A Construction Method for the Process Expression of Petri Net Based on Decomposition

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Abstract: Process expression is one of the most usefil tools to describe the process semantics of a Petri net. However, it is usually not easy to obtain the process expression of a Petri net directly. In this research, a construction method is proposed to obtain the process expressions of all kinds of Petri nets based on decomposition. With this decomposition method, a Petri net is decomposed into a set of S-Nets. The processes properties of the decomposition nets are easy to analyze since they are well-formed and their process expressions are easier to obtain. With the process expressions of the decomposition nets, an algorithm to obtain the process expression of the original Petri net is presented, which is expressed by the synchronization shuffle operation between processes.

Key words: Petri net, S-Net, process, process expression, decomposition, synchronization shuffle

INTRODUCTION

To use Petri net for analyzing the properties of the physical systems, several methods have been presented in the net theory, among which process is a powerful tool for describing dynamical behaviors of net systems. Petri net processes are very convenient for analyzing concurrent phenomena and system properties relating to concurrency. This is an advantage of process method compared to other analysis methods about Petri net (Yuan, 2005; Garg and Ranganath, 1992; Murata, 1989). However, a process of a Petri net only gives one possible running case for the net system. There are usually many (sometimes maybe infinite) cases during a Petri net system running. It is difficult to obtain all the running cases, which brings difficulties for the analysis of Petri nets using their processes. Lu (1992a, b) has developed a new kind of nets, P/R nets, as a substitute for occurrence nets to describe executions of elementary net systems. A P/R net consists of a family of occurrences nets (called pages by Lu) and a contact situation of elementary net system is resolved by using several pages. Zeng and Wu (2002c) introduce the concept of process net system of a Petri net. The process net system is a reconstruction Petri net based on the set of the basic process sections, which describes the process behaviors of the original system very well. Wu (1996) and Wu et al. (2000) present the concept of process expression for bounded Petri net and unbounded fair Petri net. Roughly, any process of a bounded Petri net or unbounded fair Petri net is really a composition of several basic process sections (called subprocess by Wu). Zeng and Wu (2003) have proved the one-to-one corresponding relation between the transition firing sequences of the process net system and the processes of the original Petri net and presented a method to obtain the process expression of any unbounded Petri net based on its process net system.

This study proposes a new method to construct the process expression of a Petri net based on a kind of decomposition methods. This decomposition method is very useful for property analysis of structure-complex Petri nets since the structure of the decomposition net is well-formed (Zeng and Wu, 2002a, 2004a, b). We analyze the language relations during the decomposition and present a method to obtain the language of a Petri net (Zeng and Wu, 2004a). In Zeng and Wu (2004b), we discuss the conditions to keep states and behaviors invariant during the decomposition and present the judgment algorithm. In this study, we analyze the process properties of the decomposition nets and present the methods to obtain their process expressions. By discussing the process relations during the decomposition, an algorithm is proposed to present the process expression of the original Petri net based on the process expressions of the decomposition nets. This method suits obtaining the process expression for bounded or unbounded, fair or unfair Petri nets and it is not necessary to construct the process net system.

RELATED TERMINOLOGY AND NOTATIONS ABOUT PETRI NET PROCESS

To save space, it is assumed that the readers are familiar with the basic definitions of Petri nets (Yuan, 2005; Murata, 1989; Zeng and Wu, 2002c; Wu, 1996). Some of the essential terminology and notations related
to this research are defined as follows. Convenient to
define, we only consider the process of P/T system and
suppose that K = \omega and W = 1. We also assume that
the Petri net discussed in this work is finite and connected
and is not an S-Net (Yuan, 2005).

**PROCESS AND BASIC PROCESS 
SECTION PETRI NET**

**Definition 1:** A net \( N = (B, E, G) \) is an occurrence net if

1. \( \forall b \in B: |b| \leq 1 \land |b| = 1 \) and
2. \( \forall x, y \in B \cup E: (x, y) \in G' \rightarrow (y, x) \in G' \),

where, \( G' \) is the transitive closure of the flow relation \( G \).

**Definition 2:** Let \( N_1 = (S, T, F) \) be a net and \( N_2 = (B, E, G) \) be an occurrence net. A mapping \( \varphi: B \cup E \rightarrow S \cup T \) is a mapping from \( N_2 \) to \( N_1 \), denoted by \( \varphi: N_2 \rightarrow N_1 \), if it satisfies:

1. \( \varphi(b) \subseteq S; \varphi(e) \subseteq T \),
2. \( \forall x, y \in B \cup E: (x, y) \in G \rightarrow (\varphi(x), \varphi(y)) \in F \),
3. \( \forall e \in E: \varphi^*(e) = \varphi(e) \land \varphi^*(e) = \varphi(e) \)

**Definition 3:** Let \( \Sigma = (N_0, M_0) = (S, T, F, M_0) \) be a Petri net
and \( N = (B, E, G) \) be a occurrence net. A process of \( \Sigma \) is a couple \( (N, \varphi) \), where \( \varphi \) is a mapping from \( N \) to \( N_1 \) and satisfies the following conditions:

1. \( \forall b_1, b_2 \in B: (b_1 \neq b_2) \land \varphi(b_1) = \varphi(b_2) \rightarrow (b_1 \neq b_2) \land \varphi^*(b_1) \neq \varphi^*(b_2) \)
2. \( \forall s \in S \in \{ b \mid \varphi(b) = s \land \varphi^*(b) = \phi \} \leq M_0(s) \)

**Definition 4:** Let \( \varphi \) be a mapping from a occurrence net
\( N = (B, E, G) \) to a Petri net \( \Sigma = (S, T, F, M) \). If:

1. \( \forall b_1, b_2 \in B: (b_1 \neq b_2) \land \varphi(b_1) = \varphi(b_2) \rightarrow (b_1 \neq b_2) \land \varphi^*(b_1) \neq \varphi^*(b_2) \)
2. \( \forall s \in S \in \{ b \mid \varphi(b) = s \land \varphi^*(b) = \phi \} \leq M_0(s) \)

then \( P = (N, \varphi) \) is said to be a surjective process of \( \Sigma \).

**Definition 5:** Let \( P = (N, \varphi) \) be a surjective process of \( \Sigma \),
where \( N = (B, E, G) \). Let \( u_i \) and \( u_j \) be two S-cuts of \( N \) such
that \( u_i \leq u_j \). We define \( N_i = (B_i, E_i, G_i) \) such that:

1. \( B_i \subseteq \{ b \mid b \in b_i, \varphi(b) \leq \varphi^*(x, y, b_i \in G^*) \} \)
2. \( E_i \subseteq \{ e \mid e \in \varphi^*(b), \varphi(b) \leq \varphi^*(x, y, b_i \in G^*) \} \)
3. \( G_i \subseteq \varphi^*(b_i) \subseteq \{ b \mid b \in b_i \land \varphi(b) \leq \varphi^*(x, y, b_i \in G^*) \} \)

If \( \varphi: N_i \rightarrow \Sigma \) satisfies \( \forall x \in B_i \cup E_i: \varphi_i(x) = \varphi(x), (N_i, \varphi_i) \) is said to be a section (between \( u_i \) and \( u_j \)) of process \( P \),
sometimes it is also called a process section of \( \Sigma \), denoted by \( (N[u_i, u_j], \varphi) \) (Zeng and Wu, 2002c).

**Definition 6:** Let \( P = (N, \varphi) \) be a surjective process of \( \Sigma \) and \( P_i = (N[u_i, u_j], \varphi) \) be a process section of \( \Sigma \). If any two s-cuts \( u_i \) and \( u_j \) (\( i, j \neq 1, 2 \)) in \( N[u_i, u_j] \) satisfies:

\( u_i \neq u_j \rightarrow (\varphi(u_i) \neq \varphi(u_j)) \land (\varphi(u_i) \neq \varphi(u_j)) \land (\varphi(u_i) \neq \varphi(u_j)) \)

then \( P_i = (N[u_i, u_j], \varphi) \) is said to be a basic process
section of \( \Sigma \).

The set of all the basic process sections of a Petri net \( \Sigma \) is denoted by \( BP(\Sigma) \) (Zeng and Wu, 2003).

**PROCESS NET SYSTEM OF PETRI NET**

The concept of process net system of a Petri net was
first introduced by Zeng and Wu (2002c). The process net
system \( \Sigma_p \) of a Petri net \( \Sigma \) describes the process behaviors of \( \Sigma \) very well. Every transition firing sequence
of \( \Sigma_p \) corresponds to a surjective process of \( \Sigma \). If the last process section of a given surjective process is complete, there is a transition firing sequence corresponding to it.

**Definition 7:** Let \( P = ([u_i, u_j], \varphi) \) be a basic process section
of a Petri net \( \Sigma = (S, T, F, M_0) \) and \( \varphi \) be the mapping from the occurrence net \( N = (B, E, G) \) to \( \Sigma \).

1. The set of \( P = \{ b \mid s = e \forall \varphi \} \) and \( P = \{ b \mid e \forall \varphi \} \) are respectively named as the input and the output place set of \( P \).
2. The bag \( B(P) = \{ b \mid e \forall \varphi \} \) is the input place bag of \( P \), where

\[ \forall s \in B(P), \#(s, B(P)) = |\{ b \mid b \in u_i \land \varphi(b) = s \} | \]

Similarly, the bag \( B(P) = \{ b \mid e \forall \varphi \} \) is the output bag of \( P \), where

\[ \forall s \in B(P), \#(s, B(P)) = |\{ b \mid b \in u_i \land \varphi(b) = s \} | \]

**Definition 8:** Let \( BP(\Sigma) \) be the set of the basic process
sections of \( \Sigma \) and \( S(P) \subseteq BP(\Sigma) \). We define

1. \( \forall s \in S(P), \forall s \in S(P) = \bigcup \forall s \in S(P) \)
2. \( B(P) = \sum \forall s \in S(P), B(S(P)) = \bigcup \forall s \in S(P) \)

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**Definition 9**: Let $\Sigma = (S, T; F, M_0)$ be a Petri net. For all $s \in S$, the projection of $M$ on $S_s$ (denoted by $\Gamma_{s \leftarrow s}(M)$) is defined by $\Gamma_{s \leftarrow s}(M)(s) = M(s)$ for all $s \in S$.

**Definition 10**: Let $\phi$ be a mapping from the occurrence net $N = (B, F; E)$ to the Petri net $\Sigma = (S, T; F; M_0)$ and BP($\Sigma$) be the set of basic process sections. Petri net $\Sigma_p = (S_p, T_p; F_p, M_0)$ is defined as the process net system of $\Sigma$ iff

1. $s_p = \phi \cdot BP(\Sigma) \cup BP(\Sigma)$;
2. There is a one-to-one mapping $\xi: T_p \rightarrow BP(\Sigma)$ such that $\forall t \in T_p, \xi(t) \in BP(\Sigma)$;
3. $F_p = \bigcup_{t \in T_p} (\{t\} \times B(\xi(t))) \cup B(\xi(t)) \times \{t\}$;
4. $M_{op} = \Gamma_{s \leftarrow s}(M_0)$

**PROCESS EXPRESSION OF PETRI NET**

The process expressions of a bounded Petri net and unbounded fair Petri net are defined in Wu (1996) and Wu et al. (2000), respectively. The following definition for process expression of a Petri net is an extension of the cases in Wu (1996) and Wu et al. (2000).

**Definition 11**: Let $BP(\Sigma)$ be the set of the basic process sections of a Petri net $\Sigma = (S, T; F; M_0)$, $Exp(P(\Sigma))$ be an expression whose alphabet is isomorphic to $BP(\Sigma)$ and $CRE(P(\Sigma))$ be the set expressed by $Exp(P(\Sigma))$: $Exp(P(\Sigma))$ is defined as the process expression of $\Sigma$, if each surjective process of $\Sigma$ is an element of the set $\text{Pref}[\text{Exp}(P(\Sigma))] = \bigcup_{P \in \text{CRE}(P(\Sigma))} \text{Pref}(P)$

**DECOMPOSITION BASED ON THE INDEX OF PLACE**

Here, we introduce the decomposition method for a structure-complex Petri net based on the indexes of places (Zeng and Wu, 2002a, 2004a, b).

**Definition 12**: Let $\Sigma = (S, T; F, M_0)$ be a Petri net, a function $f: S \rightarrow \{1, 2, ..., k\}$ is said to be an index function defined on the place set if $\forall s_i, s_j \in S$,

$(s_i \cap s_j = \phi) \lor (\ast s_i \cap \ast s_j \neq \phi) \rightarrow f(s_i) \neq f(s_j)$

$f(s)$ is named as the index of place $s$ (Zeng and Wu, 2002a).

**Definition 13**: Let $\Sigma = (S, T; F, M_0)$ be a Petri net, $f: S \rightarrow \{1, 2, ..., k\}$ be the index function on the places of $\Sigma$. Petri net $\Sigma_0 = (S_0, T_0; F_0, M_0)$ (i.e., $f(i) \in \{1, 2, ..., k\}$) is said to be the decomposition net of $\Sigma$ based on the index function $f$ if $\Sigma$ satisfies the following conditions (Zeng and Wu, 2002a).

1. $S_i = \{s \in S | f(s) = i\}$
2. $T_i = \{t \in T | \exists s \in S_i, t \in ^s \cup ^s\}$
3. $F_i = (S_i \times T_i) \cup (T_i \times S_i) \cap F$
4. $M_{0i} = \Gamma_{s \leftarrow s}(M_0)$

Simply, $\Sigma_0$ is said as the index decomposition net of $\Sigma$.

More discussions about this decomposition method can be found in Zeng and Wu (2002a, 2004a) and an algorithm of decomposition for a structure-complex Petri net is also given in Zeng and Wu (2004a). The decomposition results with the method in definition 13 are usually not unique. With the results discussed in Zeng and Wu (2002a, 2004a), it is usually to take the case that $k$ is with the minimal value as the best result.

**Definition 14**: A Petri net $\Sigma = (S, T; F, M_0)$ is an S-Net if $f$: $\forall t \in T$, $|t| \leq 1$ and $|t| \\leq 1$ (Yuan, 2005).

**Theorem 1**: Let $\Sigma_i = (S_i, T_i; F_i, M_0)$ (i.e., $\{1, 2, ..., k\}$) be the index decomposition net of a Petri net $\Sigma = (S, T, F, M_0)$, then $\Sigma_i$ is an S-net (Zeng and Wu, 2002a).

According to Theorem 1, each of the index decomposition nets is well-formed and it is easier to analyze the properties of the index decomposition nets than the original Petri net. This is one of the main reasons to decompose a structure-complex Petri net using this decomposition method.

**Example 1**: A Petri net $\Sigma$ is shown in Fig. 1. We use the decomposition method in Definition 12 to decompose $\Sigma$.

A function $f$ is first defined on the place set such that $f(s_1) = f(s_2) = 1$, $f(s_3) = f(s_4) = 2$, $f(s_5) = f(s_6) = f(s_7) = 3$. It

![Fig. 1: A Petri net $\Sigma$](image-url)
can prove that $f$ satisfies all the conditions in Definition 11. Based on the method of Definition 13, three index decomposition net systems $\Sigma_{1\text{r}}, \Sigma_{2\text{r}}$ and $\Sigma_{3\text{r}}$ are obtained and shown in Fig. 2. Obviously, $\Sigma_{1\text{r}}, \Sigma_{2\text{r}}$ and $\Sigma_{3\text{r}}$ are S-Nets.

**PROCESS EXPRESSIONS OF THE DECOMPOSITION NET SYSTEMS**

According to Theorem 1, each index decomposition net system is an S-Net. We have analyzed the language and liveness characteristics of an S-Net, respectively in Zeng and Wu (2002b) and Duan and Zeng (2004). Here, we analyze the process characteristics of an S-Net (an index decomposition net system) and present the method to obtain its process expression so as to construct the process expression of the original Petri net.

The following two propositions can be obtained easily following the related definitions.

**Proposition 1:** Let $P = (N, \varnothing)$ be a surjective process of an S-Net $\Sigma$, where $N = (B, E, G)$. For any $e \in E$, $\|e\| \leq 1$ and $\|e^*\| \leq 1$.

**Proposition 2:** Let $\Sigma_p = (S_p, \mathcal{T}_p, F_p, M_{i0})$ be the process net system of an S-Net $\Sigma = (S, \mathcal{T}, F, M_0)$. $\Sigma_p$ is also an S-Net.

**Definition 15:** Let $\Sigma = (S, \mathcal{T}, F, M_0)$ be an S-Net. For $t \in \mathcal{T}$, $t$ is a primitive transition of $\Sigma$ if $\{t\} = 0$ and $t$ is a goal transition of $\Sigma$ if $\{t\} = 0$.

In the index decomposition net, it is possible that there are primitive transitions or goal transitions. It is easy to transform an S-Net with goal transitions into one S-Net without goal transitions. The method is to add an output place for each goal transition in the original S-Net. If we delete all the added places from all the processes of the new S-Net without goal transitions, it can keep all states and behaviors of the original S-Net invariant. Thus, the process semantics can be regarded as equal between the original S-Net and the new S-Net without goal transitions. Therefore, it is assumed that for each transition $t$ in the index decomposition net discussed in this research satisfies $\{t\} \leq 1$ and $\{t^*\} = 1$.

All the S-Nets without goal transitions can be classified into four classes based on the standards of primitive transitions and initial markings. These four kinds of S-Nets, respectively are:

- S-Nets without primitive transitions and with empty markings. This kind of S-Nets has no any process, so it is not necessary to discuss.
- S-Nets without primitive transitions but with initial markings. It is obvious that this kind of S-Nets is S-graph (state machine). Without interpretations, this kind of S-Nets is named as S-graph directly in the following discussions.
- S-Nets with primitive transitions but with empty initial markings. Without interpretations, this kind of S-Nets is named as S-Net with Empty Markings directly in the following discussions.
- S-Nets with primitive transitions and with initial markings. Without interpretations, this kind of S-Nets is named as S-Net with Initial Markings directly in the following discussions.

Next, we will discuss the process characteristics of each kind of the S-Nets based on the classification and present the method to obtain their process expression.

**PROCESS EXPRESSION OF S-GRAPH**

**Lemma 1:** Each S-Graph is bounded (Yuan, 2005).

**Lemma 2:** The process expression of each S-Graph is a regular expression.

**Proof:** With the conclusions in Wu (1996), the process expression of a bounded Petri net is a regular expression.

**Theorem 2:** Let $\Sigma_i = (S_{i}, \mathcal{T}_i, F_{i}, M_{i0})$ ($i \in \{1, 2, \ldots, k\}$) be the index decomposition net of a Petri net $\Sigma = (S, \mathcal{T}, F, M_0)$. $\text{Exp}(P(\Sigma_i))$ is a regular expression if $\Sigma_i$ is an S-Graph.

**Example 2:** $\Sigma_3$ shown in Fig. 2c is an S-Net. Based on its reachability graph, we can obtain the set of the basic process sections of $\Sigma_{3\text{r}}$. The method to obtain the basic process sections can be seen in Wu (1996).\(\text{BP}(\Sigma_3) = \{P_{31}\}\) and $P_{31}$ is shown in Fig. 3. In the process section, we directly use the names of place and transition to label the corresponding elements and the same to the following discussions. The process expression of $\Sigma_3$ is $\text{Exp}(P(\Sigma_3)) = P_{31}$*. It can be shown that each surjective process of $\Sigma_3$ is the prefix of the language of $\text{Exp}(P(\Sigma_3)) = P_{31}$*.
Fig. 3: The basic process section of $\Sigma_j$

**PROCESS EXPRESSION OF S-NET WITH EMPTY MARKINGS**

**Lemma 3:** Let $\Sigma_j = (S_j, T_j, F_j, M_0)$ (i.e., $\{1, 2, \ldots, k\}$) be the index decomposition net of a Petri net $\Sigma = (S, T, F, M_0)$. If there is one and only one primitive transition $t' \in T_j$ and $\forall s \in S_j: M_0(s) = 0$ in $\Sigma_j$, $\Sigma_j$ has one and only one basic source process section.

**Proof:** Obviously.

**Lemma 4:** Let $\Sigma = (S, T, F, M_0)$ (i.e., $\{1, 2, \ldots, k\}$) be the index decomposition net of a Petri net $\Sigma = (S, T, F, M_0)$. If there is one and only one primitive transition $t' \in T$, and $\forall s \in S: M_0(s) = 0$ in $\Sigma$, there is a regular expression $RE$ such that $\exp(P(\Sigma_0)) = (RE)^*$, where $(RE)^*$ is the $\alpha$-closure of $RE$.

**Proof:** Suppose that $BP(\Sigma_i)$ is the set of the basic process sections of $\Sigma_i$. For any basic process in $BP(\Sigma_i)$, it is denoted by $P_i = (N[u_{i, j}, u_{i, j}])$, where $1 \leq j \leq |BP(\Sigma_i)|$ and $u_{i, j}$ and $u_{i, j}$ are the first and last S-Cut of $P_i$. Based on $BP(\Sigma_i)$, the process net system $\Sigma_i = (S_{\Sigma}, T_{\Sigma}, F_{\Sigma}, M_{\Sigma}(s))$ of $\Sigma_i$ can be constructed, where

1. $S_{\Sigma} = \{u_{i, j} : \{N[u_{i, j}, u_{i, j}] \in BP(\Sigma_i)\} \cup \{u_{i, j} : \{N[u_{i, j}, u_{i, j}] \in BP(\Sigma_i)\} \}
2. T_{\Sigma} = BP(\Sigma_i)$ and
3. $F_{\Sigma} = \{u_{i, j}, u_{i, j}, \{P_i \in BP(\Sigma_i) \} \cup \{P_i \in BP(\Sigma_i) \} \},$ where $P_i = (N[u_{i, j}, u_{i, j}]), P_i \in BP(\Sigma_i)$, is the source process section of $\Sigma_i$ and $M_{\Sigma}(s) = 0$.

According to Lemma 1 and 2, $\Sigma_i$ is an $S$-Net with one and only one primitive transition and with empty markings, so $L(\Sigma_i)$ can be expressed by the $\alpha$-closure of a regular expression. The proof can be seen in Zeng and Wu (2002b). With the definition of process net system, there is a one-to-one corresponding relation between $\exp(P(\Sigma_i))$ and the $L(\Sigma_i)$. So, there is a regular expression $RE$ such that $\exp(P(\Sigma_0)) - (RE)^*$, where $(RE)^*$ is the $\alpha$-closure of $RE$.

**Example 3:** An $S$-Net $\Sigma_i$, with one and only one primitive transition and has empty initial markings is shown in Fig. 4a and its two basic process sections are shown in Fig. 4b. The process expression of $\Sigma_i$ is $\exp(P(\Sigma_i)) = (P_{1, 1} P_{1, 2})^*$, which can be obtained based on the process net system of $\Sigma_i$. It can be shown that each surjective process of $\Sigma_i$ is the prefix of the language of $\exp(P(\Sigma_i)) = (P_{1, 1} P_{1, 2})^*$, where $\circ$ is the connection operation between the basic process sections. The definitions and more discussions about operations between the basic process sections can be seen in Wu (1996).

**Theorem 3:** Let $\Sigma = (S, T, F, M_0)$ (i.e., $\{1, 2, \ldots, k\}$) be the index decomposition net of a Petri net $\Sigma = (S, T, F, M_0)$. If there are primitive transitions $t_1, \ldots, t_n$ and $\forall s \in S: M_0(s)$ in $\Sigma$, there are regular expressions $RE_1, \ldots, RE_n$ such that $\exp(P(\Sigma)) = (RE_1)^* \ldots (RE_n)^*$.

**Proof:** Let $T_0 = \{t_1, \ldots, t_n\}$. To construct S-Nets $\Sigma_0 = (S, T_0, F, M_0)$ (i.e., $\{1, 2, \ldots, n\}$) to prove the conclusion, where the place set $S_i$ and initial marking $M_0$ keep invariant from $\Sigma = (S, T, F, M_0)$. $\Sigma_0 = (S, T_0, F, M_0)$, $\forall t \in \{1, 2, \ldots, n\}$, satisfies the following conditions,

1. $T_0 = (T_0 - T_i) \cup \{t_i\}$ and
2. $F_i = F - (T_0 - t_i)$) + $S_i$

It is easy to prove $\exp(P(\Sigma)) = \exp(P(\Sigma_0)) \ldots \exp(S_{\Sigma_0})$.

For each $\Sigma_0 = (S, T_0, F, M_0)$ (i.e., $\{1, 2, \ldots, n\}$), there is one and only one primitive transition and with empty initial markings. With Lemma 4, there is a regular expression $RE_i$ such that $\exp(P(\Sigma_i)) = (RE_i)^*$. So, there are regular expressions $RE_1, \ldots, RE_n$ such that $\exp(P(\Sigma)) = (RE_1)^* \ldots (RE_n)^*$.

**PROCESS EXPRESSION OF S-NET WITH INITIAL MARKINGS**

**Theorem 4:** Let $\Sigma_i = (P_i, T_i, F_i, M_i)$ (i.e., $\{1, 2, \ldots, k\}$), be the index decomposition net of a Petri net $\Sigma = (S, T, F, M_0)$. If there are primitive transitions $t_1, \ldots, t_n$ (i.e., $\{1, \ldots, t_i \in T_i\}$ and at least one place $s \in S$ such that $M_0(s) \neq 0$ in $\Sigma$, there are regular expressions $RE_1, \ldots, RE_n$ and $RE_{\Sigma_0}$ such that $\exp(P(\Sigma)) = (RE_1)^* \ldots (RE_n)^* (RE_{\Sigma_0})^*.$
Fig. 6: The basic process section of \( \Sigma_1 \)

For any language \( L \subseteq X^* \), \( \delta(L) = \{ \omega \in X^* | \exists \omega \in L : \#(x, \omega) > 0 \} \). For example, \( \delta(aab) = \delta(aa+abbb) = \{a, b\} \). Obviously, \( \delta(L) \subseteq X^* \).

**Definition 15**: Let \( \Sigma_i = (S_i, T_i, F_i, \Pi_i, M_{0i}) \) (\( i \in \{1, 2, \ldots, k\} \)) be the index decomposition net of a Petri net \( \Sigma = (S, T, F, \Pi, M_{0}) \) and \( SN(\Sigma) \) be the set of occurrence net, where \( \forall i \in \{1, 2, \ldots, k\} \). For \( \forall \Pi \in SN(\Sigma) \), if \( \Pi \subseteq \Pi \) \( \Rightarrow \Pi \subseteq \Pi \) satisfies the following conditions,

1. \( \forall b \in B \), to delete \( b \) and its input and output arcs if \( \Phi(b) \subseteq S_i \) and \( j+i \) and
2. \( \forall c \in E \), to delete \( c \) and its input and output arcs if \( \Phi(c) \subseteq E_i \) and \( j+i \),

\( \Pi \) is the projection from \( \Phi(SN(\Sigma)) \) to \( \Phi(SN(\Sigma)) \) and denoted by \( \Pi \subseteq \Pi \).

For any basic process section, it is one part of the occurrence net of a Petri net. If \( \Pi \subseteq \Pi \) in Definition 15 is extended to contain all the basic process sections and other conditions are invariant, the projection of basic process section can be also obtained.

**Definition 16**: Let \( \Sigma_i = (S_i, T_i, F_i, \Pi_i, M_{0i}) \) (\( i \in \{1, 2\} \)) be two Petri nets and \( \Phi(\Sigma_j) \) be the process expression of \( \Sigma_j \). \( \Phi(\Sigma_j) \) is the synchronizing shuffle of \( \Phi(\Sigma_i) \) and \( \Phi(\Sigma_j) \) iff:

\[
\Phi(\Sigma_i) \oplus \Phi(\Sigma_j) = (\text{Exp}(\Sigma_i)) \oplus (\text{Exp}(\Sigma_j)) = (\text{Exp}(\Sigma_i)) \oplus (\text{Exp}(\Sigma_j)), \quad \Pi \subseteq \Pi \subseteq \Pi
\]

Definition 17 only presents the synchronizing shuffle of two process expressions, which can be extended into the case of \( k \) (\( \geq 2 \) expressions), denoted by:

\[
\Phi(\Sigma_i) \oplus \Phi(\Sigma_j) = (\text{Exp}(\Sigma_i)) \oplus (\text{Exp}(\Sigma_j)) \oplus \ldots \oplus (\text{Exp}(\Sigma_j)) = \Pi \subseteq \Pi \subseteq \Pi
\]

**Process Characteristic Analysis During Decomposition**

**Lemma 5**: Let \( \Sigma = (S, T, F, M_{0}) \) (\( i \in \{1, 2, \ldots, k\} \)) be the index decomposition net of a Petri net \( \Sigma = (S, T, F, M_{0}) \) and \( BP(\Sigma) \) be the set of the basic process sections of \( \Sigma \), where \( \beta \subseteq \{\Sigma_1, \Sigma_2, \ldots, \Sigma_k\} \). For each \( BP \in BP(\Sigma) \), \( \Pi \subseteq \Pi \subseteq BP \).
Proof: It is easy to prove.

**Lemma 5.** Let $\Sigma_i = (S_i, T_i, F_i, M_0)$ (i$\in\{1, 2\}$) be the index decomposition set of a Petri net $\Sigma = (S, T, F, M_0)$. $\text{Exp}(\Sigma_i) = \text{Exp}(\Sigma_i) \cap \text{Exp}(\Sigma_i)$.

**Proof:** First with $\text{Exp}(\Sigma) = \text{Exp}(\Sigma) \cap \text{Exp}(\Sigma)$ and then $\text{Exp}(\Sigma) \cap \text{Exp}(\Sigma)$.

Now, we prove that $\text{Exp}(\Sigma)$ is a process of $\text{Exp}(\Sigma)$.

For any $\phi(N) \in \text{Exp}(\Sigma)$, let $N = (B, E, G)$. Since $\phi(N)$ is a process of $\Sigma$, according to the definition of $\Pi_{i\in\Sigma}$,

\[\begin{align*}
(1) & \quad \Pi_{i\in\Sigma} (\phi(B) \wedge \phi(E)) \subseteq T_i \wedge \Pi_{i\in\Sigma} (\phi(G)) \subseteq F_i \\
(2) & \quad \text{For any } e \in \Pi_{i\in\Sigma} (E) \subseteq E, \text{ since } \phi(e) = \\
& \quad \phi(e) \wedge \phi(e') = \phi(e') \wedge \phi(e') \cap S_i = \phi(e') \cap S_i, \wedge \phi(e') \\
& \quad \text{For any } b \in \Pi_{i\in\Sigma} (B) \subseteq B, \text{ since } \phi(b) = \phi(b') \wedge \phi(b') \cap F_i \wedge \phi(b') \\
& \quad \phi(b) \wedge \phi(b') = \phi(b') \wedge \phi(b') \cap F_i \wedge \phi(b') \\
& \quad \text{For any } s \in S \subseteq S, \text{ satisfying } |b | \phi \wedge \phi(b) = s | = M_i(s), \text{ so } \\
& \quad |\Pi_{i\in\Sigma} (b) | \phi \wedge \phi(b) = s | = M_i(s), \text{ where } M_i = \Gamma_{i\in\Sigma} (M_i) \\
& \quad \text{Thus, } \Pi_{i\in\Sigma} \phi(N) \in \text{Exp}(\Sigma)(i \in \{1, 2\}). \text{ With the definition of } \Theta, \phi(N) \in \text{Exp}(\Sigma) \cap \text{Exp}(\Sigma) \text{ and Exp}(\Sigma) \cap \text{Exp}(\Sigma) \cap \text{Exp}(\Sigma).
\end{align*}\]

Now, we prove that $\text{Exp}(\Sigma) \cap \text{Exp}(\Sigma) \cap \text{Exp}(\Sigma)$.

For any $\phi(N) \in \text{Exp}(\Sigma) \cap \text{Exp}(\Sigma)$, let $|\phi(N)|$ be the number of the basic process sections contained by $\phi(N)$. The conclusion can be proved by induction on $|\phi(N)|$.

(1) While $|\phi(N)| = 1$, $\phi(N)$ is a basic process section and $\Pi_{i\in\Sigma} \phi(N) \in \text{Exp}(\Sigma)$. With Lemma 5, $\phi(N) \in \text{BP}(\Sigma) \cap \text{Exp}(\Sigma)$, so $\text{Exp}(\Sigma) \cap \text{Exp}(\Sigma) \cap \text{Exp}(\Sigma)$.

(2) Suppose that the conclusion is correct while $|\phi(N)| = k$, i.e., $\text{Exp}(\Sigma) \cap \text{Exp}(\Sigma) \cap \text{Exp}(\Sigma)$.

Now, we prove it is also correct while $|\phi(N)| = k + 1$. With the definition of process expression, there are $\phi(N) \in \text{Exp}(\Sigma) \cap \text{Exp}(\Sigma) \cap \text{Exp}(\Sigma)$ and $\phi(N)$ such that $\phi(N) = \phi(N) \wedge \phi(N)$, where $\wedge$ is the conjunction operation between basic process sections and $|\phi(N)| = k$ and $|\phi(N)| = 1$. Now, it is proved by two cases of $\phi(N)$.

**Case 1:** If $\phi(N) \in \text{BP}(\Sigma) \cap \text{BP}(\Sigma)$, where $i, j \in \{1, 2\}$ and $i \neq j$, then

\[\begin{align*}
(\text{a}) & \quad \phi(N) = \phi(B) \wedge \phi(B) \wedge \phi(B) \subseteq S_i \cup S_j \cup S_k, \text{ so } \\
& \quad \phi(B) \subseteq S_i \cup S_j \cup S_k, \text{ i.e., } \phi(B) \subseteq F_i \cup F_j \cup F_k, \text{ and } \\
& \quad \phi(E) \subseteq T_i \cup T_j \cup T_k, \text{ so } \\
& \quad \phi(G) \subseteq F_i \cup F_j \cup F_k, \text{ and } \\
& \quad \phi(G) \subseteq F_i \cup F_j \cup F_k, \text{ i.e., } \phi(G) \subseteq F_i \cup F_j \cup F_k.
\end{align*}\]

\[\begin{align*}
(\text{b}) & \quad \text{For any } e \in E \text{ such that } E = E, \text{ and } e \in E, \\
& \quad \phi(e) = \phi(e) \wedge \phi(e) \wedge \phi(e), \text{ with the supposition.} \\
& \quad \text{While } e \in E, \text{ and } e \in E, \\
& \quad \phi(e) = \phi(e) \wedge \phi(e) \wedge \phi(e), \text{ since } \\
& \quad \phi(N) \in \text{BP}(\Sigma) \cap \text{BP}(\Sigma), \text{ and } \\
& \quad \phi(e) = \phi(e) \wedge \phi(e), \text{ and } \\
& \quad \phi(e) = \phi(e).
\end{align*}\]

\[\begin{align*}
(\text{c}) & \quad \text{For any } b, b' \in B \text{ such that } B = B, \text{ and } b = b', \text{ it can be } \\
& \quad \text{proved } \phi(b) = \phi(b'), \text{ with } \\
& \quad \text{a similar method of (b).} \\
\end{align*}\]

\[\begin{align*}
(\text{d}) & \quad \text{For any } s \in S \text{ such that } S = S_i \cup S_j, \\
& \quad \text{then } |b | \phi \wedge \phi(b) = s | = M_i(s), \text{ i.e., } \\
& \quad |b | \phi \wedge \phi(b) = s | = M_i(s).
\end{align*}\]

With (a) - (d), $\phi(N) \in \text{Exp}(\Sigma)$. So,

$\text{Exp}(\Sigma) \cap \text{Exp}(\Sigma) \cap \text{Exp}(\Sigma) \subseteq \text{Exp}(\Sigma)$.

**Case 2:** If $\phi(N) \in \text{BP}(\Sigma) \cap \text{BP}(\Sigma)$, then

Since $\phi(B) = \phi(B) \wedge \phi(B)$ and $\phi(B) \wedge \phi(B) \subseteq S_i \cup S_j \cup S_k \cup S_l$, $\phi(B) \subseteq S_i \cup S_j \cup S_k \cup S_l$, and $\phi(B) \subseteq S_i \cup S_j \cup S_k \cup S_l$, 

Thus, $\Pi_{i\in\Sigma} \phi(N) \in \text{Exp}(\Sigma)(i \in \{1, 2\})$. With the definition of $\Theta$, $\phi(N) \in \text{Exp}(\Sigma) \cap \text{Exp}(\Sigma) \cap \text{Exp}(\Sigma)$ and $\text{Exp}(\Sigma) \cap \text{Exp}(\Sigma) \cap \text{Exp}(\Sigma)$.

Now, we prove that $\text{Exp}(\Sigma) \cap \text{Exp}(\Sigma) \cap \text{Exp}(\Sigma)$.

For any $\phi(N) \in \text{Exp}(\Sigma) \cap \text{Exp}(\Sigma) \cap \text{Exp}(\Sigma)$, let $|\phi(N)|$ be the number of the basic process sections contained by $\phi(N)$. The conclusion can be proved by induction on $|\phi(N)|$. 

(1) While $|\phi(N)| = 1$, $\phi(N)$ is a basic process section and $\Pi_{i\in\Sigma} \phi(N) \in \text{Exp}(\Sigma)$. With Lemma 5, $\phi(N) \in \text{BP}(\Sigma) \cap \text{Exp}(\Sigma)$, so $\text{Exp}(\Sigma) \cap \text{Exp}(\Sigma) \cap \text{Exp}(\Sigma)$.

(2) Suppose that the conclusion is correct while $|\phi(N)| = k$, i.e., $\text{Exp}(\Sigma) \cap \text{Exp}(\Sigma) \cap \text{Exp}(\Sigma)$.

Now, we prove it is also correct while $|\phi(N)| = k + 1$. With the definition of process expression, there are $\phi(N) \in \text{Exp}(\Sigma) \cap \text{Exp}(\Sigma) \cap \text{Exp}(\Sigma)$ and $\phi(N)$ such that $\phi(N) = \phi(N) \wedge \phi(N)$, where $\wedge$ is the conjunction operation between basic process sections and $|\phi(N)| = k$ and $|\phi(N)| = 1$. Now, it is proved by two cases of $\phi(N)$.
Since \( \overline{\phi(E)} = \overline{\phi(E)} \cap \overline{\phi(E)} \) and \( \overline{\phi(E)} \),
\[ \subseteq T_1 \cup T_2 \cup T_3 \cup T_4 \]
and \( \overline{\phi(E)} \subseteq T_1 \cup T_2 \)

Since \( \overline{\phi(G)} = \overline{\phi(G)} \cap \overline{\phi(G)} \),
\[ \subseteq F_1 \cup F_2 \cup F_3 \]
and \( \overline{\phi(G)} \subseteq F_1 \cup F_2 \cup F_3 \)

Other parts can be proved like the cases in Case 1. So, \( \overline{\phi(N)} \subseteq \text{Exp}(P(S)) \cap \overline{\phi(N)} \cap \text{Exp}(P(S)) \).

With Case 1 and 2, \( \text{Exp}(P(S_1)) \cap \text{Exp}(P(S_2)) \) is proved based on the supposition.

\[ \text{Exp}(P(S_1)) = \text{Exp}(P(S_1)) \cap \text{Exp}(P(S_2)) \]
has been proved since \( \text{Exp}(P(S_1)) \cap \text{Exp}(P(S_2)) \) and \( \text{Exp}(P(S_1)) \cap \text{Exp}(P(S_2)) \).

**Theorem 5:** Let \( \Sigma = (S, T, F, M_0) \) (i.e., \( \{1, 2, ..., k\} \)) be the index decomposition net of a Petri net \( \Sigma = (S, T, F, M_0) \).

\[ \text{Exp}(P(S)) = \bigcap_{i=1}^{k} \text{Exp}(P(S_i)) \]

**Proof:** Based on Lemma 5, the theorem can be proved by induction on \( k \).

Theorem 5 is useful to obtain the process expression of a Petri net by the operation \( \Theta \) between processes based on the process expressions of the index decomposition S-Nets. The method will be presented in next section.

**A CONSTRUCTION METHOD FOR THE PROCESS EXPRESSION OF PETRI NET**

Using the decomposition method, a Petri net \( \Sigma \) can be decomposed into a set of S-Nets \( \Sigma_1, \Sigma_2, ..., \Sigma_k \). The methods to obtain the process expressions of all kinds of S-Nets are presented in earlier. Theorem 5 indicates the relation between the process expression of the original Petri net and its index decomposition of systems. \( \bigcap_{i=1}^{k} \text{Exp}(P(S_i)) \) is the process expression of \( \Sigma \), so a method to obtain the process expression of a Petri net can be presented, which is shown in Algorithm 1.

**Algorithm 1:** To obtain the process expression of a Petri net

**Input:** A Petri net \( \Sigma = (S, T, F, M_0) \)

**Output:** The process expression \( \text{Exp}(P(S)) \) of \( \Sigma \)

1. Decompose \( \Sigma \) into a set of S-Nets \( \Sigma_1, \Sigma_2, ..., \Sigma_k \). The main steps include:
   1. Define a function \( f : S \rightarrow \{1, 2, ..., k\} \) such that
      \[ \forall s_i, s_j \in S, \ (s_i \neq s_j) \Rightarrow \text{Exp}(P(S_i)) \cap \text{Exp}(P(S_j)) = \emptyset \]
   2. Decompose \( \Sigma \) into \( \Sigma_1, \Sigma_2, ..., \Sigma_k \) based on \( f \). For each \( \Sigma_i = (S_i, T_i, F_i, M_0) \), it satisfies
      \[ S_i = \{ s_i \text{if } f(s_i) = i \}, \ T_i = \{ t \in T \mid \exists s_i \in S_i \text{ and } f(s_i) = i \}, \ F_i = \{ (s_i, t) \cup (t, s_i) \mid (s, t) \in F \}, \text{ and } \]
      \[ M_i = M_0 \]
   3. **Output:** \( \text{Exp}(P(S)) = \bigcap_{i=1}^{k} \text{Exp}(P(S_i)) \)

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Fig. 7: Two new S-Nets $\Sigma_{21}$ and $\Sigma_{22}$

Fig. 8: The basic process section of $\Sigma_{123}$

earlier the process expression of $\Sigma_{123}$ is $\text{Exp} (P(\Sigma_{123})) = P_{111} \bullet^* P_{112}$, where the basic process sections $P_{111}$ and $P_{112}$ are shown in Fig. 8a. The part in $P_{112}$ containing a broken-line quadrangle should be deleted, since it is added in order to reduce the goal transition. The process expression of $\Sigma_{123}$ is $\text{Exp}(P(\Sigma_{123})) = (P_{211} \circ P_{222} \circ P_{233})^\Theta$. Similarly, the part in $P_{233}$ contained by a broken-line quadrangle should also be deleted. So, the process expression of $\Sigma_{123}$ is:

$$\text{Exp}(P(\Sigma_{123})) = \text{Exp}(P(\Sigma_{12})) \parallel \text{Exp}(P(\Sigma_{22})) = (P_{211} \bullet^* P_{212}) \parallel (P_{211} \circ P_{222} \circ P_{233})^\Theta$$

The third step in Algorithm 1 is to output the process expression of Petri net $\Sigma$.

$$\text{Exp}(P(\Sigma)) = \text{Exp}(P(\Sigma_{1})) \Theta \text{Exp}(P(\Sigma_{2})) \Theta \text{Exp}(P(\Sigma_{3}))$$

$$= (P_{111} \circ P_{112})^\Theta \parallel (P_{211} \bullet^* P_{212}) \parallel (P_{211} \circ P_{222} \circ P_{233})^\Theta P_{31}^*$$

CONCLUSION

A construction method for the process expression of a Petri net is proposed based on the index decomposition. With this decomposition method, a Petri net is decomposed into a set of S-Nets. The process properties of the S-Nets are easy to analyze since they are well-formed and their process expressions are easier to obtain than the original Petri net. With the process expressions of the index decomposition nets, the process expression of the original Petri net can be presented by the synchronization shuffle operation between processes.

There have been lots of works to obtain the process expression of a Petri net (Wu, 1996; Wu et al., 2000; Zeng and Wu, 2003). The method introduced in Wu (1996) only suits the bounded Petri net and the method of Wu et al. (2000) only suits the unbounded but fair Petri net. Zeng and Wu (2003) presents the method can be used for all kinds of Petri nets, however it is necessary to obtain the process net system first. Since the structure of the process net system is usually complex, it must decompose the process net system so as to obtain its language expression. The process of the method in Zeng and Wu (2003) is very complex. Compared with the related work, the construction method proposed in this work is easier to realize and it is useful for obtaining the process expressions of all kinds of Petri nets.

The semantics of the process expression containing $\Theta$ is not obvious. In the future, the reduction and optimization for the process expression containing $\Theta$ must be considered. The applications of the process expression for analyzing the properties such as liveness and fairness are also research work in the future.

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