^1x$, it is sufficient to prove that $A_0+M_0^1x$ is sub superalgebra of F. Since $A_0 \subseteq U(F)$, for all $r, s \in A$, it can be seen that $r \cdot sx = sx \cdot r = (rs)x$. Now the product of the elements from M^1x can be calculated:

$$\begin{aligned}rx \cdot sx &= (rx \cdot s)x-(rx, s, x) = (rs \cdot x)x-r(x, s, x)+(r, s, x)x\\ &= (rs)x^2+2(x, rs, x)-r(x, s, x) = (rs)x^2+2D(rs)-rD(s)\\ &= (rs)x^2+2D(r)s+2rD(s)-rD(s)\\ &= 2D(r)s+rD(s)+(rs)x^2\end{aligned} \quad (22)$$

Hence, $A_0+M_0^1x$ is a subsuperalgebra of F and $F = A_0+M_0^1x$, $A_0 = A = F_0$. Also Eq. 21 shows that the superalgebra F is a homomorphic image of the superalgebra B_0 (M^1, D, x^2) under the homomorphism $\pi: r+\overline{s} \mapsto r+xs$. Consider the homomorphism φ: $\Phi_0[\Gamma] \rightarrow A, \gamma_i \mapsto D^i(x^2)$. Obviously, this is a homomorphism of differential algebras and it can be extended to a homomorphism of superalgebras $\tilde{\varphi}$: $B_0(\Phi[\Gamma], D, \gamma_0) \rightarrow B_0(M^1, D, x^2)$. The composition map $\pi \circ \tilde{\varphi}$ is onto map from $B_0(\Phi[\Gamma], D, \gamma_0)$ to F. As F is a free superalgebra generated by x, this map is an isomorphism.

Lemma 3: Let F = A+M be a flexible Lie-admissible superalgebra. Then for any $x, y \in M$ the mapping $\delta_{x,y}$: $a \mapsto (a, x, y)$ is a near derivation of A.

Proof: Equation 5 and 7 yield:

$$\begin{aligned}\delta_{x,y}(rs) &= (rs, x, y) = r(s, x, y)+(r, x, y)s+(r, x \circ y, s)\\ &= r \cdot \delta_{x,y}(s)+\delta_{x,y}(r) \cdot s+(r(x \circ y))s-r((s \circ y)s)\\ &= r \cdot (\delta_{x,y}-L_{x \circ y})(s)+(\delta_{x,y}+R_{x \circ y})(r) \cdot s\end{aligned}$$

Hence, the lemma is proved.

Lemma 4: If F is simple then A is Δ-simple, that is A does not contain proper Δ-invariant ideals.

Proof: Assume that I ≠ 0 is an ideal of A such that $(I, M, M) \subseteq I$. Since F is simple, $\hat{I} = F$ where, $\hat{I}$ is the ideal generated by I in F. Hence, M = IM+MI. The $M+M^2$ is a graded ideal of B; therefore $M^2 = A$. To prove that $M^2 = (IM+MI)^2 \subseteq I$, from which I = A consider:

$$IM \cdot M \subseteq I \cdot MM+(I, M, M) \subseteq I$$

Let x, y∈M and i, j∈I; xi∘jy ≡ [x, i]∘jy = -jy∘[x, i] ≡ 0 by (**). The super-semi Jacobi identity Eq. 4 along with (**) gives:

$$\begin{aligned} xi.yj &\equiv [x,i][y,j] = [[x,j]y,j] - [[x,i],j]y - ([x,i],y,j) + ([x,i],j,y) - (j,[x,i],j) \\ &\equiv -([x,i],y,j) + ([x,i],j,y) - (j,[x,i],y) \\ &\equiv -([x,i],y,j)(j,y,[x,i]) - (y,[x,i],j) - (j,[x,i],y) \\ &= -([x,i],y,j) + ([x,i],y,j) + (j,[x,i]y) - (j,[x,i],y) = 0 \end{aligned}$$

Thus, M^2⊆I and I = A.

Remark 1: Suppose that F = A+M is a flexible Lie-admissible superalgebra with its even part A not containing non zero Δ-invariant nil ideals. Then (A, B, A) = [A, B] = 0.

Theorem 2: Let F = A+M be a flexible Lie-admissible superalgebra. Then (A, B, A) = [A, B] = 0 there exists $x_1, x_2, \ldots, x_n$∈M such that $M = Ax_1+Ax_2+\ldots+Ax_n$ and the product in M is defined by $rx_i \cdot sx_j = \gamma_{ij} \cdot rs+2D_{ij}(r)s+D_{ij}(s)r$, i, j = 1,.., n, where, γ_{ij}∈A, $D_{ij} = D_{ji}$∈DerA satisfy the conditions:

$$\gamma_{ij}x_k - \gamma_{ik}x_j = (\gamma_{jk} - \gamma_{kj})x_i \tag{23}$$

$$D_{ij}(r)x_k = D_{ik}(r)x_j \tag{24}$$

for any i, j, k = 1,..., n, r∈A.

Proof: First, to prove that A satisfies the assumptions of the above remark, by lemma 4, A does not contain proper Δ-ideals and so it is sufficient to show that A itself is not nil. Since, A satisfies the identity (**), if A were nil, it would be locally nilpotent by Eq. 4 which is impossible by Shestakov[12], lemma 5. Thus A is associative and commutative and M is an associative and commutative A-bimodule.

Next to show that (A, M, M) = A, for any x, y∈M and r∈A, (*) and (**) gives:

$$(xr)y = (rx)y = (r, x, y)+r(xy) = (r, x, y)+(xy)r = (r, x, y)+2x(ry)-(xr)y$$

which yields:

$$2(x, r, y) = (r, x, y) \tag{25}$$

Assume (A, M, M) = 0; then (M, A, M) = 0 by Eq. 25. This yields A⊆Z_0, where Z = Z(F) is the center of F. Since F is simple, Z_0 is a field and F may be treated as a superalgebra over it. Let x, y∈M and xy ≠ 0. Then, by Eq. 1:

$$(xy)z = x(yz)+(zy)x-z(yx), \text{ for any } z \in M$$

Which implies that $\dim_{z_0} M \le 2$. If M = Z_0·x then (F, F, F)⊆Z_0· (x, x, x) = Z_0·$[x^2, x]$ = 0, a contradiction. Now, let M = Z_0·x+Z_0·y, $x^2 = \alpha$, $y^2 = \beta$, xy = 0, yx = η. Hence, α = 0, η = 0 since (x, x, y) = (y, x, x,). Similarly, (x, y, y) = (y, y, x) implies that β = 0, η = 0. Hence, α = β = γ = 0 and M^2 = 0, a contradiction. Thus (A, M, M) is a nonzero Δ-subspace of A. From super-linearized Eq. 6, it follows that (A, M, M) is an ideal of A and hence (A, M, M) = A.

It is known that for any x, y∈M, the mapping $\delta_{x,y}$: $a \mapsto (a, x, y)$ is a derivation of A. Thus A is differentially simple and by Posner[13] it contains unity 1. For any derivation δ∈Δ, δ(1) = 0. Moreover, (x, y, 1) = (1, y, x) = 0 for any x, y∈M. Hence, 1∈Z(F) and so 1 is unity in F. Since 1∈(A, M, M), there exist x_i, y_i∈M and r_i∈A, i = 1,..., n, such that:

$$\sum_{i=1}^{n}(r_i, x_i, y_i) = 1 \tag{26}$$

For any r∈A and x, y, z∈M, by super Eq. 6 and the associativity of the A-bimodule M, one has:

$$(r, x, y)z = -(r, x, zy) = (r, z, y)x-(r, x, zy)+(r, z, xy) = (r, z, y)x \tag{27}$$

Substituting r = r_i, x = x_i, y = y_i and then summing the resulting equations over i, by Eq. 26 one can obtain:

$$z = 1\cdot z = \left(\sum_{i=1}^{n}(r_i, x_i, y_i)\right)\cdot z = \sum_{i=1}^{n}(r_i, z, y_i)\cdot x_i \in Ax_1 + \cdots + Ax_n$$

Thus $M = Ax_1+\ldots+Ax_n$ and hence $F = A+Ax_1+\ldots+Ax_n$. Let D_{ij} be the mapping D_{ij}: $a \mapsto (x_i, a, x_j)$. By Eq. 25, $D_{ij} = \frac{1}{2}\delta_{i,j}$ is a derivation of A. For any r, s∈A, super Eq. 5 and 25 yields:

$$\begin{aligned} rx_i \cdot sx_j &= (rx_i \cdot s)x_j - (rx_i, s, x_j) \\ &= (rs \cdot x_i)x_j - r(x_i, s, x_j) + (r, s, x_j)x_i \\ &= rs \cdot x_i x_j + 2(x_i, rs, x_j) - r(x_i, s, x_j) \\ &= rs \cdot x_i x_j + 2D_{ij}(rs) - rD_{ij}(s) \\ &= \gamma_{ij} \cdot rs + 2D_{ij}(r)s + rD_{ij}(s) \end{aligned}$$

where, $\gamma_{ij} = x_i x_j$∈A.

Equalities 23 and 24 are consequences of Eq. 1 and (*).

Remark 2: If there exists x∈M such that (A, x, M) = A or A is a local algebra, then B is isomorphic to a superalgebra B (Γ, D, γ), where $\Gamma = \Gamma_0$ is a commutative and associative D-simple algebra with 0 ≠ D∈Der Γ and γ∈Γ.

Remark 3: If A is a local algebra with a maximal ideal J, the ideal cannot contain (A, M, M) since otherwise it would be a nonzero Δ-invariant ideal in A. Hence, (A, M, M)⊄J and there exists $x\in M$ such that $(A, x, M)\not\subseteq J$. By super Eq. 5 and (*), for any $a, b\in A$ and $y\in M$ we have $(a, x, y)b = (a, b, y)x+(a, x, by)+(a, b, xy) = (a, x, by)$. Consequently, (A, x, M) is an ideal in A and $(A, x, M) = A$.

Flexible Lie-admissible superalgebra F(A, Δ, Γ): As shown in theorem 2, given $x, y\in M$, the mapping, $D_{x,y}: A\mapsto A$ defined by the rule $D_{x,y}(r) = \frac{1}{2}(x, r, y)$ is a derivation. For all $r, s\in A$ and $x, y\in M$:

$$sD_{x,y}(r) = D_{sx,y}(r) = D_{x,syy}(r),\ D_{x,y}(r)\,D_{u,v}(r) = D_{u,y}(r)\,D_{x,v}(r)$$

Since F is a flexible superalgebra, $D_{x,y} = D_{y,x}$. Also, $D_{ij} = D_{x_i x_j}$ for $i, j = 1,..., n$.

From $D_{ij}(r)x_k = D_{ik}(r)x_j$ one can deduce that:

$$D_{ij}(r)x_k\cdot x_l = D_{ik}(r)x_j\cdot x_l$$

and:

$$x_l\cdot D_{ij}(r)x_k = x_l\cdot D_{ik}(r)x_j \text{ for every } r\in A$$

Let $M^* = Hom_A(M, A)$. Given r, define $\bar{r}\in(M\otimes_A M)^*$ by setting $\bar{r}(x\otimes y) = D_{x,y}(r)$. Similarly the linear mapping $^-: A\mapsto(M\otimes_A M)^*$ given by $^-: r\mapsto\bar{r}$ is a derivation of A into the A-module $(M\otimes_A M)^*$ Zhelyabin[8].

In what follows, all rings are considered as algebras over a field of characteristic 0. Now, let A be an associative-commutative ring with unity and without zero divisors (i.e., an integral domain) and let M be a finitely generated projective A-module of rank 1. Let $^-: A\mapsto(M\otimes_A M)^*$ be a nonzero mapping. Assume that:

$$\overline{rs} = r\bar{s}+s\bar{r}$$

for all $r, s\in A$. By the definition of $^-$, each pair $x, y\in M$ gives a derivation $D_{x,y}: A\mapsto A$ by the rule $D_{x,y}(r) = \bar{r}(x\otimes y)$. Then Zhelyabin[8]:

$$D_{x,y} = D_{y,x},\ D_{rx,y} = rD_{x,y},\ D_{x,y}(r)z = D_{z,y}(r)x \tag{28}$$

for $r\in A$ and $x, y, z\in M$.

Let $x_1,..., x_n$ be some generators of the A-module M, i.e., $M = Ax_n+...+Ax_n$. Put $D_{ij} = D_{x_i,x_j}$, $i, j = 1,\cdots, n$. Then $D_{ij} = D_{ji}$.

Lemma 5: $D_{ij}(r)\,D_{kl} = D_{kl}(r)\,D_{ij}$ and $D_{ij}(r)\,D_{kl} = D_{kj}(r)\,D_{il}$ for every $r\in A$.

Proof: Let $r, s\in A$. Then:

$$D_{ij}(r)D_{kl}(s) = \bar{r}(x_i\otimes x_j)\bar{s}(x_k\otimes x_l) = \bar{r}(x_k\otimes x_l)\bar{s}(x_i\otimes x_j) = D_{kl}(r)D_{ij}(s)$$

Similarly:

$$D_{ij}(r)\,D_{kl} = D_{kj}(r)D_{il}$$

Fix some elements γ_{ij} of A, where $i, j = 1,..., n$.

Lemma 6: Assume that:

$$\gamma_{lk}D_{ij}+D_{lk}D_{ij} = \gamma_{lj}D_{ik}+D_{lj}D_{ik} \tag{29}$$

$$\gamma_{kl}D_{ij}+2D_{kl}D_{ij} = \gamma_{jl}D_{ik}+2D_{jl}D_{ik} \tag{30}$$

for all $i, j, k, l = 1,..., n$. Then Eq. 23 holds, i.e., $\gamma_{ij}x_k-\gamma_{ik}x_j = (\gamma_{jk}-\gamma_{kj})x_i$ for all $i, j, k, = 1,..., n$.

Proof: From identities Eq. 29 and 30 it can be seen that:

$$(\gamma_{kl}-2\gamma_{lk})D_{ij} = (\gamma_{jl}-2\gamma_{lj})D_{ik}$$

This implies:

$$D_{(\gamma_{kl}-2\gamma_{lk})x_j,x_i}(r) = D_{(\gamma_{jl}-2\gamma_{lj})x_k,x_i}(r)$$

for every $r\in A$. Therefore:

$$\bar{r}((\gamma_{kl}-2\gamma_{lk})x_j\otimes x_i) = \bar{r}((\gamma_{jl}-2\gamma_{lj})x_k\otimes x_i)$$

By the conditions on A and M Zhelyabin[8]:

$$(\gamma_{kl}-2\gamma_{lk})x_j = (\gamma_{jl}-2\gamma_{lj})x_k$$

By Eq. 30 and 28:

$$\begin{aligned}2D_{kl}D_{ij} &= -\gamma_{kl}D_{ij}+\gamma_{jl}D_{ik}+2D_{jl}D_{ik} = -\gamma_{kl}D_{ij}+\gamma_{jl}D_{ik}+2D_{lj}D_{ki}\\ &= -\gamma_{kl}D_{ij}+\gamma_{jl}D_{ik}-\gamma_{lj}D_{ki}+\gamma_{ij}D_{kl}+2D_{ij}D_{kl}\end{aligned}$$

Then:

$$2[D_{kl}, D_{ij}] = -\gamma_{kl}D_{ij}+(\gamma_{jl}-\gamma_{lj})D_{ik}+\gamma_{ij}D_{kl}$$

for all $i, j, k, l = 1, \cdots, n$. Hence:

$$2[D_{lk}, D_{ij}] = -\gamma_{lk}D_{ij}+(\gamma_{jk}-\gamma_{kj})D_{li}+\gamma_{ij}D_{lk}$$

Thus:

$$2[D_{kl}, D_{ij}]-2[D_{lk}, D_{ij}] = (\gamma_{lk}-\gamma_{kl})D_{ij}+(\gamma_{jl}-\gamma_{lj})D_{ki}+(\gamma_{kj}-\gamma_{jk})D_{li} = 0$$

Similarly:

$$(\gamma_{lk}-\gamma_{kl})x_j+(\gamma_{jl}-\gamma_{lj})x_k+(\gamma_{kj}-\gamma_{jk})x_l = 0$$

Then by the above:

$$-\gamma_{lk}x_j+\gamma_{lj}x_k+(\gamma_{kj}-\gamma_{jk})x_l = 0$$

Hence:

$$\gamma_{lj}x_k-\gamma_{lk}x_j = (\gamma_{jk}-\gamma_{kj})x_l$$

Therefore:

$$\gamma_{lj}x_k-\gamma_{ik}x_j = (\gamma_{jk}-\gamma_{kj})x_i$$

Lemma 7: Assume that:

$$\gamma_{lk}D_{ij}+D_{lk}D_{ij} = \gamma_{lj}D_{ik}+D_{lj}D_{ik},\ \gamma_{ij}x_k-\gamma_{ik}x_j = (\gamma_{jk}-\gamma_{kj})x_i$$

for all i, j, k, l = 1,..., n. Then Eq. 30 holds, i.e.,

$$\gamma_{kl}D_{ij}+2D_{kl}D_{ij} = \gamma_{jl}D_{ik}+2D_{jl}D_{ik}$$

for all i, j, k = 1,..., n.

Proof: By Eq. 28 and 29:

$$\gamma_{lk}D_{ij}+D_{lk}D_{ij} = \gamma_{lj}D_{ik}+D_{lj}D_{ik},\ \gamma_{ji}D_{kl}+D_{ji}D_{kl} = \gamma_{jl}D_{ik}+D_{jl}D_{ik}$$

Therefore by Eq. 28:

$$[D_{ij}, D_{kl}] = \gamma_{lk}D_{ij}-\gamma_{ji}D_{kl}+(\gamma_{jl}-\gamma_{lj})\ D_{ik}$$

Then with l = j:

$$[D_{ij}, D_{kj}] = \gamma_{jk}D_{ij}-\gamma_{ji}D_{kj}$$

By the hypothesis of the lemma:

$$[D_{ij}, D_{kj}] = D_{\gamma_{jk}x_i, x_j} + D_{-\gamma_{ji}x_k, x_j} = D_{\gamma_{jk}x_i-\gamma_{ji}x_k, x_j} = -(\gamma_{ik}-\gamma_{ki})D_{jj}$$

By proposition Eq. 5 of Zhelyabin[8]:

$$(\gamma_{ij}-\gamma_{ji})D_{lk}+D_{ij}\ D_{lk} = (\gamma_{lj}-\gamma_{jl})\ D_{ik}+D_{lj}\ D_{ik}$$

Since $\gamma_{ji}D_{kl}+D_{ji}D_{kl} = \gamma_{jl}D_{ik}+D_{jl}D_{ik}$, by summing up the last two equalities and by replacing i by k and j by l, it is obvious that:

$$\gamma_{kl}\ D_{ij}+2D_{kl}\ D_{ij} = \gamma_{jl}\ D_{ik}+2D_{jl}\ D_{ik}$$

Similarly the following lemma may be proved.

Lemma 8: Assume that:

$$\gamma_{kl}D_{ij} + 2D_{kl}D_{ij} = \gamma_{jl}D_{ik} + 2D_{jl}D_{ik},\ \gamma_{ij}x_k - \gamma_{ik}x_j = (\gamma_{jk} - \gamma_{kj})x_i$$

for all i, j, k, l = 1,..., n. Then Eq. 29 holds, i.e., for all i, j, k = 1,..., n:

$$\gamma_{lk}\ D_{ij}+D_{lk}\ D_{ij} = \gamma_{lj}\ D_{ik}+D_{lj}\ D_{ik}$$

Lemma 9: Assume that:

$$[D_{ik}, D_{kj}] = -(\gamma_{ij} - \gamma_{ji})D_{kk} \quad (31)$$

$$2[D_{ik}, D_{kj}] = \gamma_{jk}D_{ik} - \gamma_{ik}D_{kj} \quad (32)$$

for all i, j, k, l = 1,..., n. Then Eq. 29 and 30 hold.

Proof: For every r∈A., from Eq. 31 and 32:

$$-2(\gamma_{ij} - \gamma_{ji})D_{kk}(r) = \gamma_{jk}D_{ik}(r) - \gamma_{ik}D_{kj}(r)$$

Then:

$$-2(\gamma_{ij} - \gamma_{ji})\overline{r}(x_k \otimes x_k) = \gamma_{jk}\overline{r}(x_i \otimes x_k) - \gamma_{ik}\overline{r}(x_j \otimes x_k)$$

Hence:

$$-2(\gamma_{ij} - \gamma_{ji})x_k = \gamma_{jk}x_i - \gamma_{ik}x_j$$

By Eq. 31 and corollary of Zhelyabin[8]:

$$(\gamma_{ij} - \gamma_{ji})x_k = (\gamma_{ik} - \gamma_{ki})x_j + (\gamma_{kj} - \gamma_{jk})x_i$$

Therefore:

$$(\gamma_{ij} - \gamma_{ji})x_k = \gamma_{ki}x_j - \gamma_{kj}x_i$$

Let G (l.k.i, j) = $D_{lk}\ D_{ij}$-$D_{lj}\ D_{ik}$. Now for every r∈A, lemma 5 gives:

$$D_{ii}(r)G(l.k.i, j) = D_{ii}(r)(D_{lk}D_{ij} - D_{lj}D_{ik}) = D_{il}(r)(D_{ik}D_{ij} - D_{ij}D_{ik})$$
$$= -(\gamma_{kj} - \gamma_{jk})D_{il}(r)D_{ii} = -(\gamma_{kj} - \gamma_{jk})D_{ii}(r)D_{il}$$

Since, A is an integral domain:

$$G(l.k.i, j) = -(\gamma_{kj} - \gamma_{jk})D_{il} = (\gamma_{jk} - \gamma_{kj})D_{il} = \gamma_{lj}D_{ik} - \gamma_{lk}D_{ij}$$

Hence, $\gamma_{lk}D_{ij}+D_{lk}D_{ij} = \gamma_{lj}D_{ik}+D_{lj}D_{ik}$. Analogously, Eq. 30 may be proved.

Corollary 1: Equation 29 and 30 are equivalent to Eq. 31 and 32.

Proof: By lemma 9 it is sufficient to show that Eq. 29 and 30 imply Eq. 31 and 32.

Assume Eq. 29 and 30. Then lemma 6 implies:

$$\gamma_{ij}x_k - \gamma_{ik}x_j = (\gamma_{jk} - \gamma_{kj})x_i$$

As it was shown in lemma 7:

$$[D_{ij}, D_{kj}] = -(\gamma_{ik} - \gamma_{ki})D_{jj}$$

So, Eq. 31 is valid.

As it was shown in lemma 6:

$$2[D_{lk}, D_{ij}] = -\gamma_{lk}D_{ij} + (\gamma_{jk} - \gamma_{kj})D_{li} + \gamma_{ij}D_{lk}$$

Now replacing j by k and l by j:

$$2[D_{jk}, D_{ik}] = -\gamma_{jk}D_{ik} + \gamma_{ik}D_{jk}$$

This implies:

$$2[D_{ik}, D_{kj}] = \gamma_{jk}D_{ik} - \gamma_{ik}D_{kj}$$

i.e., Eq. 32 is valid.

Theorem 3: Let A be an integral domain. Let $M = Ax_1+...+Ax_n$ be a finitely generated projective A-module of rank 1. Assume that the mapping ⍰: $A \mapsto (M \otimes_A M)^*$ is a nonzero derivation of A into the A-module $(M \otimes_A M)^*$ and $\Delta = \{D_{ij} | i, j = 1,..., n\}$ is a set of derivations of A, where $D_{ij}(a) = \overline{r}(x_i \otimes x_j)$. Fix the subset $\Gamma = \{\gamma_{ij} \in A, i, j = 1,..., n\}$ of A. Consider the vector space $F(A, \Delta, \Gamma) = A \oplus M$ and equip it with the structure of a Z_2-graded algebra putting:

$$r \cdot s = rs,\ r \cdot (sx_i) = (sx_i) \cdot r = (rs)x_i,\ i = 1,\dots,n$$

$$rx_i \cdot sx_j = \gamma_{ij}rs + 2D_{ij}(r)s + rD_{ij}(s),\ i, j = 1,\dots,n$$

where, r, s∈A and rs is the product of r and s in A. Assume that the derivations D_{ij} and the elements of Γ satisfy Eq. 31 and 32 then $F(A, \Delta, \Gamma)$ is a flexible, Lie-admissible Superalgebra with even part A and odd part M.

Proof: First to show that the product in $F(A, \Delta, \Gamma)$ is correctly defined, let $r_1x_1+...+r_nx_n = 0$. Then $r_1D_{1k} + \cdots + r_nD_{nk} = D_{r_1x_1+\cdots+r_nx_n, x_k} = 0$ for every fixed k = 1,..., n.

Then by lemma 9, Eq. 29 holds. So:

$$\sum_{i=1}^{n} r_iD_{ik} \cdot sD_{kj} = \sum_{i=1}^{n} r_iD_{ik}(s)D_{kj} + \sum_{i=1}^{n} r_isD_{ik} \cdot D_{kj}$$

$$= \sum_{i=1}^{n} r_iD_{ik}(s)D_{kj} - \sum_{i=1}^{n} r_is\gamma_{ik}D_{kj} + \sum_{i=1}^{n} r_is\gamma_{ij}D_{kk} + \sum_{i=1}^{n} r_isD_{ij} \cdot D_{kk}$$

$$= \sum_{i=1}^{n} r_is\gamma_{ij}D_{kk} + \sum_{i=1}^{n} r_iD_{ij}(s)D_{kk} - \sum_{i=1}^{n} r_is\gamma_{ik}D_{kj}$$

On the other hand, by Eq. 30:

$$2sD_{kj}\sum_{i=1}^{n} r_iD_{ik} = \sum_{i=1}^{n} 2sD_{kj}(r_i)D_{ik} + \sum_{i=1}^{n} 2r_isD_{kj} \cdot D_{ik}$$

$$= \sum_{i=1}^{n} 2sD_{kj}(r_i)D_{ik} + \sum_{i=1}^{n} 2r_isD_{jk} \cdot D_{ki}$$

$$= \sum_{i=1}^{n} 2sD_{kj}(r_i)D_{ik} + \sum_{i=1}^{n} r_is\gamma_{ik}D_{kj} + \sum_{i=1}^{n} 2r_isD_{ik} \cdot D_{kj} - \sum_{i=1}^{n} r_is\gamma_{jk}D_{ki}$$

$$= \sum_{i=1}^{n} 2sD_{kj}(r_i)D_{ik} + \sum_{i=1}^{n} r_is\gamma_{ik}D_{kj}$$

Hence:

$$\sum_{i=1}^{n} r_is\gamma_{ik}D_{kj} = -\sum_{i=1}^{n} 2sD_{kj}(r_i)D_{ik} = -\sum_{i=1}^{n} 2sD_{ij}(r_i)D_{kk}$$

Therefore:

$$\sum_{i=1}^{n} r_iD_{ik} \cdot sD_{kj} = \sum_{i=1}^{n} r_is\gamma_{ij}D_{kk} + \sum_{i=1}^{n} r_iD_{ij}(s)D_{kk} + 2sD_{ij}(r_i)D_{kk} = 0$$

Since A is an integral domain:

$$\sum_{i=1}^{n} r_is\gamma_{ij} + r_iD_{ij}(s) + 2sD_{ij}(r_i) = 0$$

Hence:

$$(r_1x_1+\dots+r_nx_n) \cdot sx_j = 0$$

Therefore, the product in F(A, Δ, Γ) is correctly defined. Hence F(A, Δ, Γ) is a flexible Lie-admissible superalgebra.

DISCUSSION

Kamiya and Okubo[3] constructed some class of Lie superalgebras from nonsuper flexible Lie-admissible algebras without resorting to (-1, -1)-freudenthal-kantor tiple systems. Their motivation is a base for constructing the flexible Lie-admissible Grassmann algebra and hence the free unital flexible Lie-admissible superalgebra with one odd generator is isomorphic to the flexible Lie-admissible superalgebra of vector type B(Φ(Γ), D, γ_0) where, D(γ_i) = γ_{i+1} for i = 0, 1,... which was analogously proved for (-1, 1) superalgebras of vector type in Zhelyabin[9].

If the superalgebra is not a superalgebra of nondegenerate bilinear superform then it's even part A is differentiably simple algebra and the odd part M is a finitely generated projective A-module of rank 1. Yuan[14] restricted his studies on derivations of differentiably simple associative commutative algebra of n-torsion free (n>0). Whereas in this study remarks 2 and 3 to theorem 2 state that every simple n-torsion free (n>3) nonassociative flexible Lie-admissible superalgebra is local algebra and hence isomorphic to the superalgebra B(Γ, D, γ). Pchelintsev and Shestakov[15] proved that (-1, 1) monster constructed by Pchelintsev[16] generates the same variety of algebras as the Grassmann (-1, 1) algebra. Following Pchelintsev[16] flexible Lie-admissible monster may be constructed.

CONCLUSION

- A connection between the integral domains, finitely generated projective modules over them, derivations of an integral domain and the flexible Lie-admissible superalgebras of vector type has been established
- If A is an integral domain and $M = Ax_1+...+Ax_n$ be a finitely generated projective A-module of rank 1, then F(A, ΔΓ) is a flexible Lie-admissible superalgebra with even part A and odd part M provided that the mapping $A \mapsto (M \otimes_A M)^*$ is a nonzero derivation of A into the A-module $(M \otimes_A M)^*$, Δ = {D_{ij}|i, j = 1,..., n} is a set of derivations of A where, $D_{ij}(a) = \bar{a}(x_i \otimes x_j)$
- For F = A+M, a flexible Lie-admissible superalgebra the derivation D_{ij} is half of $\delta_{i,j}$

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