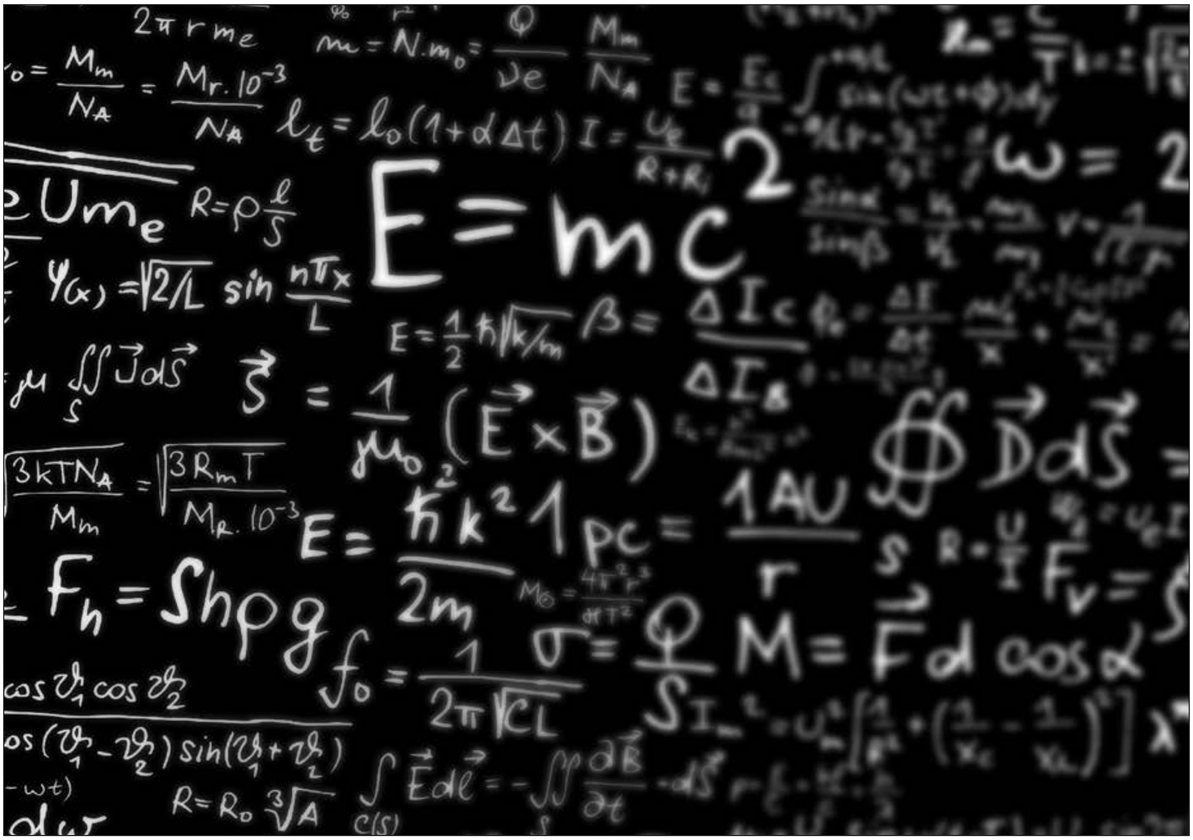

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An Approximate Solution of Some Differential Equations with New Type of Interpolation

[1]Saad A. Manna, [2]Faraidun K. Hama-Salh and [2]Hardi N. Aziz

[1]Department of Mathematics, Faculty of Science, University of Zakho, Duhok, Iraq

[2]School of Science Education, Department of Mathematics, Faculty of Science and Science Education, University of Sulaimani, Kurdistan Region, Sulaimani, Iraq

Corresponding Author: Saad A. Manna, Department of Mathematics, Faculty of Science, University of Zakho, Duhok, Iraq

ABSTRACT

In this research, we have studied the use of splines to solve lacunary interpolation problem analytically and applied to the solution of initial and boundary value problems. The analytical results of this model have been obtained in terms of convergent series with the theorems for errors estimation. Among a number of numerical methods used to solve differential equations spline methods provide an efficient tool.

Key words: Spline interpolation, boundary conditions, errors estimation, differential equations

INTRODUCTION

The given method approximates not only the solution function of the differential equations but also the convergence and errors estimation analytically. The order of the approximation coincides with that of the best possible polynomial approximation.

Many lacunary interpolation methods are used to approximate a solution of initial and boundary value problems to solve various scientific models (Al-Bayati *et al*., 2009; Bronson, 1973; Howell and Varma, 1989).

We study the convergence properties of a spline density estimate of the type introduced by Howell and Varma (1989) and discussed by Siddiqi *et al.* (2007). Many class of spline interpolation have been implemented to solve the differential equations numerically such as existence and uniqueness lacunary interpolations Al-Bayati *et al.* (2009) and interpolation by six and seven degree spline by Salh (2010), Faraj (2010) and Siddiqi *et al.* (2007), respectively. Also we are interested in the convergence analysis by two theorems to find the smallest errors bound various with problems. The method can be used to evaluate the approximating solution by the finite Taylor series and shooting procedure described by the transformed equations obtained from the original spline interpolations.

CONSTRUCTION SPLINE MODEL

We present a seven degree spline interpolation polynomial with given sufficiently smooth function f(x) defined on I = [0, 1] and the mesh point of the uniform partition of I as:

$$\Delta : 0 = x_0 < x_1 < x_2 < \ldots < x_n = 1$$

With the knot $x_i = ih+x_0$, where i = 0, 1, 2,..., n-1 and h is the distance of each subintervals. The seventh degree spline interpolation is defined:

$$S_0(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + (x-x_0)^4 a_{04} + \frac{(x-x_0)^5}{5!}y_0^{(5)} + (x-x_0)^6 a_{06} + (x-x_0)^7 a_{07}$$

On the subinterval, $[x_0, x_1]$ where a_{0j} = 4, 6, 7 are unknowns to be determined and on $[x_{i-1}, x_i]$, i = 1, 2,..., n-1 be define the expression, for $S_i(x)$ as Al-Bayati *et al.* (2009):

$$S_i(x) = y_i + (x-x_i)a_{i1} + \frac{(x-x_i)^2}{2!}y_i'' + (x-x_i)^3 a_{i3} + (x-x_i)^4 a_{i4} + \frac{(x-x_i)^5}{5!}y_i^{(5)} + (x-x_i)^6 a_{i6} + (x-x_i)^7 a_{i7}$$

where, a_{ij}, 1, n-1, j = 1, 3, 4, 6, 7 are unknowns values we need to determine it.

On the end of the interval $[x_{n-1}, x_n]$ must be as the same of the first interval:

$$S_0(x) = y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + h^4 a_{04} + \frac{h^5}{120}y_0^{(5)} + h^6 a_{06} + h^7 a_{07} \tag{1}$$

And the boundary conditions:

$$S_0(x) = y_1,\ S_0''(x_1) = y'' \text{ and } S_0^{(5)}(x_1) = y_1^{(5)} \tag{2}$$

$$S_0'(x_1) = a_{11} \text{ and } S_0'''(x_1) = 6a_{13} \tag{3}$$

IMPLEMENTATION OF THE METHOD

In this section, we study the convergence analysis of the method developed in Section 2. For this purpose we first let $y(x)\epsilon C^7[0, 1]$. Thus we can write the following:

Applied the conditions in Eq. 2 with using Eq. 1, we obtain:

$$a_{07} = \frac{1}{2520}[y_1^{(5)} - y_0^{(5)}] - \frac{2}{7h}a_{06} \tag{4}$$

and:

$$a_{06} = \frac{1}{18h^4}[y_1'' - y_0'] - \frac{1}{18h^3}y_0''' - \frac{2}{3h^2}a_{04} - \frac{1}{1080h}[y_1^{(5)} + 9y_0^{(5)}] \tag{5}$$

By substituting Eq. 4 and 5 in Eq. 1, we get:

$$a_{04} = \frac{21}{11h^4}[y_1 - y_0] - \frac{21}{11h^3}y_0' - \frac{1}{66h^2}[5y_1'' + 58y_0''] - \frac{8}{33h}y_0''' + \frac{h}{3960}[2y_1^{(5)} - 15y_0^{(5)}] \tag{6}$$

Substitute Eq. 6 in Eq. 5, we obtain:

$$a_{06} = -\frac{14}{11h^6}[y_1 - y_0] + \frac{14}{11h^5}y_0' + \frac{7}{66h^4}[y_1'' + 5y_0''] + \frac{7}{66h^3}y_0''' - \frac{1}{3960h}[5y_1^{(5)} + 23y_0^{(5)}] \quad (7)$$

Substitute Eq. 7 in Eq. 4, we get:

$$a_{07} = \frac{4}{11h^7}[y_1 - y_0] - \frac{4}{11h^6}y_0' - \frac{1}{33h^5}[y_1'' + 5y_0''] - \frac{1}{33h^4}y_0''' + \frac{1}{3960h^2}[3y_1^{(5)} + 5y_0^{(5)}] \quad (8)$$

By substituting a_{04}, a_{06} and a_{07} in Eq. 1 and using the initial condition Eq. 3, we get:

$$a_{11} = \frac{28}{11h}[y_1 - y_0] - \frac{17}{11}y_0' + \frac{h}{33}[4y_1'' - 13y_0''] - \frac{h^2}{22}y_0''' - \frac{h^4}{3960}[y_1^{(5)} - 2y_0^{(5)}] \quad (9)$$

$$a_{13} = -\frac{56}{11h^3}[y_1 - y_0] + \frac{56}{11h^2}y_0' + \frac{1}{33h}[25y_1'' + 59y_0''] + \frac{17}{66}y_0''' + \frac{h^2}{3960}[13y_1^{(5)} - 15y_0^{(5)}] \quad (10)$$

Apply the spline function in Eq. 1, with the condition in Eq. 2 on the interval $[x_i, x_{i+1}]$, where, $i = 1, 2, \ldots, n-1$ we have:

$$y_{i+1} = y_i + ha_{i1} + \frac{h^2}{2}y_i'' + h^3 a_{i3} + h^4 a_{i4} + \frac{h^5}{120}y_i^{(5)} + h^6 a_{i6} + h^7 a_{i7} \quad (11)$$

$$y_{i+1}'' = y_i'' + 6ha_{i3} + 12h^2 a_{i4} + \frac{h^3}{6}y_i^{(5)} + 30h^4 a_{i6} + 42h^5 a_{i7} \quad (12)$$

$$y_{i+1}^{(5)} = y_i^{(5)} + 720ha_{i6} + 2520h^2 a_{i7} \quad (13)$$

$$S_i'(x_{i+1}) = y_{i+1}' = a_{i+1,1} = a_{i1} + hy_i'' + 3h^2 a_{i3} + 4h^3 a_{i4} + \frac{h^4}{24}y_i^{(5)} + 6h^5 a_{i6} + 7h^6 a_{i7} \quad (14)$$

$$S_i'''(x_{i+1}) = y_{i+1}''' = 6a_{i+1,3} = 6a_{i3} + 24ha_{i4} + \frac{h^2}{2}y_i^{(5)} + 120h^3 a_{i6} + 210h^4 a_{i7} \quad (15)$$

From Eq. 13, we get:

$$a_{i7} = \frac{1}{2520h^2}[y_{i+1}^{(5)} - y_i^{(5)}] - \frac{2}{7h}a_{i6} \quad (16)$$

Substitute Eq. 16 in Eq. 12, we get:

$$a_{i6} = \frac{1}{18h^4}[y_{i+1}'' - y_i''] - \frac{1}{3h^3}a_{i3} - \frac{2}{3h^2}a_{i4} - \frac{1}{1080h}[y_{i+1}^{(5)} + 9y_i^{(5)}] \quad (17)$$

Put a_{i6} in Eq. 11, we get:

$$a_{i4} = \frac{21}{11h^4}[y_{i+1} - y_i] - \frac{21}{11h^3}a_{i1} - \frac{16}{11h}a_{i3} - \frac{1}{66h^2}[5y''_{i+1} + 58y''_i] + \frac{h}{3960}[2y^{(5)}_{i+1} - 15y^{(5)}_i] \tag{18}$$

Substitute Eq. 18 in Eq. 17, we get:

$$a_{i6} = -\frac{14}{11h^6}[y_{i+1} - y_i] + \frac{14}{11h^5}a_{i1} + \frac{7}{11h^3}a_{i3} + \frac{1}{66h^4}[7y''_{i+1} + 35y''_i] - \frac{1}{3960h}[5y^{(5)}_{i+1} + 23y^{(5)}_i] \tag{19}$$

Substitute Eq. 19 in Eq. 16, we get:

$$a_{i7} = \frac{4}{11h^7}[y_{i+1} - y_i] - \frac{4}{11h^6}a_{i1} - \frac{2}{11h^4}a_{i3} - \frac{1}{33h^5}[y'_{i+1} + 5y''_i] + \frac{1}{3960h^2}[3y^{(5)}_{i+1} + 5y^{(5)}_i] \tag{20}$$

By substituting a_{i4}, a_{i6} and a_{i7} in Eq. 14, we get:

$$a_{i+1,1} = \frac{28}{11h}[y_{i+1} - y_i] + \frac{h}{33}[4y''_{i+1} - 13y''_i] - \frac{17}{11}a_{i1} - \frac{3}{11}a_{i3} - \frac{h^4}{3960}[y^{(5)}_{i+1} - 2y^{(5)}_i]$$

By substituting a_{i4}, a_{i6} and a_{i7} in Eq. 14, we get:

$$a_{i+1,3} = -\frac{56}{11h^3}[y_{i+1} - y_i] + \frac{1}{330h}[25y''_{i+1} + 59y''_i] + \frac{56}{11h^2}a_{i1} + \frac{17}{11}a_{i3} + \frac{h^2}{3960}[13y^{(5)}_{i+1} - 15y^{(5)}_i]$$

Also for the interval $[x_{n-1}, x_n]$, clearly by the same process of the first interval.

Lemma 1: Let $y \in C^7[0, 1]$, then $|e_{i1}| \le h^6 C_i W_7(h, y)$, where $i = 1, 2, \ldots, n-1$ and e_{i1}-$y'_i(x)$ depend on the number of the intervals.

Proof: Let $|e_{i1}| = |a_{i1} - y'_i$, if $i = 1$, then from Eq. 12, we have:

$$|e_{11}| = |a_{11} - y'_1| = |\frac{h^6}{1980}y_0^7(\alpha_1) + \frac{h^6}{990}y_0^7(\alpha_2) - \frac{7h^6}{7920}y_0^7(\alpha_3) - \frac{h^6}{720}y_0^7(\alpha_4)| = \frac{h^6}{660}[y_0^7(\theta_1) - y_0^7(\theta_2)], x_0 < \alpha_1, \alpha_2, \alpha_3 < x_1$$

and:

$$x_0 < \theta_1, \theta_2 < x_1$$
$$\Rightarrow |e_{11}| \le \frac{h^6}{660}W_7(h, y) \Rightarrow |e_{11}| \le h^6 C_1 W_7(h, y)$$

Where:

$$C_1 = \frac{1}{660}$$

If i = 2, then:

$$
\begin{aligned}
&|e_{21}|=|a_{21}-y_2'|=-\frac{17}{11}[a_{11}-y_1']-\frac{h^2}{22}[6a_{13}-y_1''']+\frac{h^6}{660}W_7(h,y)\\
&=\frac{17}{7260}h^6W_7(h,y)+\frac{21}{9680}h^6W_7(h,y)+\frac{h^6}{660}W_7(h,y)\\
&\Rightarrow|e_{21}|\le\frac{35}{5808}h^6W_7(h,y)\\
&\Rightarrow|e_{21}|\le h^6C_2W_7(h,y)
\end{aligned}
$$

Where:

$$C_2=\frac{35}{5808}$$

By the same way, we can find, C_i be show that the inequality:

$$|e_{i1}|\le h^6C_iW_7(h, y), \text{ for } i = 1, 2,\ldots, n-1$$

Lemma 2: Let $y\in C^7[0, 1]$, then $|e_{i3}|\le h^4 C'_iW_7(h, y)$, for $i = 1, 2,\ldots, n-1$ and $e_{13} = 6a_{i3}-y_i'''(x)$, C'_i depends on the numbers of the intervals.

Proof: $|e_{i3} = |6a_{i3}-y_i'''|$, if i = 1, then:

$$e_{13}=-\frac{h^4}{165}y_0^{(7)}(\alpha_1)+\frac{5}{132}h^4y_0^{(7)}(\alpha_2)+\frac{13}{1320}h^4y_0^{(7)}(\alpha_3)-\frac{h^4}{24}y_0^{(7)}(\alpha_4)=\frac{21}{440}h^4[y_0^{(7)}(\theta_1)-y_0^{(7)}(\theta_2)]$$

Where:

$$x_0<\theta_1<x_1 \Rightarrow|e_{13}|\le h^4C_1'W_7(h,y)$$

Where: $C_1'=\frac{21}{440}$. If i = 2, then:

$$
\begin{aligned}
&|e_{23}|=|6a_{23}-y_2'''|\le\frac{28}{605}h^4W_7(h,y)+\frac{107}{2420}h^4W_7(h,y)+\frac{21}{440}h^4W_7(h,y)=\frac{2597}{4840}h^4W_7(h,y)\\
&\Rightarrow|e_{23}|\le\frac{2597}{4840}h^4W_7(h,y)\Rightarrow|e_{23}|\le h^4C_2'W_7(h,y)
\end{aligned}
$$

Where: $C_2'=\frac{2597}{4840}$. By the same way, we can find, C'_i can be show that the inequality:

$$|e_{i3}|\le h^4C'_iW_7(h, y), \text{ for } i = 1, 2,\ldots, n-1$$

Theorem 1: Let $y\in C^7[0, 1]$, and S(x) be a unique of degree seven which a solution of the problem 1, then for the interval $[x_0, x_1]$ the following errors bound are hold:

$$|S_i^{(7-r)}(x)-y^{(7-r)}(x)|\le\begin{cases}\frac{25}{11}W_7(h,y), \text{if} r=0\\ \frac{32}{11}hW_7(h,y), \text{if} r=1\\ \frac{39}{22}h^2W_7(h,y), \text{if} r=2\\ \frac{47}{66}h^3W_7(h,y), \text{if} r=3\\ \frac{21}{440}h^4W_7(h,y), \text{if} r=4\\ \frac{7}{132}h^5W_7(h,y), \text{if} r=5\\ \frac{1}{660}h^6W_7(h,y), \text{if} r=6\\ \frac{109}{55440}h^7W_7(h,y), \text{if} r=7\end{cases}$$

where, $W_7(h,y)$ denotes the modules of continuity of the maximum and seventh derivative of y(x).

Proof: Let $x\in[x_0, x_1]$ with using the Taylor series and take the seventh derivative in Eq. 11, we have $S_i^{(7)}(x_{i+1}) = 5040a_{i7}$:

$$\begin{aligned}&|S_0^{(7)}(x)-y^{(7)}(x)|=|y^{(7)}-5040a_{07}|\\ &=|y^{(7)}-\frac{4}{11}y_0^{(7)}(x_1)+\frac{14}{11}y_0^{(7)}(x_2)-\frac{21}{11}y_0^{(7)}(x_3)|\\ &\le\frac{25}{11}|y^{(7)}(\theta_0)-y^{(7)}(\theta_1)|=\frac{25}{11}W_7(h,y)\\ &\Rightarrow|S_0^{(7)}(x)-y^{(7)}(x)|\le\frac{25}{11}W_7(h,y)\end{aligned}$$

Also using the Taylor expansion of the function and take the sixth derivative in Eq. 11, we have:

$$\begin{aligned}&S_i^{(6)}(x_{i+1})=720a_{06}+5040ha_{07}\\ &|S_0^{(6)}(x)-y^{(6)}(x)|=|y^{(6)}-720a_{06}-5040ha_{07}|\\ &\le\frac{32}{11}h|y^{(7)}(\theta_1)-y^{(7)}(\theta_2)|\\ &\Rightarrow|S_0^{(6)}(x)-y^{(6)}(x)|\le\frac{32}{11}hW_7(h,y)\end{aligned}$$

Take the fifth derivative in Eq. 11, we have $S_i^{(5)}(x_{i+1}) = 720a_{06}+2520h^2a_{07}$:

$$\Rightarrow|S_0^{(5)}(x)-y^{(5)}(x)|\le\frac{39}{22}h^2|y^{(7)}(\theta_1)-y^{(7)}(\theta_2)|=\frac{39}{22}h^2W_7(h,y)$$

Take the fourth derivative in equation, we have:

$$\Rightarrow|S_0^{(4)}(x)-y^{(4)}(x)|\le\frac{47}{66}h^3W_7(h,y)$$

Take the third derivative in Eq. 11, we have: $S_0''' = 6a_{13}$:

$$|S_0''(x)-y''(x)|\le\frac{21}{440}h^4\,|y^{(7)}(\theta_1)-y^{(7)}(\theta_2)|=\frac{21}{440}h^4W_7(h,y)\Rightarrow|S_0''(x)-y''(x)|\le\frac{21}{440}h^4W_7(h,y)$$

Take the second derivative in Eq. 11, we have:

$$\Rightarrow|S_0'(x)-y''(x)|\le\frac{7}{132}h^5W_7(h,y)$$

Take the first derivative in Eq. 11, we have: $S_0' = a_{11}$:

$$|S_0'(x)-y'(x)|\le\frac{1}{660}h^6\,|S_0'(x)-y'(x)|=\frac{1}{660}h^6W_7(h,y)$$

$$\Rightarrow|S_0'(x)-y'(x)|\le\frac{1}{660}h^6W_7(h,y)$$

by the same way, we obtain:

$$\Rightarrow|S_0(x)-y(x)|\le\frac{109}{55440}h^7W_7(h,y)$$

Theorem 2: Let $y\epsilon C^7[x_{i-1}, x_i]$ and the approximate spline function can be find the error bound as:

$$|S_i^{(7-r)}(x)-y^{(7-r)}(x)|\le\begin{cases}\frac{1}{11}H_{1i}W_7(h,y), \text{if} r=0\\ \frac{32}{11}hH_{2i}W_7(h,y), \text{if} r=1\\ \frac{89}{22}h^2W_7(h,y), \text{if} r=2\\ \frac{h^3}{66}H_{3i}W_7(h,y), \text{if} r=3\\ \frac{h^4}{88}H_{4i}W_7(h,y), \text{if} r=4\\ \frac{7}{132}h^5W_7(h,y), \text{if} r=5\\ \frac{h^6}{2640}H_{5i}W_7(h,y), \text{if} r=6\\ \frac{109}{55440}h^7W_7(h,y), \text{if} r=7\end{cases}$$

Proof: From the last derivative of the spline model, we have:

$$\begin{aligned}&|S_i^{(7)}(x)-y^{(7)}(x)|=|5040a_{i7}-y^{(7)}(x)|\\&\Rightarrow|S_i^{(7)}(x)-y^{(7)}(x)|\le\frac{1680}{11}C_i'W_7(h,y)+\frac{20160}{11}C_iW_7(h,y)+\frac{25}{11}W_7(h,y)\\&=\frac{1}{11}[1680C_i'+20160C_i+25]W_7(h,y)\\&=\frac{1}{11}H_{1i}W_7(h,y)\\&\Rightarrow|S_i^{(7)}(x)-y^{(7)}(x)|\le\frac{1}{11}H_{1i}W_7(h,y)\end{aligned}$$

where, $H_{i1} = 1680C'_i+20160C_i+25$.

For sixth derivative, we have:

$$\Rightarrow |S_i^{(6)}(x)-y^{(6)}(x)|=|\frac{32}{11}hW_7(h,y)+\frac{10080}{11h^5}(y_i^{'}-a_{i1})+\frac{840}{11h^3}(y_i^{'''}-6a_{i3})|$$
$$\leq |\frac{32}{11}hW_7(h,y)|+|\frac{10080}{11h^5}(y_i^{'}-a_{i1})|+|\frac{840}{11h^3}(y_i^{'''}-6a_{i3})|$$
$$=\frac{32}{11}h[W_7(h,y)+315C_iW_7(h,y)+\frac{105}{4}C_i^{'}W_7(h,y)]$$
$$\Rightarrow |S_i^{(6)}(x)-y^{(6)}(x)|\leq \frac{32}{11}hH_{2i}W_7(h,y)$$

Where:

$$H_{2i}=1+315C_i+\frac{105}{4}C_i^{'}$$

For fifth derivative, we have:

$$\Rightarrow |S_i^{(5)}(x)-y^{(5)}(x)|\leq \frac{89}{22}h^2W_7(h,y)$$

For fourth derivative, we have:

$$\Rightarrow |S_i^{(4)}(x)-y^{(4)}(x)|\leq [\frac{5184}{11}C_i+\frac{204}{11}C_i^{'}+\frac{47}{66}]h^3W_7(h,y)$$
$$=\frac{h^3}{66}[30504C_i+1224C_i^{'}+47]W_7(h,y)$$
$$=\frac{h^3}{66}H_{3i}W_7(h,y)\Rightarrow |S_i^{(4)}(x)-y^{(4)}(x)|\leq \frac{h^3}{66}H_{3i}W_7(h,y)$$

where, $H_{3i} = 30504C_i+1224C'_i+47$.

For third derivative, we have:

$$\Rightarrow |S_i^{'''}(x)-y^{'''}(x)|\leq \frac{h^4}{88}H_{4i}W_7(h,y)$$

where, $H_{4i} = 2688C_i+136C'_i+19$.

For second derivative, we have:

$$\Rightarrow |S_i^{''}(x)-y^{''}(x)|\leq \frac{7}{132}h^5W_7(h,y)$$

For the first derivative, we have:

$$\Rightarrow |S_i^{'}(x)-y^{'}(x)|\leq \frac{h^6}{2640}H_{5i}W_7(h,y)$$

where H_{5i} = 4080C_i+120C'_i+29 and:

$$|S_i(x) - y(x)| \leq \frac{109}{55440} h^7 W_7(h, y)$$

Proof is completed.

NUMERICAL RESULTS

Consider two examples of initial and boundary value problems are obtained an approximate numerical solution in the first with using Taylor's expansion method and in the second example we use shooting method are referred by Bronson (1973), Jain *et al*. (2007), Omar (2005) and Russel and Shampine (2007). The problems are tested to the efficiency of the development solutions and to demonstrate its convergence computationally. Based on the numerical results, it can be concluded that most cases the execution the number of steps for solving the given problems at all tolerance (Table 1, 2).

The absolute of maximum error with respect to derivatives defined as Conte (1980).

$$AMAXE^{(j)} = \max_{1\leq i\leq n} \left\| e_i^{(j)} \right\| = \max_{1\leq i\leq n} \left\| s^{(j)}(x_i) - y^{(j)}(x_i) \right\|$$

where, j = 1, 3, 4, 5, 6, 7, be order of derivatives on whole intervals and $y(x_i)$ is the exact solution.

Problem 1: Consider the initial value problem:

$$\frac{d^3y}{dx^3} + y\frac{dy}{dx} = 0, \quad y(0) = y'(0) = 0, y''(0) = 1$$

Problem 2: Consider the boundary value problem:

$$\frac{d^2y}{dx^2} = 1 + y, \quad y(0) = 0, y'(1) = e - 1$$

Table 1: Absolute maximum error for S(x) and its derivative with different values of tolerance for the problem 1

TOL	$AMAXE^{(1)}$	$AMAXE^{(3)}$	$AMAXE^{(4)}$	$AMAXE^{(5)}$	$AMAXE^{(6)}$	$AMAXE^{(7)}$
10^{-1}	9.92×10^{-12}	5.55×10^{-8}	3.47×10^{-6}	14.01×10^{-1}	4.23×10^{-3}	5.833×10^{-1}
10^{-2}	3.46×10^{-18}	5.658×10^{-13}	3.415×10^{-10}	1.013×10^{-2}	4.35×10^{-4}	5.78×10^{-2}
10^{-3}	0	3.335×10^{-12}	1.07×10^{-8}	9.091×10^{-2}	9.09×10^{-2}	1.86×10^{0}

TOL: Tolerance, AMAXE: Absolute of the maximum error with respect to derivatives

Table 2: Absolute maximum error for and its derivative with different values of tolerance for the problem 2

TOL	$AMAXE^{(1)}$	$AMAXE^{(3)}$	$AMAXE^{(4)}$	$AMAXE^{(5)}$	$AMAXE^{(6)}$	$AMAXE^{(7)}$
0.5	3.76×10^{-1}	1.92×10^{-5}	2.409×10^{-4}	9.55×10^{-1}	2.208×10^{-2}	3.239×10^{-1}
10^{-1}	3.77×10^{-1}	6.168×10^{-9}	3.855×10^{-7}	2.21×10^{-1}	8.327×10^{-4}	6.48×10^{-2}
10^{-2}	3.774×10^{-1}	1.01×10^{-9}	3.44×10^{-7}	2.21×10^{-1}	3.17×10^{-2}	5.623×10^{0}

TOL: Tolerance, AMAXE: Absolute of the maximum error with respect to derivatives

CONCLUSION

We have considered the performance of a new spline interpolation consist of degree seven with the new boundary conditions for function, second and fifth derivatives are known on the mid interval for solving initial and boundary value problems using variable step size and order. The developed seventh spline method has shown the efficiency by solving higher order OSEs and the convergence results obtained are very encouraging better than the existing spline as Salh (2010), Faraj (2010) and Siddiqi *et al.* (2007).

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