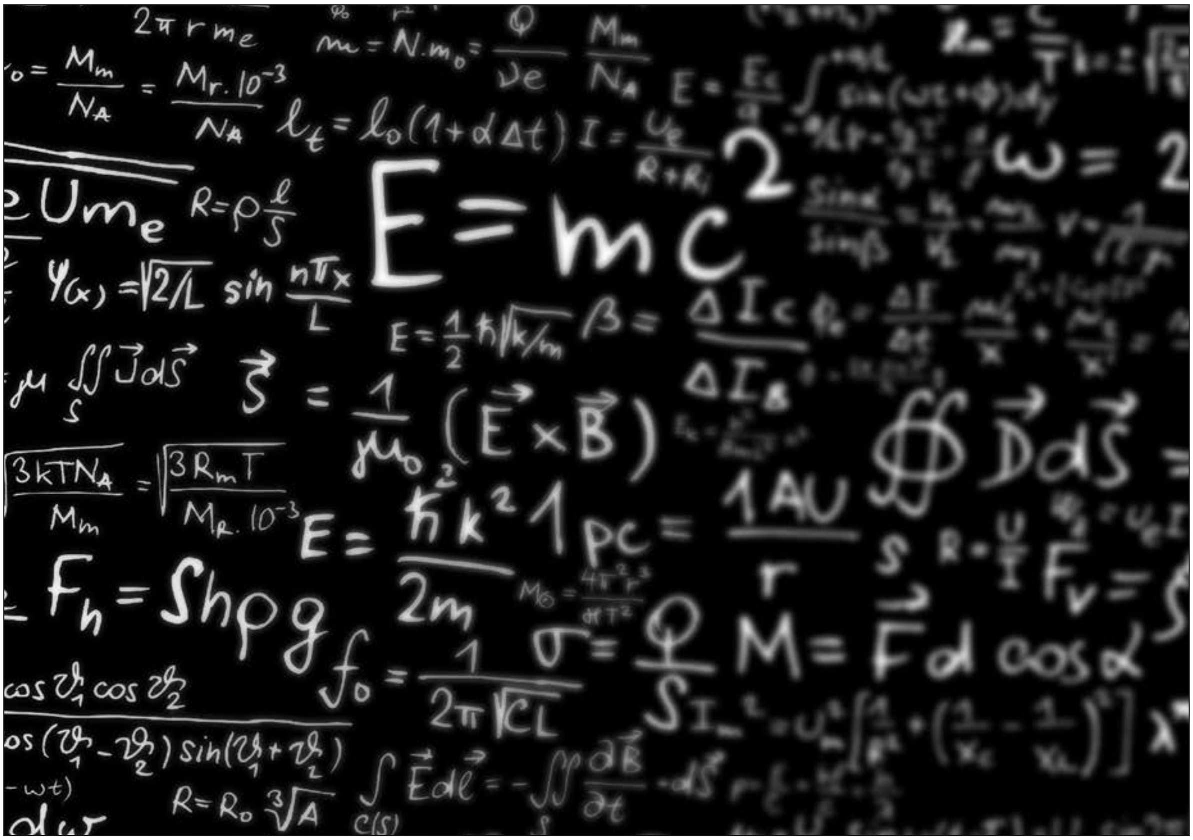

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Operators Preserving Inequalities Between Polynomials

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ABSTRACT

In this study, by combining the operators B and D_α, we investigate the dependence of $B[D_\alpha(P(Rz)-\beta P(rz))]$ on the maximum modulus of P(z) on $|z| = 1$ for every real or complex numbers α and β with $|\alpha|\geq 1$, $|\beta|\leq 1$ and $R>r\geq 1$. The present results include not only some known polynomial inequalities as special case, but also the results recently proved by Bidkham and Mezerji as a particular case.

Key words: Polynomials, B. operator, polar derivative, inequalities

INTRODUCTION

If:

$$P(z) = \sum_{j=0}^{n} a_j z^j$$

is a polynomial of degree at most n and P'(z) is its derivative, then:

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| \tag{1}$$

and:

$$\max_{|z|=R>1} |P'(z)| \leq R^n \max_{|z|=1} |P(z)| \tag{2}$$

Inequality Eq. 1 is an immediate consequence of S. Bernstein's inequality on the derivative of a trigonometric polynomial (Bernstein, 1930; Rahman and Schmeisser, 2002), where as inequality (Eq. 2) is a simple deduction from the maximum modulus principle (Riesz, 1916). In both inequalities Eq. 1 and 2 equality holds only when P(z) is a constant multiple of z^n.

If we restrict ourselves to a class of polynomials having no zero in $|z|\leq 1$, then the above inequality can be sharpened. In fact, Erdös conjectured and latter (Lax, 1944) proved that if $P(z)\neq 0$ in $|z|\leq 1$, then:

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \tag{3}$$

and:

$$\max_{|z|=R>1} |P'(z)| \le \frac{R^n+1}{2} \max_{|z|=1} |P(z)| \tag{4}$$

Turan (1939) proved that, if P(z) has all its zeros in $|z| \le 1$, then:

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{2} \max_{|z|=1} |P(z)| \tag{5}$$

Concerning the minimum modulus of a polynomial P(z) and its derivative P'(z), Aziz and Dawood (1988) proved that, if P(z) has all its zeros in $|z| \le 1$, then:

$$\min_{|z|=1} |P'(z)| \ge n \min_{|z|=1} |P(z)| \tag{6}$$

Let, α be any complex number, the polynomial $D_\alpha P(z) = nP(z)+(\alpha-z) P'(z)$ denote the polar derivative of the polynomial P(z) of degree at most n with respect to α. The polynomial $D_\alpha P(z)$ is of degree at most n-1 and it generalizes the ordinary derivative in the sense that:

$$\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$$

Aziz (1988) extended inequalities Eq. 3 and 5 to the polar derivative of a polynomial and proved that if P(z) is a polynomial of degree n which does not vanish in $|z|<1$, then for every complex number α with $|\alpha| \ge 1$:

$$\max_{|z|=1} |D_\alpha P(z)| \le \frac{n}{2}\left\{|\alpha z^{n-1}|+1\right\} \max_{|z|=1} |P(z)|, \text{ for } |z| \ge 1 \tag{7}$$

Rahman and Schmeisser (2002) introduced a class B_n of operators B that map $P\epsilon P_n$ into itself. That is, the operator B carries $P\epsilon P_n$ into:

$$B[P(z)] = \lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right)\frac{P'(z)}{1!} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2!}$$

where, λ_0, λ_1 and λ_2 are real or complex numbers such that all the zeros of:

$$u(z) := \lambda_0 + C(n,1)\,\lambda_1 z + C(n,2)\lambda_2 z^2, \quad C(n,r) = \frac{n!}{r!(n-r)!'} \tag{8}$$

lie in the half plane:

$$|z| \le \left|z - \frac{n}{2}\right|$$

Concerning this operator Shah and Liman (2008) proved.

Theorem A: If $P(z)\epsilon P_n$ and $P(z) \neq 0$ in $|z|>1$, then for $|z|\geq 1$:

$$|B[P(z)]|\geq|B[z^n]|\min_{|z|=1}|P(z)| \tag{9}$$

Theorem B: If $P(z)\epsilon P_n$ and $P(z) \neq 0$ in $|z|<1$, then for $|z|\geq 1$:

$$|B[P(z)]| \leq \frac{1}{2}\left[\{|B[z^n]|+|\lambda_0|\}\max_{|z|=1}|P(z) - \{|B[z^n]|-|\lambda_0|\}\min_{|z|=1}|P(z)|\right] \tag{10}$$

Concerning the dependence of $|P(Rz)-P(z)|$ on $|P(z)|$ Aziz and Rather (1999) proved.

Theorem C: If P(z) is a polynomial of degree n, then for every real or complex number $\boldsymbol{\beta}$ with $|\boldsymbol{\beta}|\leq 1$ and $R\geq 1$:

$$|P(Rz)-\boldsymbol{\beta}P(z)| \leq |R^n-\boldsymbol{\beta}|\,|z|^n \max_{|z|=1}|P(z)|, \text{ for } |z|\geq 1 \tag{11}$$

Theorem D: If P(z) is a polynomial of degree n which does not vanish in $|z|<1$, then for every real or complex number $\boldsymbol{\beta}$ with $|\boldsymbol{\beta}|\leq 1$ and $R\geq 1$:

$$|P(Rz)-\beta P(z)| \leq \left\{\frac{|R^n-\beta||z^n|+|1-\beta|}{2}\right\}\max_{|z|=1}|P(z)|, \text{ for } |z|\geq 1 \tag{12}$$

Recently Bidkham and Mezerji (2011) have generalized some of the above inequalities by combining B and D_α operators and proved the following results.

Theorem E: If P(z) is a polynomial of degree at most n, having all its zeros in $|z|\leq 1$, then for every complex number α with $|\alpha|\geq 1$:

$$|B[D_\alpha P(z)]| \geq n|\alpha|\,|B[z^{n-1}]|\min_{|z|=1}|P(z)|, \text{ for } |z|\geq 1 \tag{13}$$

Theorem F: If P(z) is a polynomial of degree at most n, having no zero in $|z|<1$, then for every α with $|\alpha|\geq 1$:

$$\begin{aligned}|B[D_\alpha P(z)]| \leq \frac{n}{2}\Big\{&\{|\alpha||B[z^{n-1}]|+|\lambda_0|\}\max_{|z|=1}|P(z)| \\ &-\{|\alpha||B[z^{n-1}]|-|\lambda_0|\}\min_{|z|=1}|P(z)|, \text{ for } |z|\geq 1\end{aligned} \tag{14}$$

In this study, we combine the different ideas and techniques used above and consider the operator B and D_α such that the operator B carries $D_\alpha P(z)$ into:

$$B[D_\alpha(P(z))]=\lambda_0 D_\alpha p(z)+\lambda_1\left(\frac{mz}{2}\right)D_\alpha P'(z)+\lambda_2\left(\frac{mz}{2}\right)^2\frac{D_\alpha P''(z)}{2!}$$

where, $0\leq m\leq n-1$ and λ_0, λ_1 and λ_2 are real or complex numbers such that all zeros of:

$$u(z) := \lambda_0 + C(m,1)\ \lambda_1 + Z + C(m,2)\lambda_2 z^2,\ C(m,r) = \frac{m!}{r!(m-r)!'} \tag{15}$$

lie in the half plane:

$$|z| \le \left|z - \frac{m}{2}\right|$$

and obtain compact generalizations of some well-known polynomial inequalities. We first prove the following:

Theorem 1: If P(z) is a polynomial of degree n, then for every real or complex numbers α, β with $|\alpha| \ge 1$, $|\beta| \le 1$ and $R > r \ge 1$:

$$|B[D_\alpha(P(Rz)\text{-}\boldsymbol{\beta}\ P(rz))]| \le |\alpha|n|R^n\text{-}r^n\boldsymbol{\beta}||B[z^{n-1}]|\ \max_{|z|=1}|P(z)|,\ \text{for}\ \ |z| \ge 1 \tag{16}$$

The result is sharp and equality holds in inequality Eq. 16 for $P(z) = az^n$, $a \neq 0$.

Substituting for $B[D_\alpha(P(Rz)\text{-}\boldsymbol{\beta}P(rz)]$, we have for $|z| = 1$:

$$\begin{aligned}&\left|\lambda_0 D_\alpha(P(Rz) - \beta P(rz)) + \lambda_1\left(\frac{mz}{2}\right)D_\alpha(P(Rz) - \beta P(rz))'\right.\\ &\left.+\lambda_2\left(\frac{mz}{2}\right)^2 \frac{D_\alpha(P(Rz) - \beta P(rz))''}{2!}\right|\\ &\le |\alpha||n|\left|R^n - r^n_\beta\right|\left|\lambda_0 z^{n-1} + \lambda_1\left(\frac{(n-1)z}{2}\right)(n-1)z^{n-2}\right.\\ &\left.+\lambda_2\left(\frac{(n-1)z}{2}\right)^2 \frac{(n-1)(n-2)z^{n-3}}{2!}\right| \max_{|z|=1}|P(z)|\end{aligned} \tag{17}$$

where, $0 \le m \le n\text{-}1$ and λ_0, λ_1 and λ_2 are such that all the zeros of u(z) defined by inequality Eq. 15 lie in the half plane:

$$R_e\, z \le \frac{m}{4}$$

If we choose $\boldsymbol{\beta} = 0$ and let $R \rightarrow 1$ in inequality Eq. 16 we get the following result.

Corollary 1: If P(z) is a polynomial of degree n, then for every real or complex number α with $|\alpha| \ge 1$:

$$|B[D_\alpha P(z)]| \le |\alpha|n|B[z^{n-1}]|\ \max_{|z|=1}|P(z)|,\ \text{for}\ \ |z| \ge 1$$

The result is sharp and equality holds for the polynomial $P(z) = az^n$, $a \neq 0$.

Remark 1: If we choose $\lambda_1 = 0 = \lambda_2$ with $\boldsymbol{\beta} = 0$ and letting $R \rightarrow 1$ inequality Eq. 17 will reduce to:

$$|D_\alpha P(z)| \le |\alpha| n |z^{n-1}| \max_{|z|=1} |P(z)|, \text{ for } |z| \ge 1 \quad (18)$$

Dividing both side of inequality Eq. 18 by $|\alpha|$ and letting $|\alpha| \to \infty$, inequality Eq. 18 will reduce to inequality Eq. 1.

Choosing $\lambda_0 = 0 = \lambda_2$, inequality Eq. 17 will give the following result.

Corollary 2: If P(z) is a polynomial of degree n, then for every real or complex numbers α, $\boldsymbol{\beta}$ with $|\alpha| \ge 1$, $|\boldsymbol{\beta}| \le 1$ and $R > r \ge 1$:

$$\left|\frac{m}{2} D_\alpha (P(Rz) - \beta P(rz))'\right| \le |\alpha| n |R^n - r^n \beta| \left|\left(\frac{(n-1^2)}{2}\right) z^{n-2}\right| \max_{|z|=1} |P(z) \quad (19)$$

Dividing both side of inequality Eq. 19 by $|\alpha|$ and letting $|\alpha| \to \infty$, then m = n-1 and for $\boldsymbol{\beta} = 0$ and $R \to 1$, inequality Eq. 19 will reduce to:

$$|P''(z)| \le n(n-1) |z^{n-2}| \max_{|z|=1} |P(z)|, \text{ for } |z| \ge 1 \quad (20)$$

The result is best possible and equality holds in inequality Eq. 20 for $P(z) = az^n$.

We now prove the theorem which gives the extension of (Shah and Liman, 2008, lemma (2.3) to the polar derivative.

Theorem 2: If P(z) is a polynomial of degree n, then for every real or complex numbers α, $\boldsymbol{\beta}$ with $|\alpha| \ge 1$, $|\boldsymbol{\beta}| \le 1$ and $R > r \ge 1$:

$$|B[D_\alpha(P(Rz) - \boldsymbol{\beta} P(rz))]| + |B[D_\alpha(Q(Rz) - \boldsymbol{\beta} Q(rz))]| \le n(|\alpha| |R^n - r^n \boldsymbol{\beta}| |B[z^{n-1}]| + |1 - \boldsymbol{\beta}| |\lambda_0|) \max_{|z|=1} |P(z)| \quad (21)$$

for $|z| \ge 1$, where:

$$Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$$

The result is best possible and the equality holds in inequality (Eq. 21) for $P(z) = z^n + 1$.

Substituting for $B[D_\alpha(P(Rz) - \boldsymbol{\beta} P(rz))]$ in inequality (Eq. 21), we have for $|z| \ge 1$:

$$\begin{aligned}
&\left|\lambda_0 D_\alpha(P(Rz) - \beta P(rz)) + \lambda_1 \left(\frac{mz}{2}\right) D_\alpha (P(Rz) - \beta P(rz))'\right. \\
&\left.+ \lambda_2 \left(\frac{mz}{2}\right)^2 \frac{D_\alpha(P(Rz) - \beta P(rz))''}{2!}\right| \\
&+ \left|\lambda_0 D_\alpha(Q(Rz) - \beta Q(rz)) + \lambda_1 \left(\frac{mz}{2}\right) D_\alpha (Q(Rz) - \beta Q(rz))'\right. \\
&\left.+ \lambda_2 \left(\frac{mz}{2}\right)^2 \frac{D_\alpha(Q(Rz) - \beta Q(rz))''}{2!}\right|
\end{aligned}$$

$$\leq n\left\{|\alpha|\left|R^n - r^n\beta\right|\lambda_0 z^{n-1} + \lambda_1\left(\frac{(n-1)z}{2}\right)(n-1)z^{n-2} + \lambda_2\left(\frac{(n-1)z}{2}\right)^2\frac{(n-1)(n-2)z^{n-3}}{2!}\right| + |1-\beta||\lambda_0|\right\}\max_{|z|=1}|p(z)| \tag{22}$$

where, $0 \leq m \leq n-1$ and λ_0, λ_1 and λ_2 are such that all the zeros of u(z) defined by inequality Eq. 15 lie in the half plane:

$$\text{Re } z \leq \frac{m}{4}$$

If, $\beta = 0$ was chosen and let $R \to 1$ in inequality (Eq. 21) we get the following extension of (Shah and Liman, 2008) (Lemma (2.3)) to polar derivatives.

Corollary 3: If P(z) is a polynomial of degree n, then for every real or complex numbers α with $|\alpha| \geq 1$ and for $|z| \geq 1$:

$$|B[D_\alpha P(z)]| + |B[D_\alpha Q(z)] \leq n(|\alpha|B[z^{n-1}]| + |\lambda_0|) \max_{|z|=1}|P(z)|$$

Which implies:

$$\left|B[nP(z)+(\alpha-z)P'(z)]\right| + \left|B[nQ(z)+(\alpha-z)Q'(z)]\right| \leq n(\left|B[\alpha z^{n-1} + |\lambda_0|\right| \max_{|z|=1}|P(z)|$$

Taking $\alpha = z$ in the above inequality, we get (Shah and Liman, 2008, Lemma (2.3)) that is:

$$|B[P(z)]| + |B[Q(z)] \leq (|B[z^n]| + |\lambda_0| \max_{|z|=1}|P(z)|, \text{ for } |z| \leq 1$$

If we take $\lambda_1 = 0 = \lambda_2$ with $\beta = 0$ and letting $R \to 1$ in inequality Eq. 22, we get the following result:

Corollary 4: If P(z) is a polynomial of degree n, then for every real or complex number α with $|\alpha| \geq 1$:

$$|D_\alpha P(z)| + |D_\alpha Q(z)| \leq n\{|\alpha||z^{n-1}| + 1\}\max_{|z|=1}|P(z)|, \text{ for } |z| \geq 1 \tag{23}$$

Dividing both sides by $|\alpha|$ and letting $|\alpha| \to \infty$, inequality (Eq. 23) will reduce to:

$$|P'(z)| + |Q'(z)| \leq n|z^{n-1}|\max_{|z|=1}|P(z)|, \text{ for } |z| \geq 1 \tag{24}$$

The result is best possible and equality holds in inequality (Eq. 24) for $P(z) = z^n+1$.

The above result is a special case of the result due to Govil and Rahman (1969), Inequality (3.2).

If we take $\lambda_0 = 0 = \lambda_2$ with $\beta = 0$ and letting $R \to 1$ in inequality (Eq. 22), we get the following result:

Corollary 5: If P(z) is a polynomial of degree n, then for every real or complex number α with $|\alpha|\geq 1$:

$$m\{|D_\alpha P'(z)|+|D_\alpha Q'(z)|\}\leq n|\alpha|(n-1)^2|z^{n-2}|\max_{|z|=1}|P(z), \text{ for } |z|\geq 1 \tag{25}$$

Dividing both sides by $|\alpha|$ and letting $|\alpha|\to\infty$, then m = n-1, inequality (Eq. 25) will reduce to:

$$|P''(z)|+|Q''(z)|\leq n(n-1)|z^{n-2}|\max_{|z|=1}|P(z)|, \text{ for } |z|\geq 1 \tag{26}$$

The result is best possible and equality holds in inequality (Eq. 26) for $P(z) = z^n+1$.

Next, we prove a result for the class of polynomials not vanishing in a unit disc and obtain compact generalization of inequality Eq. 7. In fact we prove:

Theorem 3: If P(z) is a polynomial of degree n which does not vanish in $|z|<1$, then for every real or complex numbers α, β with $|\alpha|\geq 1$, $|\beta|\leq 1$ and $R>r\geq 1$:

$$|B\left[D_\alpha(P(Rz)-\beta P(rz))\right]\leq\frac{n}{2}(|\alpha||R^n-r^n\beta||B[z^{n-1}]|+|1-\beta||\lambda_0|)\max_{|z|=1}|P(z) \tag{27}$$

for $|z|\geq 1$.

The result is best possible and equality in inequality (Eq. 27) holds for $P(z) = z^n+1$.

Substituting for $B[D_\alpha(P(Rz)-\beta P(rz))]$ in inequality (Eq. 27), we have for $|z|\geq 1$:

$$\begin{aligned}&\left|\lambda_0 D_\alpha(P(Rz)-\beta P(rz))+\lambda_1\left(\frac{mz}{2}\right)D_\alpha(P(Rz)-\beta P(rz))'\right.\\&\left.+\lambda_2\left(\frac{mz}{2}\right)^2\frac{D_\alpha(P(Rz)-\beta P(rz))''}{2!}\right|\\&\leq\frac{n}{2}\left\{|\alpha||R^n-r^n\beta|\left|\lambda_0 z^{n-1}+\lambda_1\left(\frac{(n-1)z}{2}\right)(n-1)z^{n-2}\right.\right.\\&\left.\left.+\lambda_2\left(\frac{(n-1)z}{2}\right)^2\frac{(n-1)(n-2)z^{n-3}}{2!}\right|+|1-\beta||\lambda_0|\right\}\max_{|z|=1}|P(z)|\end{aligned} \tag{28}$$

where, $0\leq m\leq n-1$ and λ_0, λ_1 and λ_2 are such that all the zeros of u(z) defined by inequality (Eq. 15) lie in the half plane Re $z\leq m/4$:

Remark 2: If we take $\beta = 0$ and let $R\to 1$, inequality (Eq. 27) will reduce to the following result due to Bidkham and Mezerji (2011).

If P(z) is a polynomial of degree at most n, having no zero in $|z|\pm 1$, then for every α with $|\alpha|\geq 1$:

$$\left|B\left[D_\alpha P(z)\right]\right|\leq\frac{n}{2}(|\alpha|\left|B[z^{n-1}]\right|+|\lambda_0|)\max_{|z|=1}|P(z)|, \text{ for } |z|\geq 1$$

Remark 3: If we take $\lambda_1 = 0 = \lambda_2$ with $\beta = 0$ and letting $R\to 1$, inequality (Eq. 28) reduces to inequality (7) that is:

$$\left|B[D_\alpha P(z)]\right| \le \frac{n}{2}(\left|\alpha z^{n-1}\right|+1)\ \max_{|z|=1}\left|P(z)\right|,\ \text{for } |z|\ge 1$$

On dividing both sides of above inequality by $|\alpha|$ and letting $|\alpha|\to\infty$ we get inequality (3). Choosing $\lambda_1 = 0 = \lambda_2$ with $\beta = 0$ and letting $R\to 1$ in inequality (Eq. 28), we get the following result:

Corollary 6: If P(z) is a polynomial of degree n which does not vanish in $|z|<1$, then for every real or complex number α with $|\alpha|\ge 1$:

$$|mD_\alpha P'(z)| \le \frac{n(n-1)^2}{2}|\alpha||z^{n-2}|\max_{|z|=1}|P(z)|,\ \text{for } |z|\ge 1 \tag{29}$$

Dividing both sides of inequality Eq. 29 by $|\alpha|$ and letting $|\alpha|\to\infty$ then m = n-1 and we have:

$$|P''(z)| \le \frac{n(n-1)}{2}|Z^{n-2}|\max_{|z|=1}|P(z)|,\quad \text{for } |z|\ge 1 \tag{30}$$

The result is best possible and equality in inequality Eq. 30 holds for $P(z) = z^n+1$.

We now prove the following interesting result, which provides the compact generalization of inequality Eq. 13.

Theorem 4: If P(z) is a polynomial of degree n having all its zeros in $|z|\le 1$, then for every real or complex numbers α, β with $|\alpha|\ge 1$, $|\beta|\le 1$ and $R>r\ge 1$:

$$|B[D_\alpha(P(Rz)-\beta P(rz))]\ge|\alpha|n|R^n - r^n\beta||B[z^{n-1}]|\min_{|z|=1}|P(z),\ \text{for } |z|\ge 1 \tag{31}$$

The result is sharp and equality holds in inequality Eq. 31 for $P(z) = az^n$. Substituting for $B[D_\alpha(P(Rz)-\beta P(rz))]$, we have for $|z|\ge 1$:

$$\begin{aligned}&\Bigg|\lambda_0 D_\alpha(P(Rz)-\beta P(rz)) + \lambda_1\left(\frac{mz}{2}\right)D_\alpha(P(Rz)-\beta P(rz))' \\ &\quad + \lambda_2\left(\frac{mz}{2}\right)^2\frac{D_\alpha(P(Rz)-\beta P(rz))''}{2!}\Bigg| \\ &\ge|\alpha|n|R^n-r^n\beta|\Bigg|\lambda_0 z^{n-1}+\lambda_1\left(\frac{(n-1)z}{2}\right)(n-1)z^{n-2} \\ &\quad +\lambda_2\left(\frac{(n-1)z}{2}\right)^2\frac{(n-1)(n-2)z^{n-3}}{2!}\Bigg|\min_{|z|=1}|P(z)|\end{aligned} \tag{32}$$

where, $0\le m\le n-1$ and λ_0, λ_1 and λ_2 are such that all the zeros of u(z) defined by inequality Eq. 15 lie in the half plane:

$$\operatorname{Re} z \le \frac{m}{4}$$

Remark 4: If we take $\beta = 0$ and let $R \to 1$, inequality Eq. 31 will reduce to inequality Eq. 13.

If we take $\lambda_0 = 0 = \lambda_2$ with $\beta = 0$ and letting $R \to 1$ in inequality Eq. 32, we will get the following result from which result of Aziz and Dawood (1988) follows as a special case.

Corollary 7: If P(z) is a polynomial of degree at most n having all its zeros in $|z| = 1$, then for every real or complex number α with $|\alpha| \geq 1$:

$$|D_\alpha P(z)| \geq n|\alpha||z^{n-1}| \min_{|z|=1} |P(z)|, \quad \text{for} \quad |z| \geq 1 \tag{33}$$

The result is best possible and equality holds in inequality Eq. 33 for $P(z) = az^n$.

Dividing the inequality Eq. 33 both sides by $|\alpha|$ and letting $|\alpha| \to \infty$ then $m = n-1$, we obtain the inequality Eq. 5 as a special case.

Choosing $\lambda_0 = 0 = \lambda_2$ with $\beta = 0$ and letting $R \to 1$ in inequality Eq. 32, we will get the following result.

Corollary 8: If P(z) is a polynomial of degree at most n having all its zeros in $|z| \leq 1$, then for every real or complex number α with $|\alpha| \geq 1$:

$$|mD_\alpha P'(z)| \geq n(n-1)^2 |\alpha||z^{n-2}| \min_{|z|=1} |P(z)| \tag{34}$$

Dividing both sides of the inequality Eq. 34 by $|\alpha|$ and letting $|\alpha| \to \infty$, then $m = n-1$ we obtain:

$$|P''(z)| \geq n(n-1)|z^{n-2}| \min_{|z|=1} |P(z)| \tag{35}$$

The result is best possible and the equality holds in inequality Eq. 35 for $P(z) = az^n$.

As an improvement of inequality Eq. 31 and generalization of inequality Eq. 10, we prove the following result.

Theorem 5: If P(z) is a polynomial of degree at most n which does not vanish in $|z| < 1$, then for every real or complex numbers α, β with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $R > r \geq 1$:

$$\begin{aligned} &|B[D_\alpha(P(Rz) - \beta P(rz))]| \\ &\leq \frac{n}{2}\Big[\left\{|\alpha||R^n - r^n\beta||B[z^{n-1}]| + |1-\beta||\lambda_0|\right\} \max_{|z|=1} |P(z)| \\ &- \left\{|\alpha|n|R^n - r^n\beta||B[z^{n-1}]| - |1-\beta||\lambda_0|\right\} \min_{|z|=1} |P(z)|\Big] \end{aligned} \tag{36}$$

The result is sharp and equality in inequality Eq. 36 holds for the polynomial having all the zeros on the unit disk.

Substituting for $B[D_\alpha(P(Rz) - \beta P(rz))]$ in inequality Eq. 36, we have for $|z| \geq 1$:

$$\begin{aligned}&\left|\lambda_0 D_\alpha(P(rz)-\beta P(rz))+\lambda_1\left(\frac{mz}{2}\right)D_\alpha(P(rz)-\beta P(rz))'\right.\\&\left.+\lambda_2\left(\frac{mz}{2}\right)^2\frac{D_\alpha(P(Rz)-\beta P(rz))''}{2!}\right|\\&\leq\frac{n}{2}\left\{\left[|\alpha||R^n-r^n\beta|\left|\lambda_0 z^{n-1}+\lambda_1\left(\frac{(n-1)z}{2}\right)(n-1)z^{n-2}\right.\right.\right.\\&\left.+\lambda_2\left(\frac{(n-1)z}{2}\right)^2\frac{(n-1)(n-2)z^{n-3}}{2!}\right|+|1-\beta||\lambda_0|\Big]\max_{|z|=1}|P(z)|\\&-\left[|\alpha||R^n-r^n\beta|\left|\lambda_0 z^{n-1}+\lambda_1\left(\frac{(n-1)z}{2}\right)(n-1)z^{n-2}\right.\right.\\&\left.\left.\left.+\lambda_2\left(\frac{(n-1)z}{2}\right)^2\frac{(n-1)(n-2)z^{n-3}}{2!}\right|+|1-\beta||\lambda_0|\right]\min_{|z|=1}|P(z)|\right\}\end{aligned}\tag{37}$$

where $0\leq m\leq n-1$ and λ_0, λ_1 and λ_2 are such that all the zeros of u(z) defined by inequality Eq. 15 lie in the half plane:

$$z\leq\frac{m}{4}$$

Remark 5: If we take $\beta = 0$ and letting $R\to 1$, inequality Eq. 36 will reduce to inequality Eq. 14.

Remark 6: If we choose $\lambda_1 = 0 = \lambda_2$ with $\beta = 0$ and let $R\to 1$, inequality Eq. 37 will reduce to the following result due to Aziz and Shah (1998).

If P(z) is a polynomial of degree n which does not vanish in $|z|<1$, then for every complex number α with $|\alpha|\geq 1$:

$$\max_{|z|=1}\left|D_\alpha P(z)\right|\leq\frac{n}{2}\left\{\{|\alpha|+1\}\max_{|z|=1}|P(z)|-\{|\alpha|-1\}\max_{|z|=1}|P(z)|\right\}$$

Dividing both sides by $|\alpha|$ and letting $|\alpha|\to\infty$, in the above inequality, it follows that if $P(z)\neq 0$ in $|z|<1$, then:

$$\max_{|z|=1}|P'(z)|\leq\frac{n}{2}\left\{\max_{|z|=1}|P(z)|-\max_{|z|=1}|P(z)|\right\}$$

The above result is an interesting refinement of Erdö s-Lax Theorem inequality Eq. 3 and was proved by Aziz and Dawood (1988).

If we take $\lambda_0 = 0 = \lambda_2$ with $\beta = 0$ and let $R\to 1$ in inequality Eq. 37, we get the following result:

Corollary 9: If P(z) is a polynomial of degree at most n, having no zero in $|z|\leq 1$, then for every α with $|\alpha|\geq 1$ and $|z|\geq 1$:

$$|mD_\alpha P(z)|\leq\frac{n(n-1)^2}{2}|\alpha||z^{n-2}|\{\max_{|z|=1}|P(z)|-\min_{|z|=1}|P(z)|\}$$

The result is best possible and equality holds in inequality Eq. 38 for $P(z) = z^n+1$.

Dividing both sides of the inequality Eq. 38 by $|\alpha|$ and letting $|\alpha| \to \infty$, then $m = n-1$ and we get:

$$|P''(z)| \le \frac{n(n-1)}{2}|\alpha||z^{n-2}|\left\{\max_{|z|=1}|P(z)| - \max_{|z|=1}|P(z)|\right\} \tag{38}$$

LEMMAS

For the proof of above theorems we need the following lemmas. The first lemma follows from Laguerre (1989).

Lemma 1: If all the zeros of polynomial $P(z)$ of degree n lie in $|z|\le k$, where $k\le 1$, then for $|\alpha|\ge k$, the polar derivative $D_\alpha[P(z)]$ of $P(z)$ at the point α also has all its zeros in $|z|\le k$.

The following lemma which we need is in fact implicit in (Rahman and Schmeisser, 2002, Lemma 14.5.7, p.540).

Lemma 2: If all the zeros of the polynomial $P(z)$ of degree n lie in a circle $|z|\le 1$, then all the zeros of the polynomial $B[P(z)]$ also lie in $|z|\le 1$.

As an application of lemmas 2 and 3 we have the following lemma.

Lemma 3: If all the zeros of polynomial $P(z)$ of degree n lie in $|z|\le 1$, then for $|\alpha|\ge 1$, all the zeros of the polynomial $B[D_\alpha P(z)]$ also lie in $|z|\le 1$.

Proof: From lemma 1 for $k = 1$ all the zeros of the polynomial $D_\alpha P(z)$ lie in $|z|\le 1$ and so from lemma 2 the polynomial $B[D_\alpha P(z)]$ has all its zeros in $|z|\le 1$.

The next lemma is due to Aziz and Rather (1998).

Lemma 4: If $P(z)$ is a polynomial of degree at most n having all its zeros in $|z|<k$ where $k\le 1$, then $|P(Rz)|>|P(z)|$, for $|z|\ge 1$ and $R>1$.

Lemma 5: If $P(z)$ is a polynomial of degree n which does not vanish in $|z|<1$, then for every real or complex numbers α, β with $|\alpha|\ge 1, |\beta|\le 1$ and $R\ge r\ge 1$:

$$|B[D_\alpha(P(Rz)-\beta P(rz))]| \le |B[D_\alpha(Q(Rz)-\beta Q(rz))]| \tag{39}$$

for $|z| = 1$, where:

$$Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$$

Proof: For $R = r = 1$, the result reduces to Bidkham and Mezerji (2011) [Lemma 4, p.597]. Now we will prove the result for $R>r\ge 1$. Since all the zeros of $P(z)$ lie in $|z|\ge 1$ and for every real or complex number λ with $|\lambda|>1$, the polynomial $G(z) = P(z)-\lambda Q(z)$ where:

$$Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$$

has all its zeros in $|z|\le 1$. Applying lemma 4 to the polynomial G(z) with k = 1, we get:

$$|G(rz)|<|G(Rz)|, \text{ for } |z| = 1 \text{ and } R>r\ge 1$$

Since all the zeros of G(z) lie in:

$$|z|\le \frac{1}{R}<1$$

therefore for any real or complex number $\boldsymbol{\beta}$ with $|\boldsymbol{\beta}|\le 1$, the polynomial $H(z) = G(Rz)\text{-}\boldsymbol{\beta}G(rz)$, has all its zeros in $|z|<1$, for every λ with $|\lambda|>1$ and $R>r\ge 1$, by lemma 3 all the zeros of $B[D_\alpha H(z)]$ lie in $|z|<1$. This implies:

$$B[D_\alpha(G(Rz)\text{-}\boldsymbol{\beta}G(rz))] = B[D_\alpha(P(Rz)\text{-}\boldsymbol{\beta}P(rz))]\text{-}\lambda B[D_\alpha(Q(Rz)\text{-}\boldsymbol{\beta}Q(rz))], \text{ for } |z|\ge 1 \text{ and } R>r\ge 1 \quad (40)$$

Inequality Eq. 40 implies:

$$B[D_\alpha(G(Rz)\text{-}\boldsymbol{\beta}P(rz))]|\text{-}\boldsymbol{\beta}P(rz))]|\le|B[D_\alpha(Q(Rz)\text{-}\boldsymbol{\beta}Q(rz))] \quad (41)$$

for $|z| = 1$ and $R > r\ge 1$.

For if it is not true, then there is a point $z = z_o$ with $|z_o|\ge 1$, such that:

$$B[D_\alpha(P(Rz_o)\text{-}\boldsymbol{\beta}P(rz_o))]|\ge|B[D_\alpha(Q(Rz_o)\text{-}\boldsymbol{\beta}Q(rz_o) \quad (42)$$

for $|z_0| \ge 1$ and $R>r\ge 1$.

Since all the zeros of Q(z) lie in $|z|\le 1$, therefore it follows that all the zeros of $Q(Rz)\text{-}\boldsymbol{\beta}Q(rz)$, lie in $|z|\le 1$ for every $\boldsymbol{\beta}$ with $|\boldsymbol{\beta}|\le 1$. Hence $Q(Rz_0)\text{-}\boldsymbol{\beta}Q(rz_0)\ne 0$, for $|z_o|\ge 1$. Which implies:

$$B[D_\alpha(Q(Rz_o)\text{-}\boldsymbol{\beta}Q(rz_o))]\ne 0, \text{ for } |z_o|\ge 1 \text{ and } R>r\ge 1$$

We take:

$$\lambda = \frac{B[D_\alpha(P(Rz_o)-\beta P(rz_o))]}{B[D_\alpha(Q(Rz_o)-\beta Q(rz_o))]}$$

so that $|\lambda|>1$.

Which shows that $B[D_\alpha H(z)]$ has a zero in $|z|\ge 1$. Which is contradiction to the fact that all the zeros of $B[D_\alpha H(z)]$ lie in $|z|<1$. Thus:

$$B[D_\alpha(P(Rz)\text{-}\boldsymbol{\beta}P(rz))]|\le|B[D_\alpha(Q(Rz)\text{-}\boldsymbol{\beta}Q(rz))]|$$

for $|z|\ge 1$ and $R\ge r\ge 1$.

Proof of theorems

Proof of Theorem 1: Let $M = \max_{|z|=1}|P(z)|$ then $|P(z)|\le M$, for $|z| = 1$. Therefore, by Rouche's Theorem we have all the zeros of the polynomial $G(z) = P(z)+\lambda z^n M$, lie in $|z|<1$ for every λ with $|\lambda|>1$. Now from lemma 4, we have:

$$|G(rz)|<|G(Rz)|, \text{ for } |z| = 1 \text{ and } R>r\geq 1$$

Since all the zeros of G(Rz) lie in:

$$|z|<\frac{1}{R}<1$$

therefore if β is any real or complex number with $|\beta|\leq 1$, we have all the zeros of the polynomial:

$$G(Rz)-\beta G(rz) = (P(Rz)-\beta P(rz)) + \lambda(R^n-r^n\beta)z^nM$$

also lie in $|z|<1$ for every $R>r\geq 1$ and $|\lambda|>1$.

Therefore by lemma 3, all the zeros of $B[D_\alpha(G(Rz)-\beta G(rz))]$, lie in $|z|<1$ for every $R>r\geq 1$ and $|\lambda|>1$. Which implies:

$$B[D_\alpha(G(Rz)-\beta G(rz))] = B[D_\alpha(P(Rz)-\beta P(rz)] +\lambda_\alpha n(R^n-r^n\beta)MB[z^{n-1}] \quad (43)$$

for $|z|<|$ and $R>r\geq 1$, inequality (43) implies

$$|B[D\alpha(P(Rz)-\beta(P(rz))]|\leq|\alpha|n|R^n-r^n\beta||B[z^{n-1}]M| \quad (44)$$

for $|z|\geq 1$ and $R>r\geq 1$, if this is not true, then there is a point $z = z_o$ with $|z_o|\geq 1$, such that:

$$|B[D_\alpha(P(Rz_o)-\beta P(rz_o))]|>|\alpha|n|R^n-r^n\beta||B[z_o^{n-1}]|M$$

We take:

$$\lambda = \frac{B[D_\alpha(P(Rz_o)-\beta P(rz_o))]}{\alpha n(R^n-r^n\beta)B[z_o^{n-1}]}$$

So that $|\lambda|>1$, for this choice of λ we have $B[D_\alpha(G(Rz_o)-\beta G(rz_o))] = 0$, for $|z_o|\geq 1|$. Which is a contradiction to the fact that all the zeros of $B[D_\alpha(G(Rz)-\beta G(rz))]$ lie in $|z|<1$. Thus:

$$|B[D_\alpha(P(Rz)- P(rz))]| \leq |\alpha| n|R^n -r^n\beta||B[z^{n-1}]|\max_{|z|=1}|P(z)|$$

for $|z|\geq 1$ and $R>r\geq 1$.

Proof of theorem 2: Let $M = \max_{|z|=1}|P(z)|$, then $|P(z)|\leq M$ for $|z| = 1$. Now for every real or complex number γ with $|\gamma|>1$, it follows from Rouche's Theorem, the polynomial $G(z) = P(z)+\gamma M$ does not vanish in $|z|<1$. Now applying lemma 4 and 5 to the polynomial G(z) we have for every real or complex number β with $|\beta|\leq 1$:

$$\begin{aligned}&|B[D_\alpha(P(Rz)- \beta P(rz))]+\gamma(1-\beta)M]\\ &\quad\leq|B[D_\alpha(Q(Rz)-\beta Q(rz))+\bar{\gamma}(R^n-r^n\beta)z^nM]\end{aligned} \quad (45)$$

for $|z|\geq 1$ and $R>r\geq 1$, where:

$$Q(z)=z^n\overline{P\left(\frac{1}{\overline{z}}\right)}$$

Inequality Eq. 45 implies:

$$\begin{aligned}&\left|B[D_\alpha(P(Rz)-\beta P(rz))]+n\gamma(1-\beta)M\lambda_o\right|\\&\quad\leq\left|B[D_\alpha(Q(Rz)-\beta Q(rz))+\alpha n\overline{\gamma}(R^n-r^n\beta)B[z^{n-1}]M\right|\end{aligned}\tag{46}$$

for $|z|\geq 1$ and $R>r\geq 1$.

Now choosing the argument of $\overline{\gamma}$ on the R.H.S of inequality (Eq. 46), such that:

$$\begin{aligned}&|B[D_\alpha(Q(Rz)-\beta Q(rz))]+\alpha n\overline{\gamma}(R^n-r^n\beta)B[z^{n-1}]M|\\&\quad=|\alpha|n|\gamma||R^n-r^n\beta||B[z^{n-1}]|M-|B[D_\alpha(Q(Rz)-\beta Q(rz))]|\end{aligned}\tag{47}$$

for $|z|\geq 1$ and $R>r\geq 1$.

Therefore, we get from inequality Eq. 46:

$$\begin{aligned}&|B[D_\alpha(P(Rz)-\beta P(rz)]|-|n\overline{\gamma}(1-\beta)\lambda_0 M|\\&\quad\leq|\alpha|n|\gamma||R^n-r^n\beta||B[z^{n-1}]|M-|B[D_\alpha(Q(Rz)-\beta Q(rz))]|\end{aligned}\tag{48}$$

for $|z|\geq 1$ and $R>r\geq 1$.

Therefore, inequality Eq. 48 implies:

$$\begin{aligned}&|B[D\alpha(P(Rz)-\beta P(rz))]|+|B\left[D_\alpha(Q(Rz)-\beta Q(rz))\right]|\\&\quad\leq|\alpha|n|\gamma||R^n-r^n\beta||B[z^{n-1}]|M+n|\gamma||1-\beta||\lambda_0|M\end{aligned}\tag{49}$$

for $|z|\geq 1$ and $R>r\geq 1$.

Letting $|\gamma|\to 1$, in inequality Eq. 49, we get:

$$\begin{aligned}&|B[D_\alpha(P(Rz)-\beta P(rz))]|+|B[D_\alpha(Q(Rz)-\beta Q(rz))]|\\&\quad\leq n(|\alpha||R^n-r^n\beta||B[z^{n-1}]|+|1-\beta||\lambda_0|)\max_{|z|=1}|P(z)|\end{aligned}$$

for $|z|\geq 1$ and $R>r\geq 1$. Which proves the theorem.

Proof of Theorem 3: We have from lemma 5:

$$|B[D_\alpha(P(Rz)-\beta P(rz))]|\leq|B[D_\alpha(Q(Rz)-\beta Q(rz))]|$$

for $|z|\geq 1$ and $R\geq r\geq 1$, where:

$$Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$$

Also from Theorem 2, we have:

$$|B[D_\alpha(P(Rz) - \beta P(rz))]| + |B[D_\alpha(Q(Rz) - \beta Q(rz))]|$$
$$\leq n(|\alpha||R^n - r^n\beta||B[z^{n-1}]| + |1-\beta||\lambda_0|)\max_{|z|=1}|P(z)|$$

for $|z| \geq 1$ and $R > r \geq 1$, where:

$$Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$$

Combining the above two inequalities, we get:

$$B[D_\alpha(P(Rz) - \beta P(rz))]| \leq \frac{n}{2}\{|\alpha||R^n - r^n\beta||B[z^{n-1}]| + |1-\beta||\lambda_0|\}\max_{|z|=1}|P(z)|$$

for $|z| \geq 1$ and $R > r \geq 1$, Which proves the theorem.

Proof of Theorem 4: If P(z) has a zero on $|z| = 1$, then the result is trivial. So, we suppose that P(z) has all its zeros in $|z| < 1$. If $m = \min_{|z|=1}|P(z)||$, then $m > 0$ and $m \leq |P(z)|$ for $|z| = 1$. Therefore, if γ is any complex number with $|\gamma| < 1$, we have the polynomial $G(z) = P(z) - \gamma m z^n$ of degree n has all its zeros in $|z| < 1$. Now from lemma 4, we have:

$$|G(rz)| < |G(Rz)| \quad \text{for} \quad |z| = 1 \quad \text{and} \quad R > r \geq 1$$

Since all the zeros of G(Rz) lie in:

$$|z| < \frac{1}{R} < 1$$

therefore, for any real or complex number $\boldsymbol{\beta}$ with $|\boldsymbol{\beta}| \leq 1$ and $R > r \geq 1$, it follows from Rouche's Theorem, the polynomial $H(z) = G(Rz) - \boldsymbol{\beta} G(rz)$ has all its zeros in $|z| < 1$. Therefore, from Lemma 3, all the zeros of $B[D_\alpha H(z)]$ lie in $|z| < 1$. This implies:

$$B[D_\alpha(G(Rz) - \beta G(rz))] = B[D_\alpha(P(Rz) - \beta P(rz))] - \alpha n\gamma(R^n - r^n\beta)B[z^{n-1}]m \tag{50}$$

for $|z| \geq 1$ and $R > r \geq 1$.

Inequality Eq. 50 implies for $|z| \geq 1$ and $R > r \geq 1$:

$$|B[D_\alpha(P(Rz) - \beta P(rz))]| \geq |\alpha|n|R^n - r^n\beta||B[z^{n-1}]|m \tag{51}$$

If inequality Eq. 51 is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$, such that:

$$|B[D_\alpha(P(Rz_0) - \beta P(rz_0))]| < |\alpha| n |R^n - r^n\beta| |B[z_0^{n-1}]| m$$

We take:

$$\gamma = \frac{B[D_\alpha(P(Rz_0) - \beta P(rz_0))]}{\alpha n(R^n - r^n\beta)B[z_0^{n-1}]m}$$

so that $|\gamma|<1$.

For this choice of $|\gamma|$, we have $B[D_\alpha H(z)] = 0$, for $|z| \geq 1$. Which is a contradiction to the fact that all the zeros of $B[D_\alpha H(z)]$ lie in $|z|<1$.

Thus, we have:

$$|B[D_\alpha(P(Rz) - \beta P(rz))]| \geq |\alpha| n |R^n - r^n\beta| |B[z^{n-1}]| \left|\min_{|z|=1}\right| P(z)|$$

for $|z| \geq 1$ and $R>r\geq 1$.

Hence, the theorem follows.

Proof of Theorem 5: Since the polynomial P(z) does not vanish in $|z|<1$, therefore if $m = \min_{|z|=1} |P(z)|$, then $m \leq |P(z)|$, for $|z| \leq 1$. Now for any real or complex number λ with $|\lambda| \leq 1$, the polynomial $G(z) = P(z)+\lambda mz^n$ does not vanish in $|z|<1$. For if this is not true, then there is a point $z = z_0$, with $|z_0|<1$, such that $G(z_0) = P(z_0)+\lambda mz_0^n = 0$. Which implies $|P(z_0)| = |\lambda m z_0^n| \leq m$, contradicting the fact that $m \leq |P(z)|$ for $|z| \leq 1$. Thus G(z) has no zero in $|z|<1$ for every λ with $|\lambda| \leq 1$. Applying lemma 5 to the polynomial G(z) we have for $|\beta| \leq 1$ and $R>r\geq 1$:

$$\begin{aligned}&\left|B[D_\alpha(P(Rz) - \beta P(rz) + (R^n - r^n\beta)\lambda mz^n]\right| \\ &\leq |B[D_\alpha(Q(Rz) - \beta Q(rz) + (1-\beta)\bar{\lambda} m]|\end{aligned} \tag{52}$$

for $|z| \geq 1$ and $R>r\geq 1$. Where:

$$Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$$

Inequality Eq. 50 implies:

$$\begin{aligned}&\left|B[D_\alpha(P(Rz) - \beta P(rz))] + \alpha(R^n - r^n\beta)\lambda mnB[z^{n-1}]\right| \\ &\leq |B[D_\alpha(Q(Rz) - \beta Q(rz))] + n\lambda_0(1-\beta)\bar{\lambda} m]|\end{aligned} \tag{53}$$

for $|z| \geq 1$ and $R>r\geq 1$. Choosing λ in inequality Eq. 53 such that:

$$\left|B[D_\alpha(P(Rz)-\beta P(rz))]+\alpha n\lambda(R^n-r^n\beta)B[z^{n-1}]]m\right| \\ \le |B[D_\alpha(P(Rz)-\beta P(rz))]+|\alpha|n|\lambda||R^n-r^n\beta||B[z^{n-1}]|m \tag{54}$$

for $|z|\ge 1$ and $R>r\ge 1$. Inequality Eq. 54 implies:

$$\left|B[D_\alpha(P(Rz)-\beta P(rz))]|+|\alpha|n|\lambda||R^n-r^n\beta||B[z^{n-1}]\right|m \\ \le |B[D_\alpha(Q(Rz)-\beta Q(rz))]]+n|\lambda_0||1-\beta||\lambda|m \tag{55}$$

for $|z|\ge 1$ and $R>r\ge 1$. Inequality (Eq. 55) implies:

$$\left|B[D_\alpha(P(Rz)-\beta P(rz))]\right|\le \left|B[D_\alpha(Q(Rz)-\beta Q(rz))]\right| \\ +n|\lambda_0||1-\beta||\lambda|m-|\alpha|n|\lambda||R^n-r^n\beta||B[z^{n-1}]|m \tag{56}$$

for $|z|\ge 1$ and $R>r\ge 1$.

Letting $|\lambda|\to 1$, we have for $|z|\ge 1$ and $R>r\ge 1$:

$$2|B[D_\alpha(P(Rz)-\beta P(rz))]|\le |B[D_\alpha(P(Rz)-\beta P(rz))]| \\ +|B[D_\alpha(Q(Rz)-\beta Q(rz))]|+n|\lambda_0||1-\beta|m-|\alpha|n||R^n-r^n\beta||B[z^{n-1}]|m \tag{57}$$

for $|z|\ge 1$ and $R>r\ge 1$.

Applying Theorem 2, we get from inequality (Eq. 57):

$$|B[D_\alpha(P(Rz)-\beta P(rz))] \\ \le \frac{n}{2}\Big[\{|\alpha||R^n-r^n\beta||B[z^{n-1}]|+|1-\beta||\lambda_0|\}\max_{|z|=1}|P(z) \\ -\{|\alpha|n|R^n-r^n\beta||B[z^{n-1}]|-|1-\beta||\lambda_0|\}\min_{|z|=1}|P(z)|] \tag{58}$$

for $|z|\le 1$. Hence, the Theorem follows.

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