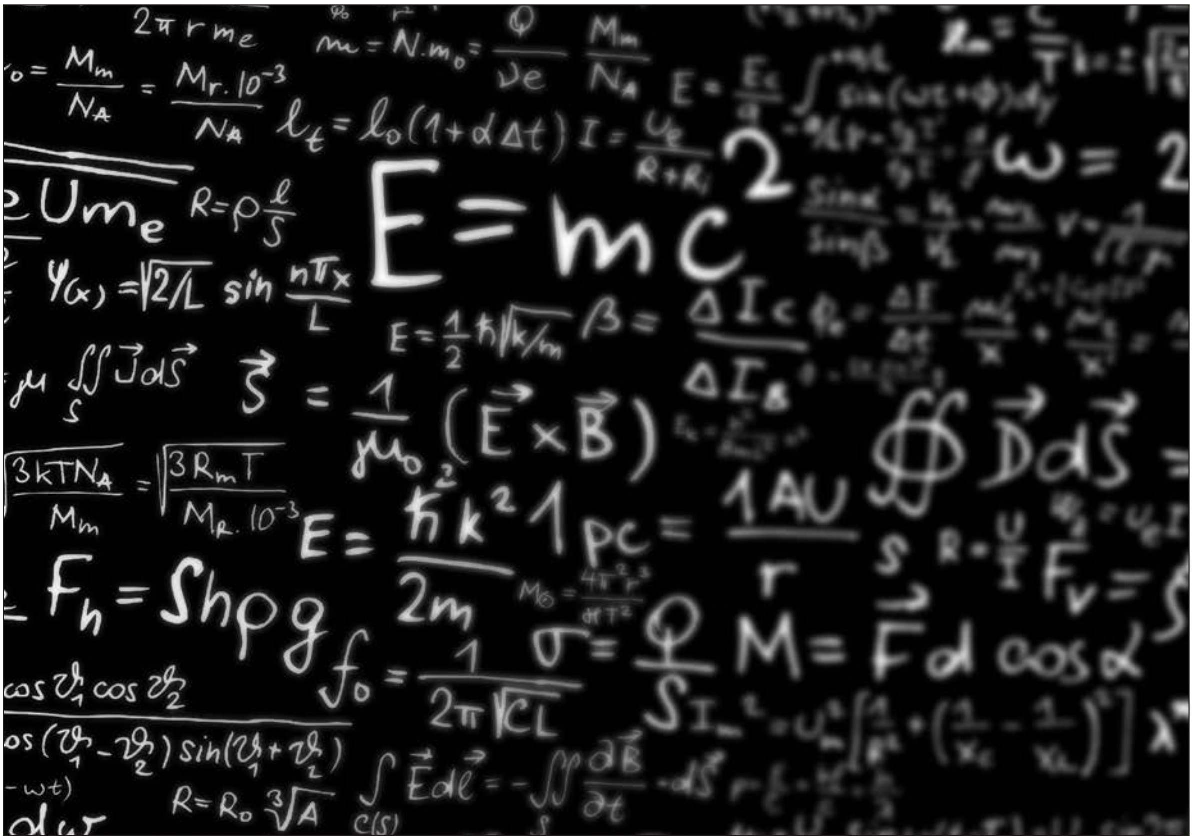

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The Zeta Function with a 2-Dimensional Shift of Finite Type via Left and Upward Shifts

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ABSTRACT

In this study, we studied the zeta function with a 2-dimensional shift of finite type via left and upward shifts, on a subspace of the space of doubly indexed sequences over a finite abelian compact group and new way to calculate periodic point numbers based on the study of linear algebra of F_2 the field and index subgroup. We showed the periodic point data of 2-dimensional shift of finite type via left and upward shifts.

Key words: Periodic point, shift of finite type, zeta function, two dimension, linear algebra

INTRODUCTION

Let G be a finite abelian group. We consider our study here when G = {0,1}, $X_G = G^{Z2}$ be the space of doubly indexed sequences over G and it is subspace

$$X_G = \left\{ \begin{array}{l} (x_{s,t}) \in \hat{X}_G : x_{s,t+1} = x_{s,t} + s + 1, t \\ \text{such that } s, t \in Z \end{array} \right\}$$

The two-dimensional shift of finite type α: $X_G \to X_G$ defined by:

$$\alpha_{\binom{m}{n}}(x_{s,t}) = (x_{s+m,t+n}) \quad \forall (m,n) \in Z^2$$

of the space G^{Z2} of doubly indexed sequences over a finite abelian compact group G. The space X_G was introduced by Ward (1993).

To fined the Zeta function, it requires from us study the periodic points in X_G by α: $X_G \to X_G$.

The group Z^2 acts natural on the space X_G via left and upward shifts we show the periodic point data of X_G determine the group G.

This allows us to view X_G as a Z^2-space for every subgroup U of Z^2, we define U-periodic point to be those x ϵ X_G by the action of the subgroup U. This is natural generalization of the notion of periodic points in case of any ordinary Z-action. If U has finite index in Z^2, the number F_U of U-periodic points x ϵ X_G is finite.

Compare this situation to the full shift, i.e., the shift action on the space X_G. If the index of U in Z^2 is m, then obviously $F_U = |G|^m$ for any group G.

It was also introduced by Lind (1996), Roettger (2005) and Al-Refaei and Noorani (2007) new way to calculate periodic point numbers based on the study of linear algebra of F_2 the field and get the zeta function which generality by Lind (1996).

Huffman code has been widely used in text, image and video compression. For example, it is used to compress the result of quantization stage in JPEG (Hashemain, 1995). The simplest data structure used in the Huffman decoding is the Huffman tree. Array data structure (Chen *et al.*, 1999) has been used to implement the corresponding complete ternary tree for the Huffman tree.

Modeling and simulation of thermal dynamic system has been studied by Bhatti *et al.* (2006). Whereas, Shabbak *et al.* (2011) proposed multivariate control chart which is less sensitive to the sustained shift in mean process. Another related work has been proposed by Abu-Shawiesh *et al.* (2009) and Goegebeur *et al.* (2005).

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Here our study will be about the compact group G = {0,1}. And so we observe that there are several of subgroups U of Z^2 with finite indexes but in our work here we can classify the periodic points for subgroups U of Z^2.

CALCULATION FOR THE ZETA FUNCTION

Here, we will introduce the formula for the zeta function of left and upward shifts α according to the quality of the subgroups which are used in the following propositions:

Proposition 1: The zeta function of a 2-dimensional shift of finite type via left and upward shifts α according to his subgroups U of Z^2 in the formula (2) which defined as follows:

$$U = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} Z^2 : \quad k \geq 0 \right\} \tag{2}$$

Will be given by ζ_x (Z) = 1/1-Z

Proof: It was introduced by Ward (1993) and Lind (1996) that for finite abilean $(x_{s,I}) \in X_G$ is a U-periodic point if and only if:

$$\left(2^k - 1\right) x_{s,t} = 0 \forall s, t \in Z \tag{3}$$

The number of periodic points for the special subgroups U as in (2) is exactly the No. of solutions to Eq. 3.

In this case at G = {0,1} we observe that for $(x_{s,I}) \in X_G$, $x_{s,i}$ will be 0 or 1. And for appositive integer k = 0, 1, 2. We observe that 2^k-1 is odd No. Then (2^k-1) 0 = 0 and (2^k-1). That is, for any subgroup U of Z^2 in the formula (2), the zero-element will be only periodic point, i.e., F_U = 1.

Hence the zeta function will define by:

$$\zeta_X(Z) = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} Z^k \right)$$

$$\zeta_x(Z) = \exp(-\log(1-Z)) = 1/1-Z$$

Proposition 2: The zeta functions of a 2-dimensional shift of finite type via left and upward shifts α according to the subgroups U of Z^2 which defined as follows:

$$U = U_{a,b,d} = \left\{ \begin{matrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} Z^2 \text{ such that} \\ a, d > 0, 0 \le b < a \end{matrix} \right\} \quad (4)$$

Will be given by:

$$\zeta_X(Z) = 1 + Z + 2Z^2 + 4Z^3 + 6Z^4 + 9Z^5 + 16Z^6 + 24Z^7 + ...$$

Proof: It was introduced by Ward (1998) that $(x_{s, I}) \in X_G$ is an U-periodic point if and:

$$x_{s, i} = x_{s+a, I} \quad (5)$$

$$x_{s-b,t} = \sum_{i=0}^{d} \binom{d}{i} x_{s+i,t} \forall s, t \in Z \quad (6)$$

The No. of periodic points for the special subgroups U as in (4) is exactly the No. of solutions to Eq. 5, 6 above.

When G = {0, 1}, we observe that for $(x_{s, I}) \in X_G$ will be 0 or 1. Hence, for appositive integer m = 0, 1, 2... we observe

$$mx_{s,t} = \begin{cases} x_{s,t} & m \text{ is odd} \\ 0 & m \text{ is even} \end{cases}$$

From definition of X_G, for $(x_{s, I}) \in X_G$ must satisfy the following equation :

$$x_{s, i+1} = x_{s, I} + x_{s+1, i} \quad (7)$$

We observe that the equation has four solutions:

$$x_{s, i+1} = x_{s, I} + x_{s+1, I}$$

Sol. 1 0 0 0
Sol. 2 1 0 1
Sol. 3 1 1 0
Sol. 4 0 1 1

Hence, if the System for Eq. 5, 6 has an unique solution, then for any subgroup U of Z^2 in the formula (4), the zero-element will be only periodic point, i.e., $F_U = 1$. And if the system for Eq. 5, 6 has more than one solution, then by the solutions for Eq. 7, for any subgroup U of Z^2 in the formula (4), $F_U = 4$.

Now we will take some special cases in this formula as follow:

- If a = 1, c = 1, b = 0 then from Eq. 5, 6 we get:

$$x_{s,\, i+1} = x_{s,\, I} + x_{s+1,\, I}$$

$x_{s,i} = x_{s+1,\, t}$ and $x_{s,\, I} = x_{s,\, I} + x_{s+1,\, t}$ this implies to

$x_{s,i} = 0$, hence $F_u = 1$.

- If a = 2, b = 1, d = 1 then from Eq. 5 and 6, we get

$$x_{s,i} = x_{s+2,\, t} \tag{8}$$

$$x_{s-1,\, t} = x_{s,\, t} + x_{s+1,\, t} \tag{9}$$

From Eq. 8 $x_{s-1,\, t}$ and from 9 $x_{s,\, I}$ this implies to $F_U = 1$

- If a = 3, b = 4, d = 1 then from Eq. 5 and 6, we get

$$x_{s,t} = x_{s+3,\, t}$$

$x_{s-i,\, t} = x_{s,\, t}\ 4x_{s+1,\, t} 6_{xs+2,\, t} + 4_{xs+3,\, t}\ x_{s+4,\, t}$ then

$x_{s-1,t} = x_{s,t} + 0 + 0 + 0 + x_{s+4,\, t}$ This implies to

$$x_{s-1,t} = x_{s+4,\, t}$$

Then this equation has more then one solution therefore $F_U = 4$

We can continuous by the same way to calculate the periodic points for some special cases again as showed in Table 1.

Where, a, d>0, $0 \le b < a$ and F_U is the periodic point.

Hence, we can define the zeta function by:

Table 1: Periodic points for some special cases

a	d	b	F_U
1	1	0	1
1	d>1	0	1
2	d≥3	1	1
2	2	1	1
3	1	1	4
3	2	1	1
3	4	1	4
3	d≥5	1	4
3	6	2	1

Proposition 3: The zeta function of a 2-dimensional shift of finite type via left and upward shifts α according to the subgroups U of Z^2 which defined as follows:

$$\zeta_X(Z) = \exp\left(\sum_{a=1}^{\infty}\sum_{d=1}^{\infty}\sum_{b=0}^{a-1}\frac{F_U}{ad}Z^{ad}\right) = 1 + Z + 2Z^2 + 4Z^3 + 6Z^4 + 9Z^5 + 16Z^6 + 24Z^7 + \dots$$

$$U = U_{a,d} = \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} Z^2 \text{such that} \\ \text{for each } a, d > 0\} \right\} \tag{10}$$

will be given by:

$$\zeta_X(Z) = 1 + Z + 2Z^2 + 4Z^3 + 6Z^4 + 9Z^5 + 16Z^6 + 24Z^7 + \dots$$

Proof: We observe that the formula (10) is special case from the formula (4) by make b = 0. Hence, we can put b = 0 in the Eq. 6 to get that:

$$\sum_{i=1}^{d}\binom{d}{i} x_{s+i,t} = 0 \forall s, t \in Z \tag{11}$$

Now we will take some special cases in this formula as follow:

- We observe that if a = 1 and d is even No. then from Eq. 5 and 6 we get:

$$x_{s,t} = x_{s+a,t} \text{ And}$$

$$0 = \sum_{i=1}^{d} C_i^d x_{s+i,t} \tag{12}$$

This is implies to:

$$0 = C_1^d x_{s+1,t} + C_2^d x_{s+2,t} + \dots + C_{d-1}^d x_{s+(d-1),t} + C_d^d x_{s+d,t}$$

We observe that C_i^d is even number for 1≤ I < d Hence, from (12) we get:

$$0 = 0 + \dots + 0 + C_d^d x_{s+d,t} = x_{s+d,t}$$

$$x_{s+d,t} = 0 \text{ then } F_U = 0$$

- If a = 1 and d = 1 then from Eq. 5 and 6 we get:

$$x_{s,t} = x_{s+1,t} \tag{13}$$

$$0 = C_1^d x_{s+1,t} + C_2^d x_{s+2,t} + \ldots + C_d^d x_{s+d,t} \tag{14}$$

From Eq. 13 we observe that:

$$\begin{aligned} &x_{s,t} = x_{s+1,t} = x_{s+2,t} \\ &x_{s+2,t} = x_{s+3,t} \\ &\vdots \\ &x_{s+(d-1),t} = x_{s+d,t} \end{aligned}$$

Hence from Eq. 14 we get:

$$0 = (C_1^d + C_2^d + \ldots + C_d^d) x_{s,t} \text{ implies } x_{s,\,t} = 0 \text{ then } F_U = 1$$

- If a = 2 and d = 1 then from Eq. 5 and 6 we get:

$$x_{s,\,t} = x_{s+2,\,t} \tag{15}$$

$$0 = C_1^d x_{s+1,t} + C_2^d x_{s+2,t} + \ldots + C_{d-1}^d + C_d^d x_{s+d,t} \tag{16}$$

If d is even then from (1) we get that $F_U = 1$
If d is odd then from (15) we observe that:

$$\begin{aligned} &x_{s,t} = x_{s+2,t} = x_{s+4,t} = \ldots. = x_{s+(d-1),t} \\ &x_{s+1,t} = x_{s+3,t} = x_{s+5,t} = \ldots = x_{s+d,t} \end{aligned}$$

Hence, from Eq. 16 we get that $x_{s+(d-1),\,t}$, this implies to $F_U = 1$.
Then e we can define the zeta function by:

$$\zeta_X(Z) = \exp\left(\sum_{a=1}^{\infty}\sum_{d=1}^{\infty} \frac{F_U}{ad} Z^{ad}\right) = 1 + Z + 2Z^2 + 4Z^3 + 6Z^4 + 9Z^5 + 16Z^6 + 24Z^7 + \ldots$$

We can also compute F_U using some linear algebra over G (we can consider G a field). A point $x \in X_G$ that is U-invariant must have horizontal periodic a, so is determined by an:

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_a \end{pmatrix}$$

element from the vector space $F^a{}_2$.

Let I_a be a×a identity matrix. Let P_a be a×a permutation matrix corresponding to the cyclic shift of elementary basis vectors $F^a{}_2$.

The condition of U-periodicity of $(x_{s,t})$ translates to the condition:

$$(I_a + P_a)^c y = P_a^{-b} y \tag{17}$$

Hence:

$$F_U = \left|\ker\left((I_a + P_a)^c - P_a^{-b}\right)\right|$$

Then the zeta function in this case equal:

$$\zeta_X(Z) = \left(\sum_{a=1}^{\infty} \sum_{d=1}^{\infty} \sum_{b=0}^{a-1} \frac{\left|\ker\left(\left(I_a + P_a\right)^c - P_a^{-b}\right)\right|}{ad} Z^{ad} \right)$$

We will take some cases in this form

- If a = 1, c = 1, b = 0, then,

$$P_a = 1, \quad I_a = 1, \quad P_a^{-0} = 1$$

$P_a + I_a = 2$ Then y = y ⇒ y = 0 because
2*1 = 0 ≠ 1 not allow so 2*0 = 0. Hence $F_U = 1$

- If a = 1, d = r, b = 0, then $P_a = 1, \quad I_a = 1, \quad P_a^{-0} = 1$

$(Pa + 1a)^r = 2^r$

2^r y = y ⇒ y = 0. Then $F_U = 1$.

- If a = 2, d = 1, b = 0 Then:

$$I_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, P_a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$P_a^{-0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence:

$$(I_a + P_a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Let $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

Hence:

$$(I_a + P_a)^1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 + y_2 \\ y_1 + y_2 \end{pmatrix} P_2^{-0} y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$y_1 \, y_2 = y_1 + y_2 = y_2$$

There fore $y_1 = y_2$,

Either $y_1 = 0$, $y_2 \Rightarrow y_1 \, y_2 = 0$ is possible or $y_1 = 1$ is impossible because $y_2 \Rightarrow y_1 \, y_2 = 0$ but $y_1 \, y_2 = y_1$ is not zero. Hence $F_U = 1$.

- If a = 2, d = 1, b = 0. Then:

$$I_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_a^{-0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(I_a + P_a)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^2$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$P_2^{-0} y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Then $y_1 = y_2 = 0$. Hence $F_U = 1$.

Let a = 2 , d = 2, b = 0:

$$I_a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_a = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$P_a^{-0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(I_a + P_a)^1 y = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 + y_3 \\ y_1 + y_2 \\ y_2 + y_3 \end{pmatrix}$$

$$P_a^{-0}y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

The we get that $y_1 = 0$, $y_2 = 0$. Hence $F_U = 1$.
And when a = 2, d = r, b = 0, then $F_U = 1$. When a = 3, d = 1, b = 0 we get that:

Then:

$$\begin{aligned} y_1 + y_3 &= y_1 \\ y_1 + y_2 &= y_2 \\ y_2 + y_3 &= y_3 \end{aligned}$$

So $y_1 = y_2 = y_3 = 0$, therefore, $F_U = 1$.
Also similarly, when a = 3, d = 1, b = 1, we get that:

$$I_a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_a = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$P_a^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P_a^{-1}y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ y_1 \end{pmatrix}$$

$$\begin{aligned} y_1 + y_3 &= y_2 \\ y_1 + y_2 &= y_3 \\ y_2 + y_3 &= y_1 \end{aligned}$$

Let $y_1 = 0 \Rightarrow y_3 = y_2$ then $y_3 = 1$ and $y_3 = 0$, $y_2 = 0$ and let:

$$y_1 = 1 \Rightarrow 1 + y_3 = y_2 \; 1 + y_2 = y_3$$

$$\text{and } y_2 + y_3 = 1$$

So:

$$\begin{aligned} y_1 = 0 &\Rightarrow y_2 = y_3 = 1 \\ y_1 = 0 &\Rightarrow y_2 = y_3 = 0 \\ y_1 = 1 &\Rightarrow y_2 = 0 = y_3 = 1 \\ y_1 = 0 &\Rightarrow y_2 = 1, \; y_3 = 1 \end{aligned}$$

Hence, $F_U = 4$.

Also when a = 3, d = 1, b = 2, similarly, $F_U = 1$.

In the same way, we can get in all theses cases (a = 3, d = 2, b = 0), (a = 3, d = 2, b = 1) that $F_U = 1$ and in these cases (a = 3, d = 2, b = 2), (a = 3, d = 3, b = 0) that $F_U = 1$..

By using Mathematica, we will get this result:

$$\zeta_X(Z) = 1 + Z + 2Z^2 + 4Z^3 + 6Z^4 + 9Z^5 + 16Z^6 + 24Z^7 + \ldots$$

CONCLUSIONS

The periodic point was calculated by using two methods namely, the index subgroup and the linear algebra of the field F_2. According to these two methods, the Zeta function for two dimensional shift of finite type by left and upward shifts α has been studied. For validation of the present results, three cases study have been presented to get the Zeta function.

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