A Stochastic Analysis of the Effect of Sudden Increase in the Income of Individuals on the Economy

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Abstract: In this study, we investigate the effect of sudden increase in the income of individuals on the economy. Such situation arises in countries where upward adjustment of salaries increases the income of individuals by a certain percentage. We show that accumulated wealth of individuals through capital investment follows the power law distribution. We quantify the effect of low and high propensity, respectively, of an individual to invest on the economy using an empirical illustration.

Keywords: Power-law distribution, stochastic multiplicative process, accumulated wealth (savings)

INTRODUCTION

It is well known that the distribution of many natural and artificial systems could be closely approximated by the power law distribution (Levy and Solomon, 1996).

This phenomenon was first noticed in 1897 by the Italian Economist V Pareto who discovered that the wealth w of individuals in the high income range is distributed according to the power-law distribution given by the following pdf:

\[ p(w) = Kw^{-(1+\alpha)} \]  

(1)

Where:
K = Normalization factor
\( \alpha \) = Parameter which characterizes the profile of the income distribution (this power law behaviour is known as Pareto law), while at the lower-middle income range R

Gibrat (1937) has shown that the probability distribution function can be approximated by the lognormal distribution:

\[ p(w) = \frac{1}{w\sqrt{2\pi} \sigma^2} \exp \left[ -\frac{\log^2 \left( \frac{w}{w_0} \right)}{2\sigma^2} \right] \]  

(2)

Where:
w_0 = The geometric mean value of w
\( \sigma^2 \) = Geometric variance

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Attempts have been made by several researchers to derive a single distribution that accommodates both income ranges (Aiyagari, 1994). Indeed, the analysis of the personal income distribution is one of the important subjects in econophysics (Mateo and Stanley, 2000).

Since it can be easily deduced from the form of the pdf(1) that the distribution of the low-middle range has a thin tail and the entire distribution shifts along with mean income so that the changes on the distribution curve are driven by a stochastic multiplicative process which is the capital investment process that underlies changes in the income owned by individuals, our first objective in this paper is to show that the wealth of an individual is power-law distributed.

Our next objective is to derive a wealth accumulation process (savings model) which is also power-law.

Our main goal in this study is to propose and justify empirically, using illustrative examples from Nigerian, the effect of sudden increase in individual income on the economy using illustrative examples from the Nigerian case study.

**Preliminary Results**

**Definition 1:** If \( \phi \in L^1(\mathbb{R}^n) \), \( \int \phi(x)dx = 1 \) and \( \phi_e(x)dx = e^{-\phi_e(x)} \) then the family of functions \( e > 0 \) is called a mollifier. Note that

\[
\int \phi_e(x)dx = 1
\]

**Definition 2:** Let \( \phi_e \) be a mollifier, \( 1 \leq p < \infty \) and \( f \in L^p(\mathbb{R}^n) \), then for each \( e > 0 \)

\[
\|f * \phi_e - f\|_p \leq \int \|T_{p}f - f\|_p \|\phi(y)\|dy.
\]

Let \( \phi_e \) be a Friedrichs’ mollifier. If \( f \in L^1(\mathbb{R}^n, \text{loc}) \) then the convolution

\[
f * \rho_e(x) = \int f(x - y)\rho_e(y)dy = \int f(x - ey)\rho(y)dy
\]

exists for each \( x \in \mathbb{R}^n \). Moreover

(i) \( f * \rho_e \in C^\infty(\mathbb{R}^n) \)

(ii) \( \text{Supp}(f * \rho_e) \leq \) closed \( e \) - neighborhood of \( \text{supp} f \).

(iii) If \( 1 \leq p < \infty \) and \( f \in L^p(\mathbb{R}^n) \) then \( f * \rho_e \rightarrow f \) in \( L^p(\mathbb{R}^n) \) as \( e \rightarrow 0 \). Indeed

\[
\|f - f * \rho_e\|_p \leq \sup|\rho_e|e \leq \|f - T_{p}f\|_p
\]

(iv) If \( k \) is a compact set of points of continuity of \( f \) then uniformly on \( k \) as \( e > 0 \).

**Definition 3:** An \( \mathbb{R}^n \)-valued stochastic process indexed by an interval \( T \) of \( \mathbb{R}^n \), the real line is a map from \( T \) to \( L^2(\Omega, f, \mathbb{R}^n) \).

**Definition 4:** A stochastic multiplicative process is a process in which the value of each element is multiplied by a random variable with each time step.

**Definition 5:** A probability distribution is a function \( \rho: Z \rightarrow [0, 1] \) such that \( \rho(z) \geq 0 \forall z \in Z \) and

\[
\sum_{z \in Z} \rho(z) = 1.
\]

**Lemma 1**

Suppose that \( X \) has the exponential density function

\[
f_x(x) = \begin{cases} e^{\alpha x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}
\]
then $y = e^x$ has the pareto distribution:

$$f_y(y) = \begin{cases} \alpha y^{-(\alpha+1)}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

**Proof**

Consider $y = e^x$. Note that $g(x) = e^x$ is strictly increasing and differentiable and that $y = e^x$ if and only if $x = h(y) = \log y$ (Olkin et al., 1980).

Hence using the transformation

$$f_y(y) = \frac{dh(y)}{dy} f_x(h(y))$$

The probability density function of $y$ gives

$$f_y(y) = \begin{cases} \frac{\alpha y^{-\alpha - 1}}{\log y}, & 0 < y \leq \infty \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \alpha y^{-(\alpha+1)}, & 0 < y \leq \infty \\ 0, & \text{otherwise} \end{cases}$$

**Lemma 2**

Suppose $X$ has a uniform distribution on $[0, 1]$ then $Y = X^{-\frac{1}{\alpha}}$, $\alpha > 0$ has a Pareto distribution.

**Proof**

Since $X$ has a uniform distribution

$$f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Then $y = x^{-\frac{1}{\alpha}}$ if $x = h(y) = y^{-\alpha}$ which implies that $g(x) = x^{-\frac{1}{\alpha}}$ is differentiable. Therefore, we have

$$f_y(y) = \begin{cases} \frac{\alpha y^{-\alpha}}{\log y}, & 0 < y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$= \alpha y^{-(\alpha+1)}$$

**Lemma 3**

Let $(X, A, \mu)$ be a measure space and let $\tau \in F(X, A, \mu)$. Let $h$ be a bounded $A$ measurable function on $X$ such that the set $A = \{x \in X : h(x) \neq 0\}$ is locally $\mu$-null. Then $\int h d\tau = 0$ (Hewitt and Stromberg, 1960).

**PROBLEM FORMULATION**

Consider a system consisting of a set of investors $i = 1, \ldots, N$ each owing a wealth $w_i$. Assume that the typical variations of $w$ are characterized effectively by a multiplicative stochastic law.
\[ W_t(t+1) = \lambda(t)W_t(t) \] (3)

With \( \lambda \) a stochastic variable with a finite support distribution of probability \( \pi(\lambda) \), does not generate a stationary PLD for \( W_t(t) \) instead, it gives rise to a time-dependent log-normal distribution. The effective transition probability distribution \( \pi(\lambda) \) is assumed not to depend on \( i \) or on the actual value of \( w \). However, if the shape of \( \pi(\lambda) \) varies in time during the process, our conclusion is not affected (Levy and Solomon, 1996).

Let us consider a discrete-time stochastic multiplicative process \( w_t(t) \), added with reset events in the following way: At each time step, \( n \) is reset with probability distribution \( P(w_n) \) if the reset event does not occur, \( w_t \) is multiplied by a stochastic positive factor \( \lambda \) with probability distribution \( \pi(\lambda) \). So that we have

\[ W_t(t+1) = \begin{cases} \frac{w_t(t+1)}{\lambda(t)W_t(t)} \text{ with probability } \pi(\lambda) & \text{with probability } 1-q \end{cases} \] (4)

between two consecutive reset events. \( w_t(t) \), then, is a pure multiplicative process. When one of such events occurs, the multiplication sequence starts again.

To gain insight in the dynamics of Eq. 4, let us consider the simplest case where, \( w_0(t) \) and \( \lambda(t) \) are constant for all \( t \).

Since an arbitrary factor in the initial value of \( w_0 \) is irrelevant to its subsequent evolution, we take \( w_0 = 1 \) without loss of generality, (Mannutia and Zanette, 1999). Thus we have,

\[ W_t(t+1) = \begin{cases} 1 & \text{with probability } 1-q \end{cases} \] (5)

Now suppose the wealth \( w_t(t) \) follows geometric Brownian motion with killing rate where, the killing rate is \( \mu \), we denote the reset point by \( w_\epsilon(t) \) and write the dynamics of \( w_t(t) \) by:

\[ \frac{dw}{w} = \mu dx + \sigma dz(t) \]

\[ \frac{dw(t)}{w(t)} = \mu dx + \sigma dz(t) \] (6)

\[ w_\epsilon(t) \geq w(0) \] (7)

Where, \( z(t) \) is a Brownian motion process and the inequality (7) means that \( w(t) \) is the reflective lower bound (Makoto and Wataru, 2002). Furthermore, suppose \( \sigma = 0 \) that is if the lower bound \( w_\epsilon(t) \) grows as fast as the drift of risky wealth, then:

\[ \frac{dw}{w} = \mu dx \] (8)

This is because the riskling capital income obeys a multiplicative process. Observe that the capital grows exponentially, that is,

\[ x = \ln w \] (9)

or

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Now we will be primarily concerned with the distribution of Eq. 5 which is essentially done in Levy and Solomon (1996). Here an alternative approach is considered which adopts the methods in Levy and Solomon (1996) with some novel modification.

**Theorem 1**

The probability distribution of the wealth process given by Eq. 5 is approximated by a power law distribution (Pareto distribution)

**Proof**

Let \( f(w, t) \) and \( f(w, t+1) \) be the wealth distribution at time \( t \) and \( t+1 \), respectively. Then, because the wealth of the \( i \)th investor changes from \( w_i(t) \) at time \( t \) to \( \lambda w_i(t) \) at time \( t+1 \) that is \( f(t) \), the wealth distribution at \( t+1 \) (Levy, 2001) is given by:

\[
F(w, t+1) = \int \frac{\pi(\lambda) F(\frac{w}{\lambda}, t) d\lambda}{\int \pi(\lambda) F(w, 0) d\lambda}
\]

Then

\[
F(w, t+1) - F(w, t) = \int \frac{\pi(\lambda) F(\frac{w}{\lambda}, t) d\lambda}{\int \pi(\lambda) d\lambda} - F(w, t) \int \pi(\lambda) d\lambda
\]

Let \( \chi = \ln w \) and \( \mu = \ln \lambda \), then the corresponding probability distributions \( \rho \) and \( \pi \) become in the new variable:

\[
\begin{align*}
F(x) &= e^x F(e^x) \\
\rho(\mu) &= e^\mu \pi(e^\mu)
\end{align*}
\]

In terms of the \( f, \chi, \rho, \mu \) and by definition 2, Eq. 12 becomes

\[
f(x, t+1) - f(x, t) = \int \rho(\mu) f(x - \mu, t) d\mu - f(x, t) \int \rho(\mu) d\mu
\]

At each time step where no reset occurs we can write at \( t = 0 \)

\[
\Delta f(x) = \int \rho(\mu) f(x - \mu) d\mu + f(x) \Delta t - \int \rho(\mu) d\mu
\]

and by definition 1, Eq. 15 becomes for \( \int \rho(\mu) d\mu = 1 \)

\[
\Delta f(x) = \int \rho(\mu) f(x - \mu) d\mu
\]

Since \( x \) and \( \mu \) are independent random variables, suppose that both follow exponential distribution such that

\[
\rho(\mu) = e^{-\alpha \mu}
\]

and

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\[ f(x - \mu) = e^{-\lambda(x - \mu)} \quad (18) \]

(Manrubia and Zanette, 1999), then

\[ \Delta f(x) = \int e^{-m} e^{-m(x - \mu)} d\mu \quad (19) \]

\[ f(x) = \frac{1}{\Gamma(m-n)} x^{m-n} e^{-x} \quad (20) \]

Putting \( n = \frac{1}{T} \), we have

\[ f(x) \propto e^{-\frac{x}{T}} \quad (21) \]

We can also see that \( T \to \infty \) for \( n = 0 \).

Translating back to the original variables we get using Eq. 13a

\[ f(w) = w^{-\theta(n)} \quad (22) \]

The wealth accumulation process which we shall call saving, in the sequel, is that part of disposable income not spent on current consumption. It is amount of money set aside for future use.

In recent years, attempt had been made to model the distribution of savings. Kaldor model in (1960) is an attempt to make the saving-income ratio a variable in the growth process. It is based on the classical saving function which implies that saving equals the ratio of profits to national income i.e

\[ S = \frac{P}{Y} \]

Where, \( s, p, y \) stand for savings, profit and income (Jhingan, 2003).

The object of this part is to present a savings model and show that it has a power-law distribution.

We shall make the following assumptions:

A1: The state size of disposable income is known
A2: The savings process consist of risky ones (profit; also know as money multiplication) and risk free ones (wages; also know as money addition). The wages comprise of manual labor and salaries, while profits include capital and human capital
A3: The marginal propensity to consume is much less then marginal propensity to save
A4: The investment-income ratio is an independent variable

**MODEL**

Given these assumptions, let \( S_i(t) \) and \( S_i(t+1) \) be the of ith individual at time \( t \) and \( t+1 \), respectively where, \( t = 0, 1, 2, 3, \ldots \). Let us now introduce the aggregate savings as a sum of additive random and multiplicative terms:

\[ S_i(t+1) = \lambda(t)S_i(t) + R(t) \quad (23) \]

(Aoyama et al., 2000), where, \( \lambda(t) \) and \( R(t) \) are assumed independent random variables.
Furthermore, by A3, we add that

\[ R(t) = \gamma + \beta S_t(t + 1) \]  

(24)

Where, \( \gamma \) and \( \beta \) are the marginal propensity to consume and marginal propensity to save, respectively so that Eq. 23 becomes:

\[ S_t(t + 1) = \lambda(t)S_t(t) + \gamma + \beta S_t(t + 1) \]  

(25)

or

\[ S_t(t + 1) = \frac{\lambda(t)S_t(t) + \gamma}{1 - \beta} \]  

(26)

We shall now show

**Theorem 2**

Let, \( S_t(t + 1) = \frac{\lambda(t)S_t(t) + \gamma}{1 - \beta} \), then the distribution of \( S_t(t + 1) \) is closely approximated by the Pareto (Power Law) distribution.

**Proof**

Let \( \lambda(t) \) be the total growth of saving \( i \) such that Eq. 26 holds. If the growth rates are independently and identically distributed random variables with density function \( f(\lambda) \) and given that the average normalized size must stay constant, that is \( \int \lambda f(\lambda) d\lambda = 1 \), then Eq. 26 expressed in term of the cumulative distribution function of \( S_t(t) \), \( G(s, t) \) gives (Ioannides and Overman, 2000).

\[ G(s, t + 1) = \frac{1}{1 - \beta} \int G(\frac{s}{\lambda}, t) f(\lambda) d\lambda \]  

(27)

Where, all values of \( \lambda \) such that the savings at \( t + 1 \) is equal, is \( \frac{s}{\lambda} \). \( S \) are accounted for and \( \mu(\lambda) = \int \varphi(\lambda) d\lambda = 0 \) (as in lemma 3). For the limiting stationary savings distribution we have \( G(s, t + 1) = G(s, t) - G(\tilde{s}) \) (Lévy, 2001), so that Eq. 27 becomes:

\[ G(s) = \frac{1}{1 - \beta} \int G(\frac{s}{\lambda}) f(\lambda) d\lambda \]  

(28)

This equals

\[ g(x) = \frac{1}{1 - \beta} \int f(\lambda) g(x - \lambda) d\lambda \]

(using Eq. 11, 12 and 13a) and gives:

\[ g(x) = \frac{e^{\gamma x}}{1 - \beta} \]  

(29)
Using Eq. 19. Translating back to original variable \( S = \alpha \), Eq. 29 becomes:

\[
G(s) = \frac{S^{(1+s)}}{1 - \beta}
\]  

(30)

Equation 30 compared Eq. 1 gives:

\[
C = \frac{1}{1 - \beta}
\]  

(31)

Equation 31 is the slope of the saving line which reflects the rate of change in saving relative to a change in disposable income.

**INVESTMENTS RETURN VARIABLE GROWTH, DATA FITTING AND PARAMETER ESTIMATION**

Here, we attempt to fit the expected return rate of investments made by investors base on the knowledge of state of income to Eq. 30. For this purpose, we define the lower bound \( w \), as:

\[
w = \frac{\sigma}{N} \sum_{i=1}^{N} w_i(t)
\]  

(32)

Where:

\( N = \) No. of investors

\( \sigma = \) Threshold given in absolute terms \((\sigma < 1)\).

We convert the vertical axis to the accumulated probability \( P(z \leq w) \), the probability that a given investor has wealth equal to or greater than \( w \), and the horizontal axis to \( w \).

In terms of the normalized wealth, the probability that a given individual has wealth \( W \) is given by simple rescaling:

\[
P(W) = \frac{W_i}{W_k}
\]  

(33)

Where,

\[
W_k = \frac{1}{N} \sum_{i=1}^{N} W_i
\]  

(34)

Since Eq. 30 is paretosian, we employ the maximum likelihood method and estimate our parameter as follows; for \( W_k \), \( W_m \), we conclude that

\[
W_m = \min W_i
\]  

(35)

\[
\alpha = \frac{n}{\sum_{i=1}^{n} (\ln W_i - \ln W_m)}
\]  

(36)
Fig. 1: Savings distributions G(s) of (30) for different values of α and β. These different values are due to the implementation of salary adjustment and investment ability, actual data from Nigeria (Source: Federal office of statistics (FOS), Owerri, 1977-82)

Fig. 2: The non-uniform savings distributions in percentage. The income of individual is non-uniform and the propensity to save for individuals varies

\[ C = \alpha W_m \]  

(37)

Using Eq. 31 and 37 we have

\[ \beta = 1 - \frac{1}{\alpha W_i} \]  

(38)

Present analysis implies (and the experimental available data collected from the FMS Owerri confirm) that if the individual income distribution is fitted by Eq. 30, then the savings of an individuals increase, as β increases (Fig. 1, 2). This is due to \( A_\alpha \), where high amount spent on investment can increase the rate of return. On the other hand, as β increases, the savings decreases. With \( \beta = 1 \), the saving increases without bound. For \( \beta < 0 \), the individual is fairly alright due to the implementation of salary adjustment policy. But this behaviour affects the economic indicator as individuals tend to relax their investment activity and so, the per Capita Gross National Product (GNP) in such nation will be very low (Wolken and Glcker, 1988).
Fig. 3: Simulated savings distributions $G(s)$ of (30). The negative $\beta$ shows the negative effect of salary adjustment to the economy as individuals tend to relax their investment ability due to high income.

**GENERATING A RANDOM SAMPLE FROM EQUATION 30**

Equation 30 is paretosian and Pareto distribution is not yet recognized by many programming languages. In actuarial field, pareto distribution is widely used to estimate portfolio costs. As a matter of fact. It can be quite demanding to get data from this particular probability distribution.

One can easily generate a random sample from Pareto distribution by making use of transformation given in lemmas 1 and 2. Figure 3 confirms the different values of $\beta$, given the simulated available data generated by lemmas 1 and 2.

**CONCLUSION**

Individuals can use their income either for consumption or (financial investment). Essentially, economy can grow only if the individuals are able to find ways to produce a little extra over an extended period of time and produce enough to exceed their immediate need so that they can consume less than the labour income they earn.

The ability to invest the surplus to finance future consumption is crucial to growth. Many of the poor nations of the world have the people (labour), but they have been unable to accumulate savings in quantities sufficient enough to purchase tools and machinery. Unless money is available, no new machinery, tools, or factories will be produced and as the already existing capital goods begin to wear out, they will not be replaced.

Therefore, with proper implementation of the government policy on salary adjustment and proper investment, the economy grows.

**REFERENCES**


