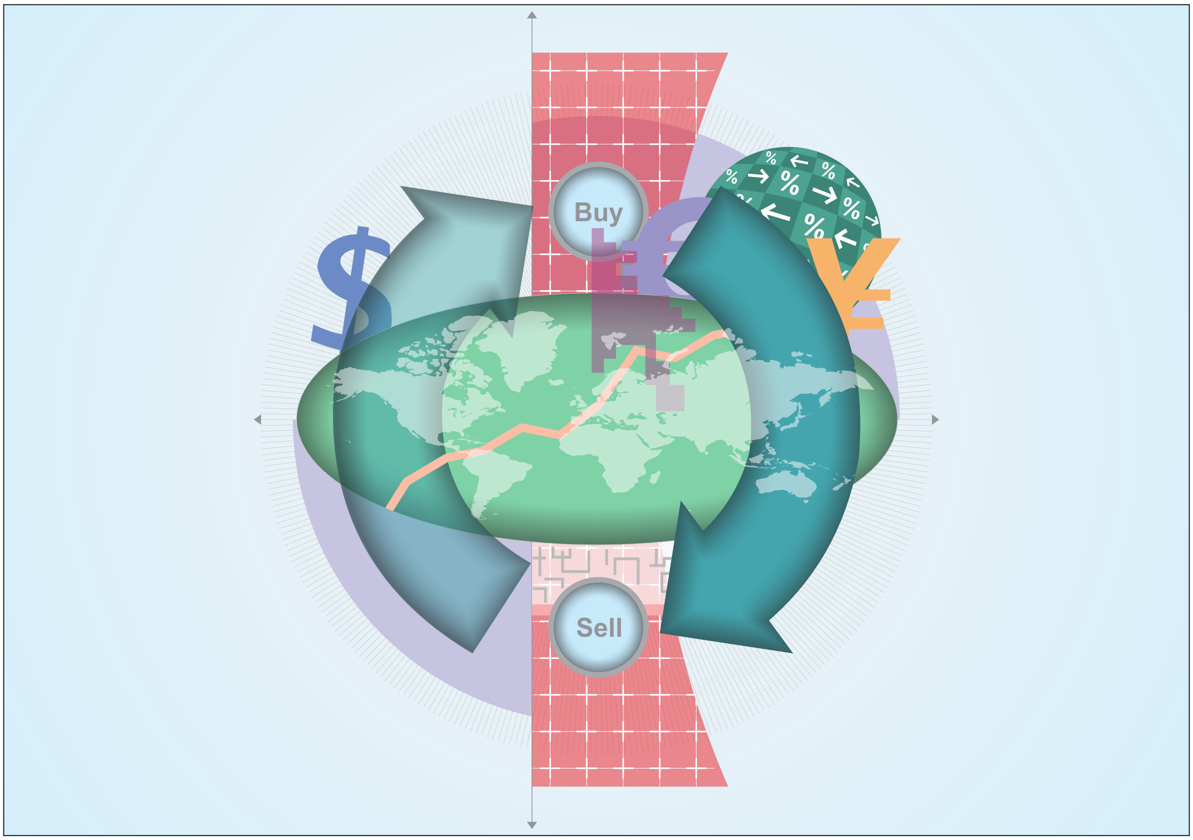

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Forecasting Performance of Constant Elasticity of Variance Model: Empirical Evidence from India

V.K. Singh and N. Ahmad
Department of Mathematics, Jamia Miilia Islamia, Central University, New Delhi-110025, India

Corresponding Author: Vipul Kumar Singh, Department of Mathematics, Jamia Miilia Islamia, Central University, New Delhi-110025, India Tel: +91-01296454518/+91-9212320495

ABSTRACT

This study tested the forecasting performance of Constant Elasticity of Variance (CEV) and benchmark Black-Scholes (BS) option pricing model for pricing S&P CNX Nifty 50 Index options of India. This study adopts a common method of evaluating the performance of an option pricing model that involves calculating the error metrics, Percentage Mean Error (PME) and Mean Absolute Percentage Error (MAPE). For the purpose of this research we used the Non-Linear Least Square (NLLS) loss function to imply option-related parameters while estimating the structural parameters that governs the underlying asset distribution purely from the underlying asset option data and placed options in one of 15 moneyness-maturity groups. The optimal set of parameters is then used to compute the models price. The prices are compared analytically by updating the parameters of two models continuatiously by using cross-sectional option data almost every day. Aim of this study is to first find out parameters of two models analytically then to show that the parameters of the models estimated from option prices can be used to produce reliable predictions of the day-ahead relationship between option prices and index volatility. Constant Elasticity of Variance model, introducing only one more parameter compared with Black-Scholes formula, improves the performance notably in 9 out of 15 PME and 12 out of 15 MAPE moneyness-maturity groups and also increases the stability of implied volatility. Therefore, with much less implementational cost and faster computational speed, the CEV option pricing model can be a better candidate than Black-Scholes model. JEL Classification: C01, C13, C52, C53, G17.

Key words: Call options, Chi-square, maturity, moneyness, MAPE, PME, volatility

INTRODUCTION

Option pricing is one of the most important areas of modern finance. Black and Scholes (1973) developed a model from which a closed-form solution for a European option price can be obtained. Their model assumes that the volatility of returns is constant through time. However, the model has certain well-known deficiencies. For example, when calibrated to accurately price at-the-money options the Black-Scholes model often misprices deep-out-of-the-money and deep-in-the-money options. This model-anomalous behavior gives rise to what options professionals call volatility smiles. A volatility smile is the skewed pattern that results from calculating implied volatilities across a range of strike prices for an option series. This phenomenon is not predicted by the black-scholes model, since volatility is a property of the underlying instrument and the same implied volatility value should be observed across all options on that instrument. Volatility smiles are generally thought to result from the parsimonious assumptions used to derive the BS model. Empirical evidence shows that the constant volatility assumption is inappropriate in real market situations. The constant variance assumption has been tested and rejected in early studies.

The one parameter in the Black-and-Scholes equation that can not be observed directly is the volatility (σ) of the underlying asset. The Black-Scholes model assumes that this volatility to be constant over the life of the option. Volatility is the most sensitive input parameter in pricing terms. The constant volatility assumption of Black-Scholes (henceforth BS) was soon challenged. Estimating the volatility is not a straightforward procedure. There are several approaches to get information about volatility. The use of historical data is one way although volatility measured over long terms might be quite different from the volatility observed during the lifetime of the option. A more commonly used way is to measure implied volatility. This is done by using the BS formula backwards, taking present option prices and calculating the volatility, option traders expect for the future. This value is often referred to as the implied (or implicit) volatility, since it is implied by the option price. Under BS assumption, implied volatilities from options should be the same regardless of which option is used to compute the volatility. However, this is not the case in practice. Implied volatility appears to be dependent on option maturities as well as strike prices (Glauche, 2001).

Therefore, a remedy for this shortcoming of the BS model is needed. The Constant Elasticity of Variance (henceforth CEV) model introduced by Cox and Ross (1976) is an example. Rather than assuming constant volatility, Cox expressed the volatility as a function of the price of the underlying asset. In the study of MacBeth and Merville (1980), they compared the performance of the BS and CEV models through simulations and real examples. Their results show that the CEV model has a better performance, which underprices in-the-money call options and overprices out-of-the-money call options. Although, the CEV model can be a better alternative to the BS model, the estimation of parameters appears to be more difficult. National Stock Exchange (NSE) of India is one of the largest stock exchange of Asia is also using benchmark Black-Scholes model for fixing the base price of Nifty 50 index options. In this study our aim is to do the empirical analysis of CEV and BS model to find the better alternate to minimize the price bias between market and model price.

In this study, we estimate the parameters of the CEV model jointly by the method of least-squares. An empirical study on the S&P CNX Nifty 50 Index option is carried out to compare the performance of the BS and CEV models. This study contributes to the volatility literature by reporting direct estimates of the parameters of a true volatility process obtained as implied parameters from observed option prices. The procedures followed here represent a substantial generalization of the widespread practice of obtaining an implied standard deviation from observed option prices. In order to reduce the empirical biases of BS in 1973 option pricing model, succeeding option pricing models relaxed the restrictive assumptions made by the BS model: the underlying price process (distribution), the constant interest rate and dynamically complete markets. The tradeoff is, however, more computational cost.

DATA DESCRIPTION

This investigation required the collection of historical data for the S&P CNX Nifty 50 index and index option contract. Historical data of index options contract and the Indian risk-free interest rate data which is equal to yield of 91 Day T-Bill collected from respective websites of National Stock exchange (www.nseindia.com) and Reserve Bank of India (www.rbi.org). The data from January

1, 2008 to December 31, 2008 collected for S&P CNX Nifty 50 Index Option. For the option contract: the date, time, contract month, option type, strike price and traded prices were obtained. Data were collected manually. For the purpose of this research we have considered daily data (option strike prices for which number of contracts are non zero for trading days). The final set of data after screening procedure contained 7455 call option.

Data screening procedure: All the call option prices taken from the market are checked whether they satisfy the lower boundary condition:

$$S_t - Ke^{-r(T-t)} \leq C(S_t, t) \tag{1}$$

where, S_t is the current asset price, K is the strike price, r is the risk-free interest rate and $C(S_t, t)$ is the call price at time t. If a call price from the market does not satisfy the lower boundary condition, it is considered as an invalid observation and discarded. Very deep out-of-the-money and very deep in-the- money options are not traded actively on NSE and their price quotes may not reflect the true option value so we have excluded Data of moneyness greater than 15%. Option strike prices having maturity greater than 90 days is less actively traded on NSE so we have excluded options data having maturity period of greater than 90 days. Option strikes with less than five days of maturity are also excluded amid liquidity related biases. The final data set contained 7455 call option.

Option categories: Each market option price that remains after the screening procedure is placed in one of 15 categories depending on their time to expiration and ratio of the asset price to the strike price. Three ranges of time to expiration are distinguished:

- Short maturity (0-30 days or below 1 months)
- Medium maturity (30 to 60 days or between 1 and 2 months)
- Long maturity (60 to 90 days or between 2 to 3 months)

Unlike their price behavior because option prices are very sensitive to their exercise prices and their times to maturity. We divide the option data into several categories according to either moneyness or the time to maturity. A call option is said to be at-the-money (ATM) if the moneyness is ε (-5%, 5%), in-the-money (ITM) if the moneyness ε (5%, 10%), out-of-the-money (OTM) if the moneyness ε (-10%, -5%) and deep in-the-money (DITM) if the moneyness is greater than 10% and deep out-of-money (DOTM) if the moneyness is less than -10%.

Method: For this study we divided the option data into two categories according to moneyness and time to maturity. Volatility is the only unknown parameter in the BS model; the implied volatility can also be used to justify the accuracy of the option-pricing model. We test the out-of sample forecast performance by comparing the one-day ahead forecasting accuracy of the CEV and BS model with Market value. Models parameters is computed analytically by optimization techniques and use as an input to figure out the effectiveness of the CEV model against benchmark black-scholes model for pricing S&P CNX Nifty 50 index option contract with market value. This study adopts a common method of evaluating the performance of an option pricing model that involves calculating the error metrics Percentage Mean Error (PME) and Mean Absolute Percentage Error (MAPE). To see how well a model performs, we look at the relative error generated by the model.

Percentage Mean Error (PME)

$$PME = \frac{1}{K}\sum_{i=1}^{k}(C_i^{Model} - C_i^{Market})/C_i^{Market} \tag{2}$$

A negative relative error means that the model underprices the specific option, whereas a positive relative error means that the model overprices the specific option.

Mean Absolute Pricing Error (MAPE)

$$MAPE = Mean \quad \left[\left|C_i^{Model} - C_i^{Market}\right|\right] \tag{3}$$

where, C_i^{Model} is the predicted price of the option and C_i^{Market} is the actual price for observation i and k is the number of observations.

If the relative error (expressed in percentage) is small, it means the model gives a good approximation to the market. Conversely, if the relative error is big, then the model is considered to be a poor approximation to the market.

OPTION PRICING MODELS

The BS option pricing model: The BS model hardly needs introduction. It is the most popular option pricing model, due to its simplicity, closed-form solution and ease of implementation. The Black and Scholes (1973) price at time t for a European call option with maturity at time t + T with strike price K on a stock paying no dividends is:

$$C_{BS} = SN(d_1) - Ke^{-rt}N(d_2) \tag{4}$$

where:

$$d_1 = \frac{\ln[S/K] + [r + 0.5\sigma^2]t}{\sigma\sqrt{t}}$$

$$d_2 = \frac{\ln[S/K] + [r - 0.5\sigma^2]t}{\sigma\sqrt{t}} = d_1 - \sigma\sqrt{t}$$

where, C denotes the price of a call option, S denotes the underlying Index price, K denotes the option exercise price, t is the time to expiry in years, r is the risk free rate of return, N(d) is the standard normal distribution function and σ^2 is the variance of returns on the Index. In general, the pricing relationships explained for options on stocks also apply to options on Index; including the underlying assumptions concerning log normally distributed prices, perfect and continuous markets and interest rate certainty.

Similarly put option prices can be found with the formula:

$$P = Ke^{-rt} N(-d_2) - S.N(-d_1) \tag{5}$$

where d_1 and d_2 are defined as above.

The Constant-Elasticity-of-Variance (CEV) option pricing model: The CEV model proposed by Black (1975) and Cox and Ross (1976) is complex enough to allow for changing volatility and simple enough to provide a closed form solution for options with only two parameters. The CEV diffusion process also preserves the property of nonnegative values of the state variables as does in the lognormal diffusion process assumed in the BS model (Lee and Chen, 1993). The early research of the CEV model was conducted by MacBeth and Merville (1980) and Emanuel and MacBeth (1982) to test the empirical performance and compared with the BS model.

An important issue in option pricing is to find a stock return distribution that allows returns to stock and its volatility to be correlated with each other. There is considerable empirical evidence that the returns to stocks are heteroscedastic and the volatility of stock returns changes with stock price. Black (1975), Cox (1996) and Cox and Ross (1976) proposed the CEV model. The CEV model assumes the diffusion process for the stock is:

$$dS = \mu dt + \delta S^{\beta/2} dz$$

and the instantaneous variance of the percentage price change or return, σ^2, follows deterministic relationship:

$$\sigma^2(S,t) = \delta^2 S^{(\beta-2)} \tag{6}$$

If $\beta = 2$, prices are lognormally distributed and the variance of returns is constant. This is the same as the well-known BS model. If $\beta<2$, the stock price is inversely related to the volatility. Cox originally restricted $0\le\beta<2$. Emmanuel and MacBeth (1982) extended his analysis to the case $\beta>2$ and discuss its properties. However, Jackwerth and Rubinstein (2001) find that typical values of the β can fit market option prices well for post-crash period only when $\beta<0$ and they called the model with $\beta<0$ unrestricted CEV. In their empirical study, the difference of pricing performance of restricted CEV model ($\beta\ge0$) and BS model is not significant.

The main feature of the CEV model is that it allows the volatility to change with the underlying price. The CEV closed-form pricing formula involving the evaluation of the non-central chi-square distribution function and the analytic approximation method using the standard normal distribution function are only for European options, which can only be exercised at maturity and not for American options, which can be exercised earlier. The rationale behind the CEV model is that the model can explain the empirical bias exhibited by the BS model, such as the volatility smile. The option pricing formula when the underlying process follows the CEV model is derived by Cox and Ross (1976) and the formula is further simplified by Schroder (1989).

Schroder (1989) shows that the CEV option pricing formula can be expressed in terms of the noncentral chi-square distribution functions. In this study, the CEV formula in terms of the noncentral chi-square distribution expressed by Schroder (1989) and Dyrting (2004) is adopted to compute option prices. Schroder (1989) expressed the CEV call option pricing formula in terms of the noncentral chi-square distribution:

When β<2,

$$C = S_t Q(2y; 2 + 2/(2-\beta), 2x) - e^{-r\tau}K(1 - Q(2x; 2 + 2/(2-\beta), 2y)) \tag{7}$$

When β>2,

$$C = S_t Q(2x; 2 + 2/(2-\beta), 2y) - e^{-r\tau}K(1 - Q(2y; 2 + 2/(2-\beta), 2x)) \tag{8}$$

Q(z,v,k) is a complementary noncentral chi-square distribution function with z, v and k being the evaluation point of the integral, degree of freedom and noncentrality, respectively, where:

$$k = \frac{2r}{\delta^2(2-\beta)(e^{r(2-\beta)\tau} - 1)}$$

$$x = kS_t^{2-\beta}e^{r(2-\beta)\tau}$$

$$y = kK^{2-\beta}$$

where, C is the call price; S, the stock price; τ, the time to maturity; r, the risk-free rate of interest; K, the strike price and β and δ, the parameters of the formula.

Although, the CEV formula can be represented more simply in the terms of noncentral chi-square distributions that are easier to interpret, the evaluation of the infinite sum of each noncentral chi-square distribution can be computationally slow when neither z or k are too large. This study uses the approximation derived by Sankaran (1963) to compute the complementary noncentral chi-square distribution Q(2z,2v,2k) when z and k are large as follows:

$$Q(z; v, k) \sim \frac{1 - hp[1 - h + 0.5(2-h)mp] - [z/(v+k)]^h}{h\sqrt{2p(1+mp)}} \tag{9}$$

where:

$$h = 1 - (2/3)(v+k)(v+3k)(v+2k)^{-2}$$

$$p = \frac{v + 2k}{(v+k)^2}$$

$$m = (h-1)(1-3h)$$

When neither z or k are too large (i.e., z<200 and k<200 and no underflow errors occur), the exact CEV formula is used. Otherwise, the approximation CEV formula is used.

STRUCTURAL PARAMETER ESTIMATION PROCEDURE

The BS implied volatilities are extracted using the BS formula. This must be done numerically because the formula cannot be solved for σ in terms of the other parameters. If σ_{iv} denotes the implied volatility, $C_{obs}(K, T)$ denotes the observed call price with strike price K and time to maturity T and $C_{BS}(\sigma, K, T)$ denotes the BS price of the call with same strike price and maturity, then σ_{iv} is the value of volatility in the BS formula such that $C_{obs}(K, T) = C_{BS}(\sigma_{iv}, K, T)$. To find implied volatilities numerically, the objective function f (σ) is defined by the squared loss function:

$$f(\sigma) = \min_{\sigma} \sum_{i=1}^{n} \left[C_{obs,i}(K,T) - C_{BS,i}(\sigma) \right]^2 \tag{10}$$

The deterministic volatility process assumed by CEV has two parameters β and δ that must be estimated. We estimate these parameters using a simultaneous equations method based on minimizing the following sum of squares.

$$\min_{\beta,\delta} \sum_{j=1}^{N} \left[C_{Market,j} - C_{CEV}(\beta,\delta) \right]^2 \tag{11}$$

Above steps results is an estimate of the implied spot variance and the structural parameter values, for date t. Go back to Step 1 until the two steps have been repeated for each day in the sample. These out-of-sample parameter values are then used to calculate theoretical option prices for all option price observations. We then compare these theoretical option prices of every model with their corresponding market-observed prices.

EMPIRICAL RESULTS

In this section, we report the empirical comparison of model performances based on implied volatility stability and out-of-sample forecasting performance in Table 1.

The CEV parameter β is generally less than two for the entire sample, which explains the empirical evidence for the negative relationship between the sample variance of returns and stock price. CEV model under prices short term OTM, ATM, ITM and DITM call options and medium term DOTM call options while overprices short term DOTM options. On the other side, BS model over prices medium and long term OTM and DOTM call options and under prices short term DOTM, OTM, ITM and DITM call options. It also overprices long term call options for all moneyness groups with percentage mean error in between 4-9%. For CEV model degree of pricing bias is least for medium term OTM, ATM, ITM and DITM options. CEV model, introducing only one more parameter compared with BS formula, improves the performance notably in most of the out-of-sample moneyness-maturity price and implied volatility category. The CEV option pricing model performs better than the BS model in 9 out of 15 PME moneyness-maturity groups and 12 out of 15 MAPE moneyness-maturity groups. In terms of model misspecification, the volatility of CEV model is ranging between 26% and 34% in all maturity. For those options with less than 60 days to expiration, the volatility of CEV model is more stable than BS models. For longer-maturity options, the volatility smile of the CEV model is lower to BS model and stable compared to BS with around 6% fluctuation. The prices generated by the CEV model appears to be a very good approximation for ITM and ATM options of short and medium maturities, with relative errors between -5% and +2%.

Earlier empirical investigations of MacBeth and Merville (1979), Beckers (1980), Emmanuel and MacBeth (1982), Lee *et al.* (2004) and Lu and Hsu (2005) has reported results similar to us, stability of implied volatility and out performance of CEV model in comparison to BS model for

Table 1: BS and CEV implied volatility, Percentage Mean Error (PME), Mean Absolute Pricing Error (MAPE) moneyness- maturity bias for call option

			Time to maturity											
			Implied volatility				PME				MAPE			
Moneyness (X=S/K-1)	Model	Statistical value	Short term (T≤30)	Medium term (30<T≤60)	Long term (T>60)	Overall	Short term (T≤30)	Medium term (30<T≤60)	Long term (T>60)	Overall	Short term (T≤30)	Medium term (30<T≤60)	Long term (T>60)	Overall
DOTM	BS	Average	0.39	0.39	0.40	0.39	-0.27	0.10	0.15	-0.10	6.36	13.58	26.26	11.01
(X<-0.10)		SD	0.12	0.12	0.13	0.12	0.53	0.34	0.44	0.50	10.34	17.10	30.51	17.28
	CEV	Average	0.32	0.32	0.33	0.32	-0.47	-0.08	0.04	-0.28	6.19	10.54	20.06	9.19
		SD	0.11	0.11	0.11	0.11	0.39	0.26	0.38	0.41	9.96	12.22	23.33	13.51
		Total	772	491	153	1416	772	491	153	1416	772	491	153	1416
OTM	BS	Average	0.37	0.37	0.36	0.36	-0.07	0.10	0.11	0.02	8.75	15.98	22.51	13.07
(-0.10≤X<-0.05)		SD	0.12	0.12	0.11	0.12	0.42	0.35	0.24	0.39	10.13	19.13	23.06	16.57
	CEV	Average	0.30	0.30	0.29	0.30	-0.21	0.01	0.06	-0.10	8.36	12.64	19.95	11.37
		SD	0.10	0.11	0.10	0.10	0.34	0.30	0.21	0.33	9.25	15.59	20.08	14.02
		Total	938	632	235	1805	938	632	235	1805	938	632	235	1805
ATM	BS	Average	0.35	0.35	0.33	0.35	-0.03	0.00	0.07	-0.01	14.24	16.78	26.45	16.74
(-0.05≤X≤0.05)		SD	0.11	0.12	0.09	0.11	0.17	0.11	0.50	0.23	15.86	21.8	31.74	21.08
	CEV	Average	0.28	0.28	0.26	0.28	-0.05	0.00	0.09	-0.01	13.48	15.72	27.02	16.05
		SD	0.10	0.10	0.08	0.10	0.15	0.10	0.48	0.22	14.97	20.31	30.84	20.12
		Total	1697	1085	434	3214	1697	1085	434	3214	1697	1085	434	3214
ITM	BS	Average	0.36	0.37	0.38	0.36	-0.04	-0.01	0.00	-0.03	17.25	20.76	34.26	19.03
(0.05<X≤0.10)		SD	0.12	0.12	0.10	0.12	0.05	0.07	0.10	0.06	19.65	20.23	26.85	20.35
	CEV	Average	0.29	0.30	0.31	0.30	-0.02	0.02	0.04	0.00	14.74	19.86	31.85	17.09
		SD	0.10	0.11	0.09	0.10	0.06	0.07	0.09	0.06	17.97	20.99	27.05	19.72
		Total	414	233	24	669	414	233	24	669	414	233	24	669
DITM	BS	Average	0.37	0.39	0.42	0.38	-0.03	-0.02	0.04	-0.02	17.75	19.71	43.22	19.07
(X>0.10)		SD	0.13	0.14	0.11	0.13	0.04	0.05	0.12	0.05	21.31	24.37	44.19	23.44
	CEV	Average	0.30	0.32	0.34	0.31	-0.01	0.02	0.08	0.00	13.89	19.99	54.18	17.01
		SD	0.11	0.12	0.10	0.11	0.04	0.05	0.12	0.05	19.17	22.81	39.3	22.11
		Total	222	120	9	351	222	120	9	351	222	120	9	351
Overall	BS	Average	0.36	0.37	0.35	0.36	-0.09	0.04	0.09	-0.02	11.96	16.47	25.71	15.08
		SD	0.12	0.12	0.11	0.12	0.34	0.25	0.42	0.33	15.14	20.4	29.47	19.62
	CEV	Average	0.30	0.30	0.29	0.30	-0.16	-0.01	0.07	-0.08	11.05	14.54	24.23	13.75
		SD	0.10	0.11	0.10	0.10	0.30	0.20	0.39	0.30	14.01	18.33	27.32	17.99
		Total	4043	2561	851	7455	4043	2561	851	7455	4043	2561	851	7455

pricing options on different indexes. Therefore, with much less implementational cost and faster computational speed, the CEV option pricing model can be a better candidate than BS option pricing model for pricing S&P CNX Nifty 50 index options.

CONCLUSION

This study uses the S and P CNX Nifty 50 index options to find the out-of-sample forecasting performance of CEV and BS option pricing formula. We find that CEV option pricing formula provides a significant improvement BS constant volatility option pricing formula. The CEV option pricing model performs better than the BS model in most of the moneyness-maturity groups. The empirical evidence shows that for out-of-sample performance, the mean absolute errors and percentage errors of the CEV model performs better than the BS model in medium and long term

OTM cases. In addition, the CEV model is even better than the BS model in a few cases in these categories. In terms of model misspecification, by using implied volatility graph introduced by Rubinstein (1985), the volatility of CEV model is ranging between 33 and 26% in all maturity groups. For those options with less than 60 days to expiration, the volatility of CEV model is more stable than all the other models. For longer-maturity options, the volatility smile of the CEV model is similar to BS model with around 2% fluctuation. In summary, the CEV model, introducing only one more parameter compared with BS formula, improves the performance notably in all the tests of out-of-sample and the stability of implied volatility. Furthermore, with a much simpler model, the CEV model can still performs better than the BS model in short term and OTM categories. Therefore, with much less implementational cost and faster computational speed, the CEV option pricing model can be a better candidate than benchmark BS model. Further empirical work is required to prove that CEV model works better than much more complex stochastic option pricing models or not. National Stock Exchange of India can switch from Black-Scholes Models to Constant Elasticity of Variance Model for fixing the base price of index/stock options traded on NSE more efficiently. Also traders can use CEV model to predict the next day options more accurately compared to BS model for trading purpose which could help them in maximizing their returns on Index options.

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