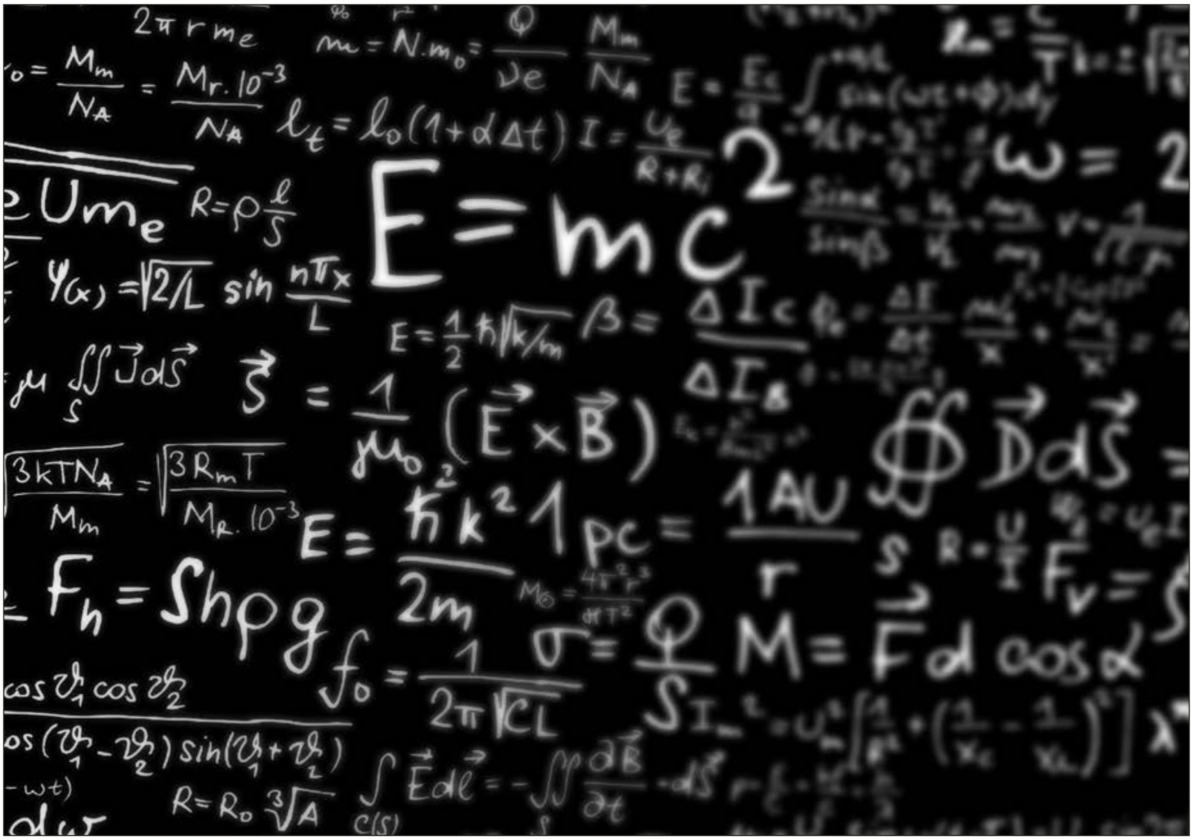

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On Subclass of Univalent Functions with Fixed Second Negative Coefficients

[1]Ghassan H. Esa Al-Shabeeb and [2]Maslina Darus
[1]Centre of Ionic, Faculty Science, University Malaya, 50603 KL, Malaysia
[2]Faculty of Science and Technology, School of Mathematical Sciences, Universiti Kebangsaan Malaysia, Bangi 43600 Selangor D. Ehsan, Malaysia

Corresponding Author: Ghassan H. Esa Al-Shabeeb, Centre of Ionic, Faculty Science, University Malaya, 50603 KL, Malaysia

ABSTRACT

In this study, we consider the class $Q_n(\alpha, \gamma, \mu, c)$ consisting of analytic functions with fixed second coefficients. The object is to show coefficient estimates, convex linear combinations, some distortion theorems and radii of starlikeness and convexity for f (z) in the class $Q_n(\alpha, \gamma, \mu, c)$.

Key words: Univalent functions, analytic functions, coefficient inequalities, distortion and growth theorem, radii of starlikeness and convexity, convex linear combination

INTRODUCTION AND DEFINITIONS

Let S denote the class of functions:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (ak = 0) \tag{1}$$

which are analytic and univalent in $U = \{z : z\in\mathbb{C} \text{ and } |z| < 1\}$.

Let T denote the subclass of S consisting of functions of the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \tag{2}$$

A function f (z) of T is in $Q_n(\gamma)$ $(0\le\gamma<1,\ n\in\mathbb{N}\cup\{0\},\ \mathbb{N} = \{1, 2, 3, ...\})$ if:

$$\mathrm{Re}(D^n f(z))' > \gamma,\ z\in U \tag{3}$$

where, D^n f(z) denote usual n-th order derivative introduced by Ruscheweyh (1975). The class Q_n (γ) was introduced and studied by Uralegaddi and Sarangi (1988). We note that, such type of classes consisting of functions of the form Eq. 1 was introduced and studied by Darus (2003).

Esa and Darus (2007) were studied the class $Q_n(\alpha, \gamma, \mu)$ which consists of functions f $(z)\in T$ and satisfy the conditions:

$$\left|\frac{(D^n f(z))'-1}{\alpha(D^n f(z))'+(1+\alpha)}\right|<\mu,\ z\in U \quad (4)$$

for $0\le\alpha<1$, $0\le\gamma<1$, $0<\mu<1$.

The aim of this study is to estimate the coefficients, study the distortion property and the radii of starlikeness and convexity for f (z) in the class $Q_n(\alpha, \gamma, \mu, c)$, finally study the convex linear combinations for f(z).

We need the following result throughout the work.

PRELIMINARY RESULT

Theorem 1: Let f (z) be defined by Eq. 1. Then f (z) ϵ $Q_n(\alpha, \gamma, \mu)$ if and only if:

$$\sum_{k=2}^{\infty} k(1+\mu\alpha)\delta(n,k)a_k \le \mu(\alpha+(1-\gamma)) \quad (5)$$

where, $0\le\alpha<1, 0\le\gamma<1,\ 0<\mu\le 1, n\in N_0\,(N_0=\mathbb{N}/\{0\})$ and:

$$\delta(n,k)=\binom{n+k-1}{n}$$

In view of Theorem 1, we can see that the function f defined by Eq. 1 is in the class $Q_n(\alpha; \gamma; \mu)$ satisfy:

$$a_k \le \frac{\mu(\alpha+(1-\gamma))}{k(1+\mu\alpha)\delta(n,k)} \quad (6)$$

Let $Q_n(\alpha, \gamma, \mu, c)$ denote the class of functions f in $Q_n(\alpha, \gamma, \mu)$ of the form:

$$f(z)=z-\frac{c\mu(\alpha+(1-\gamma))}{2(n+1)(1+\mu\alpha)}z^2-\sum_{k=3}^{\infty}a_k z^k,\ (\text{with } 0\le c<1) \quad (7)$$

COEFFICIENT INEQUALITIES

Theorem 2: A function f defined by Eq. 1 is in the class $Q_n(\alpha, \gamma, \mu, c)$, if and only if:

$$\sum_{k=3}^{\infty} k(1+\mu\alpha)\delta(n,k)a_k \le (1-c)\mu(\alpha+(1-\gamma)) \quad (8)$$

The result is sharp for function defined by Eq. 7.

Proof: By putting:

$$a_k \le \frac{c\mu(\alpha+(1-\gamma))}{k(1+\mu\alpha)\delta(n,k)},\ 0\le c<1 \quad (9)$$

in Eq. 5, we have the result. The result is sharp for function:

$$f(z) = z - \frac{c\mu(\alpha + (1-\gamma))}{2(n+1)(1+\mu\alpha)} z^2 - \frac{(1-c)\mu(\alpha + (1-\gamma))}{k(1+\mu\alpha)\delta(n,k)} z^k, \quad k \geq 3. \tag{10}$$

Corollary 1: Let the function defined by Eq. 7 be in the class $Q_n(\alpha, \gamma, \mu, c)$, then:

$$a_k \leq \frac{(1-c)\mu(\alpha + (1-\gamma))}{k(1+\mu\alpha)\delta(n,k)}, \quad k \geq 3 \tag{11}$$

DISTORTION AND GROWTH THEOREM

A distortion property for function f in the class $Q_n(\alpha, \gamma, \mu, c)$ is given as follows:

Theorem 3: If the function f defined by Eq. 7 is in the class $Q_n(\alpha; \gamma; \mu; c)$ for $0<|z| = r < 1$, then we have:

$$\begin{aligned} & r - \frac{c\mu(\alpha + (1-\gamma))}{2(n+1)(1+\mu\alpha)} r^2 - \frac{(1-c)\mu(\alpha + (1-\gamma))}{3(n+1)(n+2)(1+\mu\alpha)} r^3 \\ & \leq |f(z)| \\ & \leq r + \frac{c\mu(\alpha + (1-\gamma))}{2(n+1)(1+\mu\alpha)} r^2 + \frac{(1-c)\mu(\alpha + (1-\gamma))}{3(n+1)(n+2)(1+\mu\alpha)} r^3 \end{aligned} \tag{12}$$

with equality for:

$$f_3(z) = z - \frac{c\mu(\alpha + (1-\gamma))}{2(n+1)(1+\mu\alpha)} z^2 - \frac{(1-c)\mu(\alpha + (1-\gamma))}{3(n+1)(n+2)(1+\mu\alpha)} z^3 \quad (z = \pm r) \tag{13}$$

Proof: Since, $f \in Q_n(\alpha; \gamma; \mu; c)$, Theorem (2) yields the inequality (11)

$$a_k \leq \frac{(1-c)\mu(\alpha + (1-\gamma))}{k(1+\mu\alpha)\delta(n,k)}, \quad k \geq 3$$

Thus, for $0<|z| = r<1$ and making use of (11), we have:

$$\begin{aligned} |f(z)| & \leq |z| + \frac{c\mu(\alpha + (1-\gamma))}{2(n+1)(1+\mu\alpha)} z^2 + \sum_{k=3}^{\infty} |a_k| z^k, \quad (|z| = r) \\ & \leq r + \frac{c\mu(\alpha + (1-\gamma))}{2(n+1)(1+\mu\alpha)} r^2 + r^3 \sum_{k=3}^{\infty} |a_k| \\ & \leq r + \frac{c\mu(\alpha + (1-\gamma))}{2(n+1)(1+\mu\alpha)} r^2 + \frac{(1-c)\mu(\alpha + (1-\gamma))}{3(n+1)(n+2)(1+\mu\alpha)} r^3 \end{aligned} \tag{14}$$

and

$$\begin{aligned} |f(z)| & \geq |z| - \frac{c\mu(\alpha + (1-\gamma))}{2(n+1)(1+\mu\alpha)} z^2 - \sum_{k=3}^{\infty} |a_k| z^k, \quad (|z| = r) \\ & \geq r - \frac{c\mu(\alpha + (1-\gamma))}{2(n+1)(1+\mu\alpha)} r^2 - r^3 \sum_{k=3}^{\infty} |a_k| \\ & \geq r - \frac{c\mu(\alpha + (1-\gamma))}{2(n+1)(1+\mu\alpha)} r^2 - \frac{(1-c)\mu(\alpha + (1-\gamma))}{3(n+1)(n+2)(1+\mu\alpha)} r^3 \end{aligned} \tag{15}$$

and the proof is complete.

RADII OF STARLIKENESS AND CONVEXITY

The radii of starlikeness and convexity for the class $Q_n(\alpha, \gamma, \mu, c)$ is given by the following theorem:

Theorem 4: If the function f defined by Eq. 7 is in the class $Q_n(\alpha, \gamma, \mu, c)$, then f is starlikeness of order β $(0\le\beta\le1)$ in the disk $|z|<r_1(\alpha, \gamma, \mu, c, \beta)$ where, $r_1(\alpha, \gamma, \mu, c, \beta)$is the largest value for which:

$$\frac{c\beta(\alpha+(1-\gamma))}{2(n+1)(1+\mu\alpha)}r_0^2+\frac{(1-c)(1+\beta)\mu(\alpha+(1-\gamma))}{3(n+1)(n+2)(1+\mu\alpha)}r_0^3\le 1-\beta \tag{16}$$

The result is sharp for functions f_k given by Eq. 10.

Proof: It suffices to show that:

$$\left|1-\frac{zf'(z)}{f(z)}\right|\le 1-\beta$$

for $|z|<r_1$.

We have:

$$\left|1-\frac{zf'(z)}{f(z)}\right|=\frac{\frac{c\mu(\alpha+(1-\gamma))}{2(n+1)(1+\mu\alpha)}r^2+\sum_{k=3}^{\infty}(k-1)a_k r^k}{z-\frac{c\mu(\alpha+(1-\gamma))}{2(n+1)(1+\mu\alpha)}r^2-\sum_{k=3}^{\infty}a_k r^k}\le 1-\beta \tag{17}$$

Hence, Eq. 17 hold true if:

$$\frac{c(2-\beta)\mu(\alpha+(1-\gamma))}{2(n+1)(1+\mu\alpha)}z^2+\sum_{k=3}^{\infty}(k-\beta)a_k z^k\le 1-\beta$$

and it follows that from Eq. 8, we may take:

$$|a_k|=\frac{(1-c)\mu(\alpha+(1-\gamma))}{k(1+\mu\alpha)\delta(n,k)}\lambda_k,\ k\ge 3$$

where $\gamma_k\ge0$ and

$$\sum_{k=3}^{\infty}\lambda_k\le 1$$

For each fixed r, we choose the positive integer $k_0 = k_0 r$ for which:

$$\frac{(k-\beta)}{k(1+\mu\alpha)\delta(n,k)}$$

is maximal.

Then it follows that:

$$\sum_{k=3}^{\infty}(k_0-\beta)a_k r_0^k \le \frac{(1-c)(k_0-\beta)\mu(\alpha+(1-\gamma))}{k_0(1+\mu\alpha)\delta(n,k_0)}r^{k_0}$$

Hence, f is starlike of order β in $|z|<r_1(\alpha, \gamma, \mu, c, \beta)$ provided that:

$$\frac{c\beta\mu(\alpha+(1-\gamma))}{2(n+1)(1+\mu\alpha)}r^2+\frac{(1-c)(1+\beta)\mu(\alpha+(1-\gamma))}{3(n+1)(n+2)(1+\mu\alpha)}r^3 \le 1-\beta$$

We find the value $r_0 = r_0(\alpha, \gamma, \mu, c, k, \beta)$ and the corresponding integer $k_0 r_0$ so that:

$$\frac{c\beta\mu(\alpha+(1-\gamma))}{2(n+1)(1+\mu\alpha)}r_0^2+\frac{(1-c)(1+\beta)\mu(\alpha+(1-\gamma))}{3(n+1)(n+2)(1+\mu\alpha)}r_0^3 \le 1-\beta$$

Then this value r_0 is the radius of starlike-ness of order β for the function f belonging to the class $Q_n(\alpha, \gamma, \mu, c)$.

Theorem 5: If the function f defined by Eq. 7 is in the class $Q_n(\alpha, \gamma, \mu, c)$, then f is convexity of order β $(0\le\beta\le1)$ in the disk $|z| < r_2(\alpha, \gamma, \mu, c, \beta)$ where, $r_2(\alpha, \gamma, \mu, c, k, \beta)$ is the largest value for which:

$$\frac{c\beta(\alpha+(1-\gamma))}{2(n+1)(1+\mu\alpha)}r^2+\frac{(1-c)(1+\beta)\mu(\alpha+(1-\gamma))}{3(n+1)(n+2)(1+\mu\alpha)}r^2 \le 1-\beta \quad (18)$$

The result is sharp for functions f_k given by Eq. 10.

Proof: By using the same technique in the proof of Theorem 4. We can show that:

$$\left|\frac{zf''(z)}{f'(z)}\right| \le 1-\beta$$

for $|z| \le r_2$ with the aid of Theorem 2. Thus, we have the assertion of Theorem 5.

CONVEX LINEAR COMBINATION

The next result involves a linear combinations of function of the type (3.3).

Theorem 6: If:

$$f_2(z) = z-\frac{c\mu(\alpha+(1-\gamma))}{2(n+1)(1+\mu\alpha)}z^2 \quad (19)$$

and

$$f_k(z) = z - \frac{c\mu(\alpha + (1-\gamma))}{2(n+1)(1+\mu\alpha)} z^2 - \frac{(1-c)\mu(\alpha + (1-\gamma))}{k(1+\mu\alpha)\delta(n,k)} z^k \quad (k \geq 3) \tag{20}$$

Then f ϵ $Q_n(\alpha, \gamma, \mu, c)$ if and only if it can be expressed in the form:

$$f(z) = \sum_{k=3}^{\infty} \lambda_k f_k(z) \tag{21}$$

where, γ_k and

$$\sum_{k=3}^{\infty} \lambda_k \leq 1$$

Proof: From (19-21), we have:

$$f(z) = \sum_{k=2}^{\infty} \lambda_k f_k(z)$$

so that

$$f(z) = z - \frac{c\mu(\alpha + (1-\gamma))}{2(n+1)(1+\mu\alpha)} z^2 - \frac{(1-c)\mu(\alpha + (1-\gamma))}{k(1+\mu\alpha)\delta(n,k)} \lambda_k z^k \tag{22}$$

Since,

$$\begin{aligned} f(z) &= \sum_{k=3}^{\infty} \frac{k(1+\mu\alpha)\delta(n,k)(1-c)\mu(\alpha+(1-\gamma))}{(1-c)\mu(\alpha+(1-\gamma))k(1+\mu\alpha)\delta(n,k)} \lambda_k \\ &= \sum_{k=3}^{\infty} \lambda_k = 1 - \lambda_2 \leq 1 \end{aligned}$$

it follows from Theorem 2 that the function f ϵ $Q_n(\alpha, \gamma, \mu, c)$.

Conversely, suppose that f ϵ $Q_n(\alpha, \gamma, \mu, c)$ and since

$$a_k \leq \frac{(1-c)\mu(\alpha + (1-\gamma))}{k(1+\mu\alpha)\delta(n,k)}, \ k \geq 3$$

setting

$$\lambda_k = \frac{k(1+\mu\alpha)\delta(n,k)}{(1-c)\mu(\alpha + (1-\gamma))} a_k, \ k \geq 3$$

and

$$\lambda_2 = 1 - \sum_{k=3}^{\infty} \lambda_k$$

It follows that:

$$f(z) = \sum_{k=3}^{\infty} \lambda_k f_k(z)$$

and the proof is complete.

Theorem 7: The class $Q_n(\alpha, \gamma, \mu, c)$ is closed under linear combinations.

Proof: Suppose that the function f be defined by (10) and let the function g be defined:

$$g(z) = z - \frac{c\mu(\alpha + (1-\gamma))}{2(n+1)(1+\mu\alpha)} z^2 - \sum_{k=3}^{\infty} |b_k| z^k, \qquad (b_k \geq 0) \tag{23}$$

Assuming that f (z) and g (z) are in the class $Q_n(\alpha, \gamma, \mu, c)$, it is sufficient to prove that the function H defined by:

$$H(z) = \lambda f(z) + (1-\lambda) g(z) \quad (0 \leq \lambda \leq 1) \tag{24}$$

is also in the class $Q_n(\alpha, \gamma, \mu, c)$.
Since:

$$H(z) = z - \frac{c\mu(\alpha + (1-\gamma))}{2(n+1)(1+\mu\alpha)} z^2 - \sum_{k=3}^{\infty} |a_k \lambda + (1-\lambda) b_k| z^k$$

we observe that:

$$\sum_{k=3}^{\infty} k(1+\mu\alpha)\delta(n,k) |a_k \lambda + (1-\lambda) b_k| \leq (1-c)\mu(\alpha + (1-\gamma))$$

with the aid of theorem 2.

Thus, $H \in Q_n(\alpha, \gamma, \mu, c)$ and the proof is complete.

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