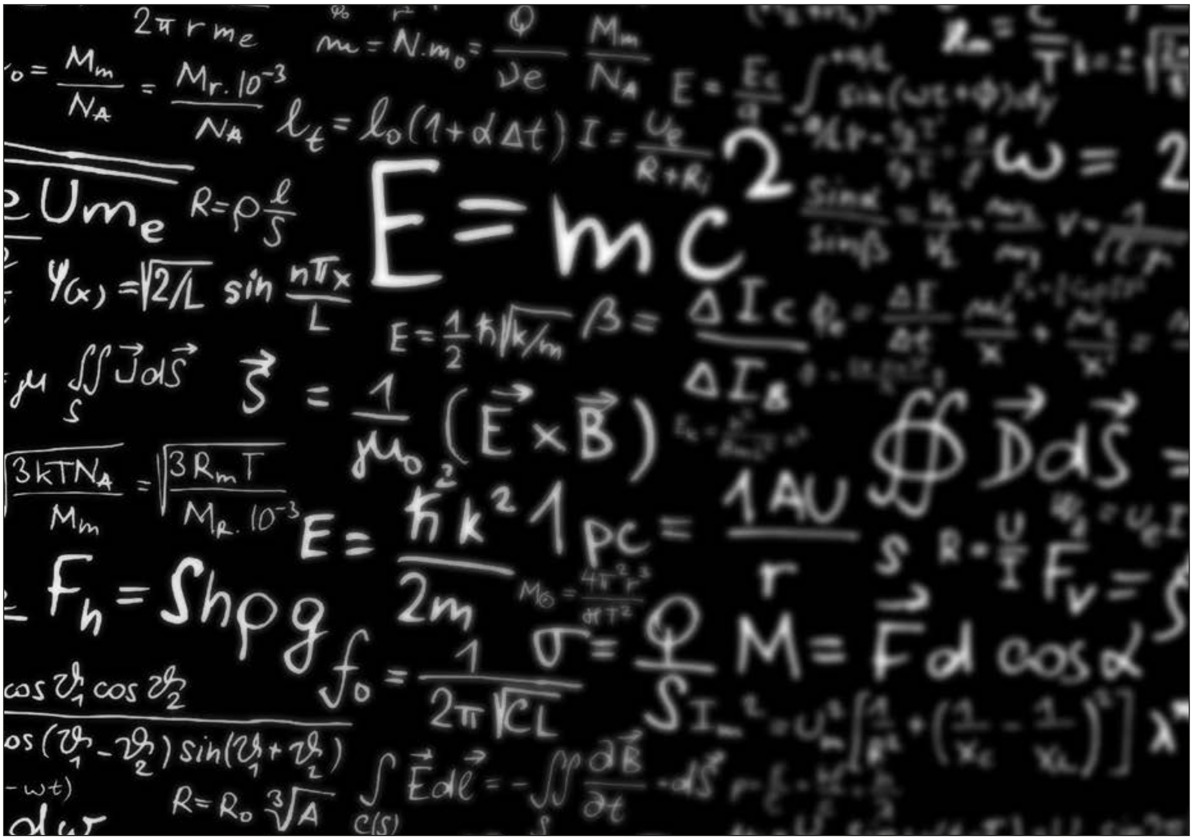

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On Unit Element's Norm in Some Banach Spaces

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ABSTRACT

In this study, we obtain Banach algebras which norm of their unit elements is not one. These Banach algebras are subsets of $\mathbb{R}^k$. Also, we present some interesting properties.

Key words: Unit element, normed algebra, Banach algebra, equivalent norms

INTRODUCTION

We know normed algebras are one of the most important subjects in functional analysis. Also we know that if a normed algebra is unitary then norm of its unit is one. Most of persons who study functional analysis in a non-professional way think that norm of unit in any algebra should be one. We wish to present algebras in which their unit element's norm is not one. We need to following definitions.

Definition 1: Kreyszig (1987) (Normed algebra): An algebra over F (the real field $\mathbb{R}$ or the complex field C) is a linear space A over F together with a mapping $(x, y)\rightarrow xy$ of $A\times A$ into A that satisfies the following axioms (for all $x,y,z\in A$, $\alpha\in F$):

(i) $x(yz) = (x,y)z$
(ii) $x(y+z) = (xy)+xz$, $(x+y)z = xz+yz$
(iii) $(\alpha x)y = \alpha(xy) = x(\alpha y)$

Furthermore if there exists a norm $\|\cdot\|$ on A such that we have for any $x,y\in A$, $\|xy\|\leq\|x\|\|y\|$ then A is a normed algebra.

As is usual for normed linear spaces a normed algebra A is regarded as a metric space with the distance function $d(x,y) = |x-y|$ $(x,y\in A)$. If A is a complete metric space with defined metric, then A is called a Banach algebra.

Definition 2: Bonsall and Duncan (1973): An element e of an algebra A is an unit element if and only if $e\neq 0$ and $ex = xe = x$ $(x\in A)$.

Definition 3: Kreyszig (1987) (Equivalent norms): A norm $\|\cdot\|$ on a vector space X is said to be equivalent to a norm $\|\cdot\|_1$ on X if there are positive numbers a and b such that for all $x\in X$ we have $a\|x\|_1\leq\|x\|\leq b\|x\|_1$.

MAIN RESULTS

In this section we introduce some subsets of $\mathbb{R}^k$ which are Banach algebras with defined norm. In these algebras the norm of unit element isn't one.

We begin with $\mathbb{R}$ (real numbers set). Consider $\mathbb{R}$ with ordinary addition and scalar product. $\mathbb{R}$ with defined operations is a vector space. We define the product of $\mathbb{R}$ as follow:

$$x.y = xy$$

And define norm on $\mathbb{R}$ by:

$$\|x\| = c|x| \quad (1)$$

where, c is a constant and c>1. clearly, $\mathbb{R}$ is a normed algebra with 1 as unit element, but $\|1\| = c|1| = c>1$.

The defined norm in Eq. 1 is equivalent with original norm on $\mathbb{R}$. The norm of 1 with original norm is one, whereas with (1) isn't one. We define, In general, for k∈N:

$$A_k = \{(0,\ldots,0,x_k, 0,\ldots,0): x_k \in \mathbb{R}\} \quad (2)$$

Clearly, A_k is a subset of $\mathbb{R}^k$ for k≥1. A_k 's with following operations are vector spaces on $\mathbb{R}$:

$$(0,\ldots,0, x_k, 0,\ldots,0)+(0,\ldots,0, y_k,0,\ldots,0) = (0,\ldots,0, x_k+y_k,0,\ldots,0)$$
$$\alpha. (0, \ldots,0, x_k, 0,\ldots,0) = (0,\ldots,0,\alpha x_k,0,\ldots,0)$$

We define The product of A_k 's as follow:

$$(0,\ldots,0, x_k, 0,\ldots,0)(0,\ldots, 0,y_k,0,\ldots,0) = (0,\ldots,0, x_k y_k,0,\ldots,0)$$

And define the norm on A_k's by:

$$\|(0,0, ,0, x_k,\ldots, 0)\|_{A_k} = c|x_k|$$

where, c is a constant and c>1. It is easy to verify that any A_k, k≥1, induces a metric by:

$$d((0,\ldots,0.x_k,0,\ldots,0),(0,\ldots,0,y_k,0,\ldots,0)) = \|(0,\ldots, 0, x_k-y_k, 0,\ldots, 0)\|_{A_k} = c|x_k-y_k|$$

It is easy to show that any A_k, k≥1, is a Banach algebra.

Since in Eq. 2 c>1 is arbitrary, thus we can obtain infinite many of Banach algebras.

Corollary 1: We can obtain infinite many of Banach algebras of $\mathbb{R}^k$ which norm of their unit element's norm is not one.

Note 1: According to last explanations We conclude that norm of unit element in an algebra depends on algebra norm's. For example, We have:

$$\mathbb{R}^k = Ds(A_k) \quad (3)$$

where, $Ds(A_k)$ denotes direct sum of A_k's. If we define the product in A_k by:

$$(x_1, x_2, \ldots, x_k).(y_1, y_2, \ldots, y_k) = (x_1y_1, x_2y_2, \ldots, x_ky_k)$$

And norm in $Ds(A_k)$ by:

$$\|(x_1, x_2, \ldots, x_k)\| = c \max \{|x_i|: 1 \leq i \leq k\}$$

where, $c>1$ is a constant.

It is easy to show that A_k with above definitions is a normed algebra which its unit element is (1,0,0,... ... .,0) and $\|(1, 0, 0, \ldots, 0)\| = c>1$. Also (1, 0,0,..., 0) is the unit element in R^k. But its norm is one.

Note 2: Note 1 shows that we can write R^k as direct sum of their subsets which any of them is a Banach algebra. Although, in the left side of Eq. 3 the norm of (1, 0,... ... ,0) is one where in right side is $c(>1)$.

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