Asymptotic Properties of Parameters for Linear Circular Functional Relationship Model

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Abstract: This study discusses the asymptotic properties of parameters for unreplicated linear circular functional relationship model. The model is formulated assuming both variables are circular, subject to errors and there is a linear relationship between them. The Maximum Likelihood Estimation (MLE) have been used to estimate the slope parameters (β), intercept (κ) and error concentration parameters for both variables which are κ and ν, respectively. The Fisher information matrix have been derived and based on this we estimated the asymptotic covariance matrix of κ, ν, α and β.

Key words: Asymptotic properties, linear circular functional relationship model, error of concentration parameters, Fisher information matrix, maximum likelihood estimation

INTRODUCTION

Circular data are rather special, but they do arise in surprisingly many different contexts. Mardia (1972) lists examples within the geological, meteorological, biological, astronomical and economical sciences. The difficulty in the statistical analysis of circular data stems from the disparate topologies of the circle and the straight line: if angles are recorded in radians in the range (-π, π), then directions close to the opposite end-points are near neighbours in a metric which respect the topology of the circle, but maximally distant in a linear metric. Some of the examples are the wind and wave direction data measured by two different methods, the anchored wave buoy and HF radar system as given by Sova (1995).

The problem of specifying distributional models and stochastic processes on the circle has been tackled in various ways. Consider first the problem of constructing models for a circular random variables X whose domain as (-π, π). The canonical example of this type is the von Mises distribution with probability density function given by Mardia (1972) and can be defined as f(x; κ, θ) = exp{κ cos(x-θ)}/2π∫π-π exp{κ cos(x-θ)} dx. In this model, the parameter θ represent the model direction, while κ is a concentration parameter.

The linear circular functional relationship model refers to the case when both variables are circular and as an analogy to the linear functional relationship model, we assume both observations for circular variables X and Y are observed with errors. We also assume that the errors are independently distributed and follow the von Mises distribution with mean zero and concentration parameters κ and ν, respectively.

THE MODEL

Suppose x_i and y_i are observed values of the circular variables X and Y, respectively, thus 0 ≤ x_i, y_i < 2π, for i = 1, ..., n. For any fixed X_i, we assume that the observations x_i and y_i have been measured with errors δ_i and e_i, respectively. We use the same notation here as in the unreplicated linear functional relationship model (for continuous variables or linear metric) and thus the full model can be written as
\[ x_i = x_i + \delta_i \quad \text{and} \quad y_i = Y_i + \epsilon_i, \quad \text{where} \]
\[ Y_i = \alpha + \beta X_i \mod{2\pi}, \quad \text{for} \quad i = 1, 2, \ldots, n \quad (1) \]

We also assumed \( \delta_i \) and \( \epsilon_i \) are independently distributed with von Mises distributions, that is \( \delta_i \sim \text{VM}(0, \kappa) \) and \( \epsilon_i \sim \text{VM}(0, \kappa) \). There are \((n+4)\) parameters to be estimated, i.e., \( \alpha, \beta, \kappa, \nu \) and the incidental parameters \( X_1, \ldots, X_n \) by maximum likelihood estimation. The log likelihood function is given by:

\[
\log L(\alpha, \beta, \kappa, \nu, X_1, \ldots, X_n; x_1, \ldots, x_n, y_1, \ldots, y_n) = \\
-2n \log(2\pi) - n \log L_1(\kappa) - n \log L_2(\nu) \\
+ \nu \sum \cos(x_i - X_i) + \nu \sum \cos(y_i - \alpha - \beta X_i) 
\]

Differentiating \( \log L \) with respect to parameters \( \alpha, \beta, \kappa, \nu \) and \( X_0 \) we can obtain \( \dot{\alpha}, \dot{\beta}, \dot{\kappa}, \dot{\nu} \) and \( \dot{X}_0 \). The first partial derivative of the log likelihood function with respect to \( \alpha \) is

\[
\frac{\partial \log L}{\partial \alpha} = -\sum \sin(y_i - \alpha - \beta X_i) 
\]

Setting this equal to zero and simplifying we get:

\[
\sum \sin(y_i - \beta X_i) \cos \dot{\alpha} - \sum \cos(y_i - \beta X_i) \sin \dot{\alpha} = 0 
\]

This gives,

\[
\tan \dot{\alpha} = \frac{\sum \sin(y_i - \beta X_i)}{\sum \cos(y_i - \beta X_i)} 
\]

\[
\dot{\alpha} = \tan^{-1} \left( \frac{\sum \sin(y_i - \beta X_i)}{\sum \cos(y_i - \beta X_i)} \right) 
\]

\[
- \tan^{-1} \left( \frac{S}{C} \right), \quad \text{say} 
\]

That is,

\[
\dot{\alpha} = \begin{cases} 
\tan^{-1} \left( \frac{S}{C} \right), & S > 0, C > 0 \\
\tan^{-1} \left( \frac{S}{C} \right) + \pi, & C < 0 \\
\tan^{-1} \left( \frac{S}{C} \right) + 2\pi, & S < 0, C > 0 
\end{cases} \quad (2) 
\]

The first partial derivative with respect to \( X_0 \) is:

\[
\frac{\partial \log L}{\partial X_0} = \nu \sin(y_i - X_0) + \nu \beta \sin(y_i - \alpha - \beta X_i) 
\]

If we set this equal to zero, we may solve iteratively for \( X_0 \) given some initial guesses. Suppose \( \hat{X}_0 \) is an initial estimate for \( X_0 \). We write as suggested by Mardia (1972),
\[ x_i - \hat{x}_i - x_i - \hat{x}_{i0} + \hat{x}_i - \hat{x}_{i0} = (x_i - \hat{x}_{i0}) + \Delta_i \]

where, \( \Delta_i = \hat{x}_{i0} - \hat{x}_i \). Also we have \( y_i - \hat{a} - \hat{b}\hat{x}_i = (y_i - \hat{a} - \hat{b}\hat{x}_{i0}) + \hat{b}\Delta_i \). Hence the above equation becomes,

\[ \sin(x_i - \hat{x}_{i0} + \Delta_i) + \frac{\hat{b}}{\hat{c}} \sin(y_i - \hat{a} - \hat{b}\hat{x}_{i0} + \hat{b}\Delta_i) = 0. \]

For small \( \Delta_i \), we have

\[ \cos \Delta_i = 1, \quad \cos \Delta_i = \Delta_i, \quad \text{and} \quad \sin \Delta_i = \beta \Delta_i. \]

Hence the equation is simplified (approximately) to:

\[ \hat{x}_{i0} = \hat{x}_{i0} + \frac{\sin(x_i - \hat{x}_{i0}) + \frac{\hat{b}}{\hat{c}} \sin(y_i - \hat{a} - \hat{b}\hat{x}_{i0})}{\cos(x_i - \hat{x}_{i0}) + \frac{\hat{b}}{\hat{c}} \cos(y_i - \hat{a} - \hat{b}\hat{x}_{i0})} \quad (3) \]

where, \( \hat{x}_{i0} \) is an improvement of \( \hat{x}_{i0} \).

The first partial derivative with respect to \( \hat{b} \) is:

\[ \frac{\partial \log L}{\partial \hat{b}} = \sum x_i \sin(y_i - \hat{a} - \hat{b}x_i) \]

\( \hat{b} \) may also be obtained iteratively. Suppose \( \hat{b}_0 \) is an initial estimate of \( \hat{b} \). Then

\[ y_i - \hat{a} - \hat{b}\hat{x}_i = (y_i - \hat{a} - \hat{b}_0\hat{x}_i) + \Delta_i, \quad \text{where} \quad \Delta_i = \hat{b}_0 - \hat{b}. \]

For small \( \Delta_i \), setting the partial derivative to zero gives

\[ \hat{b} = \hat{b}_0 + \frac{\sum \hat{x}_i \sin(y_i - \hat{a} - \hat{b}_0\hat{x}_i)}{\sum \hat{x}_i^2 \cos(y_i - \hat{a} - \hat{b}_0\hat{x}_i)} \quad (4) \]

where, \( \hat{b}_0 \) is an improvement of \( \hat{b}_0 \).

The first partial derivative with respect to \( \kappa \) is

\[ \frac{\partial \log L}{\partial \kappa} = -n \frac{\sum \iota'(\kappa)}{\iota(\kappa)} + \sum \cos(x_i - \hat{x}_i) \]

This give

\[ \sum \cos(x_i - \hat{x}_i) = n \frac{\sum \iota'(\kappa)}{\iota(\kappa)} = n \frac{\iota'(\kappa)}{\iota(\kappa)} = nA(\kappa) \]

where, the function \( A \) is ratio of the modified Bessel function for the first kind of order one and the first kind of order zero. Thus

\[ A(\kappa) = \frac{1}{n} \sum \cos(x_i - \hat{x}_i) \]

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Hence

\[ \hat{\kappa} = A^{-1} \left( \frac{1}{n} \sum \cos(x_i - \hat{\lambda}_i) \right) \]  \hspace{1cm} (5)

The estimate of \( \kappa \) can be obtained by using various simple approximations, for example for \( w \) near 1, the approximation is:

\[ A^{-1}(w) = \frac{9 - 8w + 3w^2}{8(1-w)} \]

Finally, the first partial derivative with respect to \( \nu \) is:

\[ \frac{\partial \log L}{\partial \nu} = \sum \cos(y_i - \alpha - \beta X_i) - nA(\nu) \]

and hence

\[ \hat{\nu} = A^{-1} \left( \frac{1}{n} \sum \cos(y_i - \hat{\alpha} - \hat{\beta} X_i) \right) \]  \hspace{1cm} (6)

\( \nu \) may also be estimated by the above approximations.

It can be shown that all the parameters above can be estimated by using the following simple procedure. Firstly, find the estimate of parameters for the model assuming equal error concentration parameters or the ratio of concentration parameters is known (Hussin, 2001). Next, these estimates will be used as the initial estimate in the iterative procedure to obtain the estimates as defined in Eq. 2-6. In the following section we consider the Fisher information matrix of the model and derived the asymptotic properties of parameters.

**FISHER INFORMATION MATRIX OF PARAMETERS**

In this section we consider the Fisher information matrix of \( \hat{\kappa}, \hat{\nu}, \hat{\alpha}, \) and \( \hat{\beta} \) for the unreplicated linear circular functional relationship model. As usual we have taken into consideration that the appropriate regularity conditions (see as an example, Cox and Hinkley, 1974) are satisfied.

The first partial derivatives of the log likelihood function are

\[ \frac{\partial \log L}{\partial \alpha} = \nu \sum \sin(y_i - \alpha - \beta X_i), \]

\[ \frac{\partial \log L}{\partial \beta} = \nu \sin(x_i - X_i) \]

\[ \frac{\partial \log L}{\partial \nu} = \nu \sum X_i \sin(y_i - \alpha - \beta X_i), \]

\[ \frac{\partial \log L}{\partial \nu} = -n \frac{I_{\nu}(x)}{I_{\nu}(x)} + \sum \cos(x_i - X_i) \]

and

\[ \frac{\partial \log L}{\partial \nu} = -n \frac{I_{\nu}(\nu)}{I_{\nu}(\nu)} \sum \cos(y_i - \alpha - \beta X_i) \]

The second partial derivatives for log likelihood function and their negative expected values are given by.
\[
\frac{\partial \log L}{\partial X_i} = -\kappa \cos(X_i - X_j) - \nu \beta \cos(Y_i - \alpha - \beta X_i), \text{ hence }
E \left[ \frac{\partial \log L}{\partial X_i} \right] = nA'(\kappa) + \nu \beta A(\nu).
\]
\[
\frac{\partial^2 \log L}{\partial X_i \partial X_j} = 0, \text{ hence } E \left[ \frac{\partial^2 \log L}{\partial X_i \partial X_j} \right] = 0, \text{ for } i \neq j.
\]
\[
\frac{\partial \log L}{\partial X_i \partial \alpha} = -\nu \beta \cos(Y_i - \alpha - \beta X_i), \text{ hence }
E \left[ \frac{\partial \log L}{\partial X_i \partial \alpha} \right] = \nu \beta A(\nu).
\]
\[
\frac{\partial \log L}{\partial X_i \partial \beta} = -\nu \beta X_i \cos(Y_i - \alpha - \beta X_i), \text{ hence }
E \left[ \frac{\partial \log L}{\partial X_i \partial \beta} \right] = \nu \beta X_i A(\nu) \text{ because } E[\sin(Y_i - \alpha - \beta X_i)] = 0.
\]
\[
\frac{\partial \log L}{\partial X_i \partial c} = \sin(Y_i - X_i), \text{ hence } E \left[ \frac{\partial \log L}{\partial X_i \partial c} \right] = 0, \text{ because }
E[\sin(Y_i - X_i)] = 0.
\]
\[
\frac{\partial^2 \log L}{\partial \alpha \partial \beta} = -\nu \sum X_i \cos(Y_i - \alpha - \beta X_i), \text{ hence }
E \left[ \frac{\partial^2 \log L}{\partial \alpha \partial \beta} \right] = -\nu A(\nu) \sum X_i.
\]
\[
\frac{\partial \log L}{\partial \alpha \partial \kappa} = 0, \text{ hence } E \left[ \frac{\partial \log L}{\partial \alpha \partial \kappa} \right] = 0.
\]
\[
\frac{\partial \log L}{\partial \beta \partial \kappa} = -\nu \sum X_i \cos(Y_i - \alpha - \beta X_i), \text{ hence }
E \left[ \frac{\partial \log L}{\partial \beta \partial \kappa} \right] = -\nu A(\nu) \sum X_i.
\]
\[
\frac{\partial \log L}{\partial \beta \partial \beta} = 0, \text{ hence } E \left[ \frac{\partial \log L}{\partial \beta \partial \beta} \right] = 0.
\]
\[
\frac{\partial \log L}{\partial \beta \partial \alpha} = \sum X_i \sin(Y_i - \alpha - \beta X_i), \text{ hence } E \left[ \frac{\partial \log L}{\partial \beta \partial \alpha} \right] = 0.
\]
\[
\frac{\partial \log L}{\partial \beta \partial c} = -nA'(\kappa), \text{ hence } E \left[ \frac{\partial \log L}{\partial \beta \partial c} \right] = nA'(\kappa).
\]

where, \[A'(\kappa) = 1 - A'(\kappa) - \frac{A(\kappa)}{\kappa}\].

\[
\frac{\partial \log L}{\partial \alpha \partial \nu} = 0, \text{ hence } E \left[ \frac{\partial \log L}{\partial \alpha \partial \nu} \right] = 0 \text{ and } \frac{\partial \log L}{\partial \beta \partial \nu} = -nA'(\nu) \text{ hence } E \left[ \frac{\partial \log L}{\partial \beta \partial \nu} \right] = nA'(\nu).
\]
Next we find the estimated Fisher information matrix, i.e., \( F \), for \( \bar{X}_i, \ldots, \bar{X}_n, \bar{k}, \bar{v}, \bar{\alpha}, \) and \( \bar{\beta} \) and given by

\[
F = \begin{bmatrix}
B & 0 & E \\
0 & C & 0 \\
E^T & 0 & D
\end{bmatrix}
\]

where, \( B \) is an \( n \times n \) matrix given by:

\[
B = \begin{bmatrix}
\bar{\kappa}A(\bar{\kappa}) + \bar{\nu}\bar{\beta}^2 A(\bar{\nu}) & 0 \\
\vdots & \ddots & \vdots \\
0 & \bar{\kappa}A(\bar{\kappa}) + \bar{\nu}\bar{\beta}^2 A(\bar{\nu})
\end{bmatrix}
\]

\( E \) is an \( n \times 2 \) matrix, given by:

\[
E = \begin{bmatrix}
\bar{\nu}\bar{\beta}A(\bar{\nu}) & \bar{\nu}\bar{\beta}\bar{X}_n A(\bar{\nu}) \\
\vdots & \ddots & \vdots \\
\bar{\nu}\bar{\beta}A(\bar{\nu}) & \bar{\nu}\bar{\beta}\bar{X}_n A(\bar{\nu})
\end{bmatrix}
\]

\( C \) is a \( 2 \times 2 \) matrix given by:

\[
C = \begin{bmatrix}
\bar{n}A'(\bar{\kappa}) & 0 \\
0 & \bar{n}A'(\bar{\nu})
\end{bmatrix}
\]

and \( D \) is a \( 2 \times 2 \) matrix given by:

\[
D = \begin{bmatrix}
\bar{\nu}\bar{\beta}A(\bar{\nu}) & \bar{\nu}\bar{\beta}\bar{X}_n A(\bar{\nu}) \\
\bar{\nu}A(\bar{\nu})\sum \bar{X}_i & \bar{\nu}A(\bar{\nu})\sum \bar{X}_i^2
\end{bmatrix}
\]

We are primarily interested in the bottom right minor of the inverse of \( F \) of order \( 4 \times 4 \), which forms the asymptotic covariance matrix of \( \bar{k}, \bar{v}, \bar{\alpha}, \) and \( \bar{\beta} \). From the theory of partitioned matrices Graybill (1961), this is given by

\[
\text{Var} \begin{bmatrix}
\bar{k} \\
\bar{v} \\
\bar{\alpha} \\
\bar{\beta}
\end{bmatrix} = \begin{bmatrix}
C^{-1} & 0 \\
0 & (D^E B^{-E} E)^{-1}
\end{bmatrix}
\]

It can be shown that

\[
C^{-1} = \begin{bmatrix}
\left(\bar{n}A'(\bar{\kappa})\right)^{-1} & 0 \\
0 & \left(\bar{n}A'(\bar{\nu})\right)^{-1}
\end{bmatrix}
\]

and

\[
(D - E^T B^{-E} E) = \begin{bmatrix}
\sum \bar{X}_i^2 - \sum \bar{X}_i \\
-\sum \bar{X}_i
\end{bmatrix}
\]

Where:

\[
H = \frac{\bar{\kappa}A(\bar{\kappa}) + \bar{\nu}\bar{\beta}^2 A(\bar{\nu})}{\bar{\kappa}A(\bar{\kappa})A(\bar{\nu})\left(\sum \bar{X}_i^2 - (\sum \bar{X}_i)^2\right)}
\]

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Therefore, the asymptotic covariance matrix for $\hat{\kappa}, \hat{\nu}, \hat{\alpha}, \text{ and } \hat{\beta}$ is given by,

$$
G = \begin{bmatrix}
(nA'(\hat{k}))^{-1} & 0 & 0 & 0 \\
0 & (nA'(\hat{\nu}))^{-1} & 0 & 0 \\
0 & 0 & H\sum \hat{X}_i^2 & -H\sum \hat{X}_i \\
0 & 0 & -H\sum \hat{X}_i & Hn
\end{bmatrix}
$$

RESULTS

This study presents the unreplicated linear circular functional relationship model. By using various approximation, matrix operations and Fisher information matrix we can find the estimated covariance matrix of parameters. In particular, we have the following results:

$$
\text{Var}(\hat{k}) = \frac{\hat{k}}{n(\hat{k} - \hat{k}A'(\hat{k}) - A(\hat{k}))}
$$

$$
\text{Var}(\hat{\nu}) = \frac{\hat{\nu}}{n(\hat{\nu} - \hat{\nu}A'(\hat{\nu}) - A(\hat{\nu}))}
$$

$$
\text{Var}(\hat{\alpha}) = \frac{\hat{\alpha}A(\hat{k}) + \hat{\beta}^2A(\hat{\nu})}{\hat{\kappa}A'(\hat{\nu})A(\hat{k})(n\sum \hat{X}_i^2 - (\sum \hat{X}_i)^2)}
$$

and

$$
\text{Var}(\hat{\beta}) = \frac{n\left(\hat{\kappa}A(\hat{k}) + \hat{\beta}^2A(\hat{\nu})\right)}{\hat{\kappa}A'(\hat{\nu})A(\hat{k})(n\sum \hat{X}_i^2 - (\sum \hat{X}_i)^2)}
$$

Further, for small $\kappa$, (less than 10), $A(\kappa)$ can be approximated, Mardia (1972) by

$$
A(\kappa) = \frac{1}{2}\kappa \left(1 - \frac{1}{8}\kappa^2 + \frac{1}{48}\kappa^4 \ldots\right)
$$

and also for large $\kappa$, $A(\kappa)$ can be approximated by

$$
A(\kappa) = 1 - \frac{1}{2\kappa} - \frac{1}{8\kappa^2} - \frac{1}{8\kappa^3} \ldots
$$

Hence for large $n$, the estimates, i.e., $\hat{\kappa}, \hat{\nu}, \hat{\alpha}$ and $\hat{\beta}$ are normally distributed with means $\kappa, \nu, \alpha$ and $\beta$ and above variances, respectively. Also we have

$$
\text{Cov}(\hat{\alpha}, \hat{\beta}) = \frac{\left(\hat{\kappa}A(\hat{k}) + \hat{\beta}^2A(\hat{\nu})\right)\sum \hat{X}_i}{\hat{\kappa}A'(\hat{\nu})A(\hat{k})(n\sum \hat{X}_i^2 - (\sum \hat{X}_i)^2)}
$$

and

$$
\text{Cov}(\hat{\alpha}, \hat{\kappa}) = \text{Cov}(\hat{\beta}, \hat{\kappa}) = 0
$$

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REFERENCES