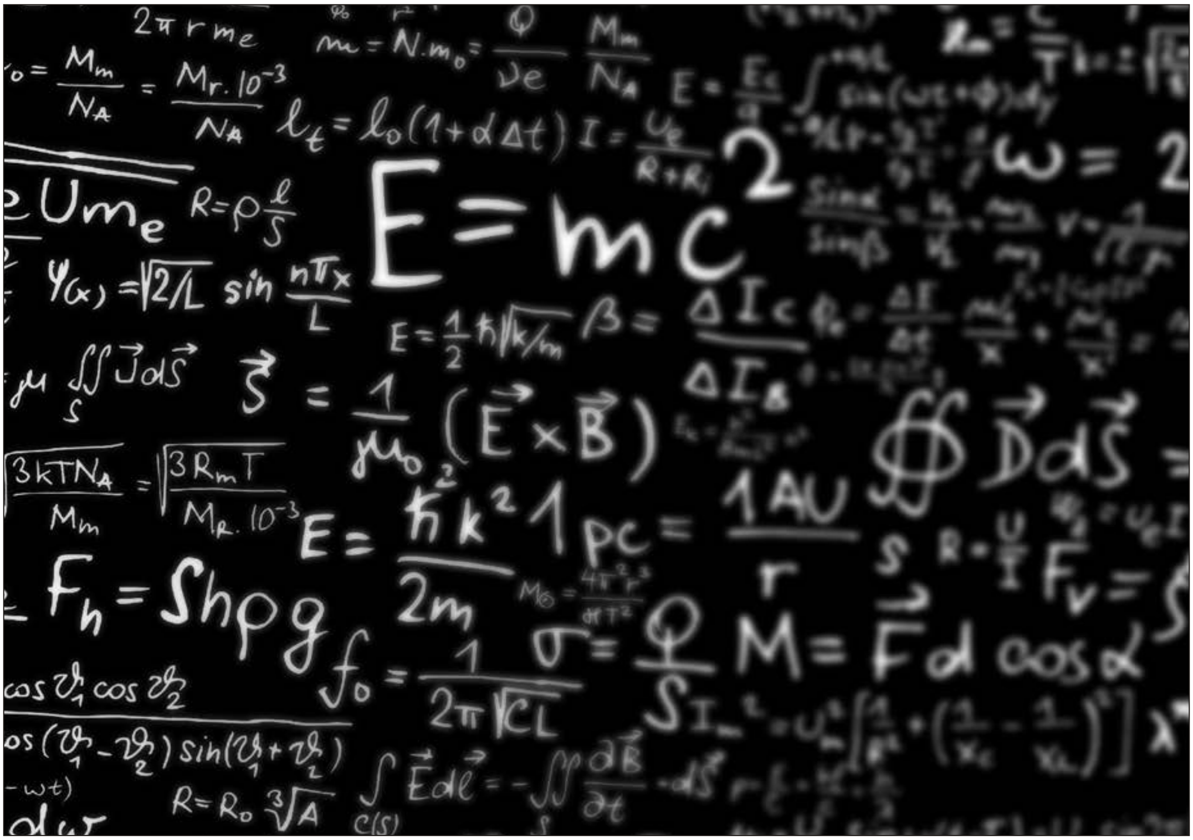

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Maximal Independent Neighborhood Set of an Interval Graph

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ABSTRACT

A graph G is an interval graph if there is a one-one correspondence between its vertices and a family I of intervals, such that two vertices in G are adjacent if and only if their corresponding intervals overlap. In this context, the family I of intervals is referred to as an interval model of G. Recently found minimum independent neighbourhood set of an interval graph. In this study, we exploit the Maximal Independent Neighbourhood Set (MLINS) of an interval graphs. This problem includes finding a maximal independent set, a shortest path between any two vertices in G in terms of directed network.

Key words: Interval family, interval graph, neighbourhood set, maximal independent set, shortest path, directed network

INTRODUCTION

The neighborhood number of a graph was introduced by Sampathkumar and Neeralagi (1985). He studied this parameter for various classes of graphs and obtained bounds and also found a Polynomial time algorithm for finding a minimum independent neighborhood set of an interval Graph (Maheswari *et al.*, 2004). A maximal independent set is also a dominating set (Ramalingam and Rangan, 1988; Keil, 1986) in the graph and every dominating set, that is independent, must be maximal independent (Furedi, 1987; Johnson and Yannakakis, 1988). So, maximal independent sets are also called independent dominating sets. A graph may have many maximal independent sets of widely varying sizes (Liang *et al.*, 1991) a largest maximal independent set is called a maximum independent set. Let G (V, E) be a graph. The neighbourhood (Hell, 1978) of a vertex v in G is defined as the set of vertices adjacent with v (including v) and is denoted by nbd [v]. A subset S of V in G is called a neighbourhood set of G if $G = \bigcup_{v \in S} < nbd[v] >$, where $<$ nbd [v] $>$ is the vertex induced sub graph of G. The neighbourhood number of G is defined as the maximum cardinality of a neighbourhood set of G. In addition, if the set S is independent then S is called an independent neighbourhood set of G.

Let $I = \{I_1, I_2, ..., I_n\}$ be given interval family. Each interval i in I is represented by (a_i, b_i),for i = 1, 2 ,..., n. Here a_i is called the left endpoint and b_i the right endpoint of the interval i. Without loss of generality we may assume that there are 2 n endpoints, which are distinct. The intervals are labelled in the increasing order of their right endpoints. Thus let I = {1, 2, ..., n} be the given interval family with right endpoint labelling and G its corresponding interval graph. For each interval i, let nbd (i) denote the set of intervals that intersect i (including i).

Let min (i) denote the smallest interval and max (i) the largest interval in nbd (i). Define NI (i) = j, if $b_i < a_j$ and there do not exist an interval k such that $b_i < a_k < a_j$. If there is no such j, then define NI (i) = null.

We now define Next (i) = min ({nbd [max (NI(i))]}\{nbd [i]}).

We may assume that there is no interval $i \in I$ that intersects all other intervals in I. For {i} itself becomes a maximum neighbourhood set. First we augment I with two dummy intervals say, I_0 and I_{n+1}, where:

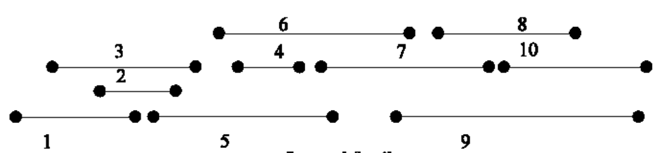
$$I_0 = [a_0, b_0] \text{ and } I_{n+1} = [a_{n+1}, b_{n+1}] \text{ such that}$$

$$b_o < \max_{1 \le k \le n} \{a_k\} \text{and } a_{n+1} > \max_{1 \le k \le n} \{b_k\}$$

Let $I_1 = I \cup \{I_0, I_{n+1}\}$. As in I the intervals in I_1 are also indexed by increasing order of their right endpoints, namely $b_0 < b_1 < ... < b_{n+1}$.

We now construct a directed network (Cockayne *et al.*, 1980), D (N, L) associated with G. For its vertices we take those intervals in I_1 which are not properly contained within other intervals. Because, if there is an interval j which contains another interval i, then the maximum neighbourhood set containing i can be changed to {MLINS\i} ∪ {j}.The lines in L are partitioned into two disjoint sets L_1 and L_2 which are defined below. For $j \in D$, there is a directed line (I_0, j) between I_0 and j that belongs to L_1 if and only if there is no interval I_h such that $b_0 < a_h < b_h < a_j$. Similarly there is a directed line (j, I_{n+1}) between j and I_{n+1} that belongs to L_1 if and only if there is no interval I_h such that $b_j < a_h < b_h < a_{n+1}$. This gives the scope to join the intervals I_0 and I_{n+1} to other intervals in I and it is obvious that all such joined directed lines, belong to L_1. Next for $i, j \in D$, there is a directed line (i, j) between i and j that belongs to L_2 if and only if j = Next (i).

Let us illustrate the construction of a directed network (Cockayne and the method of finding maximal independent neighbourhood set of an interval graph:



Interval family

MAIN THEOREMS

Lemma 1: If i and k are any two intervals which are intersecting and j is such that i<j<k , then j intersects k.

Proof: Let i, j, k are any three intervals and the intervals are labelled in increasing order of their right end points. The proof it is easy to see that when i<j<k then $b_i < b_j < b_k$. Since b_j is their right end points.

Now, i intersects k implies that $a_k < b_i$. Therefore, $a_k < b_i < b_j < b_k$, which implies that j also intersects k.

Lemma 2: If the directed line $(0, j) \in L_1$, where j is any interval of I, then the intervals between 0 and j belong to nbd [j].

Proof: Suppose '0' is any dummy interval and $0 \in I$, such that $(0, j) \in L_1$.

By the definition of lines in L_1 it follows that there is no interval I_m such that $b_0<a_m<b_m<a_j$. Soany interval between I_0 and I_j must intersect with I_j. Therefore, the intervals between 0 and j belong to nbd [j].

The proof of the following lemma follows on similar lines to that of lemma 2.

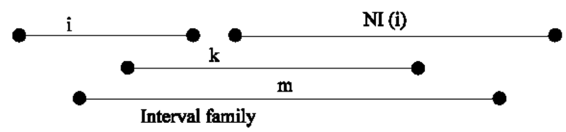
Lemma 3: If the directed line (j, n+1) $\in L_1$, where j is any interval of I, then the intervals between j and n+1 belong to nbd [j].

Thus it is clear by lemma 2 and 3 that if there is a directed line (i, j) $\in L_1$, then the intervals between I_i and I_j are adjacent with I_i or I_j.

Lemma 4: If i is any interval and k = max (NI (i)), then the intervals between i and k intersect I.

Proof: Let m be any interval between i and k. Suppose m does not intersect i.

Then $a_m>a_{NI(i)}$. Since $b_m<b_k$ and $a_m>a_{NI(i)}$, we get max (NI (i)) = m, a contradiction. Thus m must intersect i:



Lemma 5: If i is any interval in I, then i < max (NI (i)).

Proof: Let NI (i) = k and m = max (k). Now NI (i) = k implies $b_i < a_k$.

Again m intersects k implies $a_m<a_k<b_m$. Therefore, $b_i<a_k<b_m$.

That is, i<m = max (NI (i)).

Lemma 6: If the directed line (i, j) $\in L_2$, then the intervals between i and j belong to nbd [i] or nbd [j].

Proof: Let (i, j) $\in L_2$. Then j = Next (i). Let m be any interval between i and j. Then four conditions arise.

Condition 1: Suppose m intersects i. In such condition m$\in$ nbd [i].

Condition 2: Suppose m intersects max (NI (i)) and does not intersect i. Then m $\in$ nbd [(NI (i))]. Since m does not intersect i, m $\notin$ nbd [I]. So nbd [max (NI (i))]$\setminus$ nbd [i] contains m. Since j is the minimum element in nbd [max(NI(i))] and m $\in$ nbd [max (NI (i))] it follows that m must intersect j. That is m $\in$ nbd [j].

Condition 3: Assume that m does not intersect neither i nor max (NI (i)). Suppose i<m<max (NI (i)). Then i and max (NI (i)) intersect implies m and max (NI (i)) intersect. Suppose (NI (i)) <m< Next (i). Again max (NI (i)) intersects Next (i) implies m and Next (i) intersect. Therefore, m does not intersect neither i nor max (NI (i)) does not arise.

Condition 4: Suppose m intersects j. Then clearly m$\in$nbd [j]. Thus for all possibilities, the intervals between i and j belong to nbd [i] or nbd [j].

Lemma 7: Let (i, j) be any directed line in D. If D is a directed network D (N,L),then the vertex induced subgraph H on the vertex set $\{I_i, I_{i+1}, I_{i+2} \ldots, I_{j-1}, I_j\}$ is a sub graph of the induced graph $<$ nbd $[I_i] \cup$ nbd $[I_j]>$.

Proof: Let H be the induced sub graph on the vertex set $\{I_i, \ldots, I_j\}$. By lemmas 2, 3 and 6, it is clear that the vertex set $\{I_i, \ldots, I_j\} \in$ nbd $[I_i] \cup$ nbd $[I_j]$. It suffices to show that the edges of the graph H occur in $<$ nbd $[I_i] \cup$ nbd $[I_j]>$. Let I_p and I_q be any two arbitrary intervals between I_i and I_j. Without loss of generality assume that $I_p<I_q$. Now $(i, j) \in D$ implies that $(i, j) \in L_1$ or $(i, j) \in L_2$. Suppose $(i, j) \in L_1$. Then either i = 0 or j = n+1. Suppose i = 0. Then by lemma 2 the intervals between I_0 and I_j belong to nbd $[I_j]$. In particular I_p, $I_q \in$ nbd $[I_j]$.

Therefore, the edge $(I_p, I_q) \in <$nbd $[I_j]>$. Similarly when j = n+1 it follows that I_p, $I_q \in$ nbd $[I_i]$ and so the edge $(I_p, I_q) \in <$nbd $[I_i]>$.

Suppose that $(i, j) \in L_2$. Then by lemma 6, the intervals between I_i and I_j belong to nbd $[I_i] \cup$ nbd $[I_j]$. That is I_p, $I_q \in$ nbd $[I_i] \cup$ nbd $[I_j]$. If possible, let both I_p, $I_q \in$ nbd $[I_i]$. Then the edge $(I_p, I_q) \in <$ nbd $[I_i]>$. Similarly if I_p, $I_q \in$ nbd $[I_j]$ then $(I_p, I_q) \in <$nbd $[I_j]>$.

Hence assume that $I_p \in$ nbd $[I_i]$ and $I_q \in$ nbd $[I_j]$. Again it is clear that the edge $(I_p, I_q) \in <$ nbd $[I_i] \cup$ nbd $[I_j] >$. Thus for all possibilities, the edge $(I_p, I_q) \in <$nbd $[I_i] \cup$ nbd $[I_j]>$. Since I_p, I_q are arbitrary, it follows that $H \subseteq <$ nbd $[I_i] \cup$ nbd $[I_j]>$.

Lemma 8: If i, j are any two intervals in I such that j = Next (i), then i and j are non-adjacent.

Proof: By the definition of Next (i) = min({nbd [max (NI (I))]}\{nbd [i]}), the proof follows immediately.

Theorem 1: If P is a shortest directed path between the vertices 0 to n+1 in D (N, L), then the vertices in P other than 0 and n+1 correspond to a maximal independent neighbourhood set of an interval graph.

Proof: Let P be a shortest directed path from vertex 0 to n+1 in D. Define.

$S = \{I_i$: vertex i appears in P, $i \neq 0$, $i \neq n+1\}$. For each directed line (i, j) in P, by lemmas 2, 3 and 6, it follows that all intermediate intervals $I_{i+1}, I_{i+2}, \ldots, I_{j-1}$ between I_i and I_j belong to nbd $[I_i] \cup$ nbd $[I_j]$. Hence all intermediate intervals between the intervals in S belong to $<$nbd $[I_i] \cup$ nbd $[I_j]>$. Since the intervals in S correspond to the vertices in path P, the intervals in between I_0 and the first interval in S as well as the intervals in between the last interval in S and I_{n+1} also belong to $\bigcup_{I_i \in S} <$nbd $[I_i]>$.

Thus all the vertices in graph G are exhausted by the vertices in S.

That is, $V(G) = \bigcup_{I_i \in S}$ nbd $[I_i]$. But by lemma 7,

$<\{I_i, \ldots, I_j\}> \subseteq <$ nbd $[I_i] \cup$ nbd $[I_j]>$, where $I_i, I_j \in S$.

Therefore, $\bigcup_{I_i, I_j \in S} < \{I_i, \ldots, I_j\} > \subseteq \bigcup_{I_i, I_j \in S} <$nbd $[I_i] \cup$ nbd $[I_j] >$.

Since $V(G) = \bigcup_{I_i \in S}$ nbd $[I_i]$ it follows that $G = \bigcup_{I_i \in S} <$nbd $[I_i]>$.

Thus S is a neighbourhood set of G. By lemma 8 the vertices in S are non-adjacent. Therefore, S forms an maximal independent neighbourhood set of G. Since P is shortest, it follows that S is a maximal independent neighbourhood set of G.

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