Impulsive Exponential Consensus of Multi-agent Nonlinear Fuzzy Systems

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Abstract: In this study, we investigate the problem of impulsive exponential consensus of multi-agent systems, where each agent is represented by an identical T-S fuzzy model. Firstly, a fuzzy impulsive control protocol is designed for networks with fixed topology based on the local information of agents. Then sufficient conditions in terms of linear matrix equalities are given to guarantee the exponential consensus of the multi-agent fuzzy systems. Numerical simulations show the effectiveness of our theoretical results.

Key words: Multi-agent systems, fuzzy systems, T-S fuzzy model, consensus, impulsive control protocols, algebraic graph theory

INTRODUCTION

Since consensus is a kind of typical collective behaviors and basic motions in nature, the consensus problem of multi-agent systems has been intensively studied in various disciplines recently (Vicsek et al., 1995; Fax and Murray, 2004; Jadbabaie et al., 2003; Savkin, 2004; Tian and Liu, 2008).olfati-saber et al. (2007) investigated a systematical framework of consensus problem in networks of agents. Ren and Beard (2005) considered the problem of information consensus among multiple agents in the presence of limited and unreliable information exchange with dynamically changing interaction topologies. Ren (2008) proposed the consensus algorithms for double-integrator dynamics. The consensus problem for multi-agent systems was considered, in which all agents have an identical linear dynamic mode that can be of any order (Wang et al., 2008). Meanwhile, the consensus problem of multi-agent systems has also been considered by impulsive control method (Jiang and Bi, 2010; Guan et al., 2010; Wu et al., 2012; Zhang and Jiang, 2012; Jiang et al., 2011).

In the past few decades, the Takagi-Sugeno (T-S) fuzzy model has been used as an effective method to design and stability analysis of fuzzy control systems. More recently, stability analysis and synthesis of T-S fuzzy systems with impulse have gained considerable attention (Ho and Sun, 2007; Jiang et al., 2008; Zheng and Chen, 2009). Jiang et al. (2008) investigated the problem of robust fuzzy control for a class of nonlinear fuzzy impulsive systems with time-varying delay and proposed sufficient conditions for global exponential stability of systems.

Ho and Sun (2007) proposed some criteria for uniform stability and uniform asymptotic stability of T-S fuzzy time-delay systems with impulse were proposed. Zheng and Chen (2009) proposed a fuzzy impulsive controller for controlling chaotic dynamical systems by integrating T-S fuzzy model and impulsive control.

Motivated by the aforementioned discussions, in this study, we introduce fuzzy impulsive control protocols for multi-agent fuzzy systems. Firstly, a fuzzy impulsive protocol is designed for network with fixed topology based on the local information of agents. Then sufficient conditions in terms of linear matrix equalities are given to guarantee the exponential consensus of the multi-agent fuzzy systems by the theory of fuzzy systems and impulsive systems.

This study is organized as follows. In Section 2, we provide some results in matrix theory and algebraic graph theory. In Section 3, we formulate the consensus problem for multi-agent fuzzy systems and introduce an impulsive control protocol. The convergence analysis of the exponential consensus problem for network with fixed topology is discussed in Section 4. In Section 5, numerical simulations are included to show the effectiveness of our theoretical results. Some conclusions are drawn in Section 6.

Notation: Throughout this study, the superscripts ' -1' and 'T' stand for the inverse and transpose of a matrix, respectively; \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space; \( \mathbb{R}_n = [0, \infty), N = \{0, 1, 2, \ldots\}, N_+ = \{1, 2, \ldots\}; \mathbb{R}^n_{\times m} \) is the set of all \( n \times m \) real matrices; For real symmetric matrices \( X \) and \( Y \), the notation \( X \succeq Y \) (respectively, \( X \succ Y \)) means that the matrix \( X - Y \) is positive semi-definite (respectively, positive definite). In \( \mathbb{R}^n \) is an identity matrix; \( \lambda_{\min}(P) \) (\( \lambda_{\max}(P) \)) denotes the smallest (largest) eigenvalue of \( P \). For a vector \( x \in \mathbb{R}^n \), let \( ||x|| = \sqrt{x^T x} \). Let \( 1_n = (1, 1, \ldots, 1)^T \in \mathbb{R}^n \) and \( e_i \in \mathbb{R}^n \), \( e_i(i) = 1, e_j(j) = 0, j \neq i. \)

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PRELIMINARIES

In this section, we provide some results in matrix theory (Horn and Johnson, 1985, 1991) and algebraic graph theory (Godsil and Royle, 2001; Wang et al., 2008).

An undirected graph \( G \) of order \( N \) consists of a vertex set \( V = \{1, 2, \ldots, N\} \) and an edge set \( E = \{(i,j) : i,j \in V\} \). The set of neighbors of vertex \( i \) is denoted by \( N_i = \{j \in V : (i,j) \in E\} \). A path between each distinct vertices \( i \) and \( j \) is meant a sequence of distinct edges of \( G \) of the form \((i, k_1), (k_1, k_2), \ldots, (k_j, j)\). If there is a path between any two vertices of a graph \( G \), then \( G \) is connected, otherwise disconnected. A weighted adjacency matrix \( A = [a_{ij} \in \mathbb{R}^{n \times n}] \), where \( a_{ij} = 0 \) and \( a_{ij} > 0 \) if and only if there is an edge between vertex \( i \) and vertex \( j \). For an unweighted graph \( G \), \( A \) is a 0-1 matrix. The out-degree of vertex \( i \) is defined as follows:

\[
\deg_{out}(i) = \sum_{j=1}^{N} a_{ij}
\]

Let \( D \) be the diagonal matrix with the out-degree of each vertex along the diagonal and call it the degree matrix of \( G \). The Laplacian matrix of the weighted graph is defined as \( L = D - A \). For an unweighted graph \( G \):

\[
L = \{L_{ij}\}_{n \times n} \quad (1)
\]

Where:

\[
L_{ij} = \begin{cases} 
|N_i|, & i = j, \\
-1, & j \in N_i, \\
0, & \text{otherwise}. 
\end{cases}
\quad (2)
\]

By the definition, every row sum of \( L \) is zero.

**Lemma 1:** (Godsil and Royle, 2001; Wang et al., 2008) Let \( L \) be the Laplacian of an undirected graph \( G \) with \( N \) vertices, \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N \) be the eigenvalues of \( L \). Then:

- \( 0 \) is an eigenvalue of \( L \) and \( 1_n \) is the associated eigenvector, that is, \( L 1_n = 0 \).
- If \( G \) is connected, then \( \lambda_N = 0 \) is the algebraically simple eigenvalue of \( L \). 
- If \( 0 \) is the simple eigenvalue of \( L \), then it is an \( N \) multiplicity eigenvalue of \( L \) with the corresponding eigenvectors are \( 1_N \otimes e_i, i = 1, 2, \ldots, N \).

**PROBLEM FORMULATION**

Here, we consider a system consisting of \( N \) agents indexed by \( i = 1, 2, \ldots, N \). The dynamics of each agent are represented by the following T-S fuzzy model.

**Plant rule m:**

If \( \theta(t) \) is \( M_1^\mu \) and \( \ldots \), and \( \theta(t) \) is \( M_N^\mu \),

\[
\begin{align*}
\dot{x}(t) &= A_1 x(t) + u(t), & t &> t_0, \\
&= A_N x(t) + u(t), & t &\leq t_0, \\
&= 0, & t &= \infty,
\end{align*}
\]

where, \( M_i \) is the fuzzy set, \( q \) is the number of rules, \( \theta(t) = [\theta_1(t), \theta_2(t), \ldots, \theta_N(t)] \) is the premise variable, \( A_i \in \mathbb{R}^{n \times n} \) are constant matrices, \( x(t) = [x_1(t), x_2(t), \ldots, x_N(t)] \in \mathbb{R}^n \) and \( u(t) \in \mathbb{R}^n \) are the state and the control input of agent \( i \) at time \( t \), respectively.

Using the fuzzy inference method with singleton fuzzification, product inference and center average defuzzification, the overall fuzzy model of agent \( i \) has the following form:

\[
\begin{align*}
\dot{x}_i(t) &= \sum_{\mu=1}^{N} h_{\mu}(\theta(t))A_{\mu}x(t) + u(t), \\
x'_i(t) &\in \mathbb{R}^n, & i &= 1, 2, \ldots, N.
\end{align*}
\]  

Where:

\[
\begin{align*}
h_{\mu}(\theta(t)) &= \frac{w_{\mu}(\theta(t))}{\sum_{\nu=1}^{N} w_{\nu}(\theta(t))} \\
M_{\mu}(\theta(t)) &= \sum_{\nu=1}^{N} w_{\nu}(\theta(t))
\end{align*}
\]

We assume that \( \sum_{\nu=1}^{N} w_{\nu}(\theta(t)) > 0 \) and

\[
\sum_{\nu=1}^{N} w_{\nu}(\theta(t)) > 0
\]

It is clear that:

\[
h_{\mu}(\theta(t)) \geq 0, \sum_{\nu=1}^{N} h_{\nu}(\theta(t)) = 1
\]

The control input of agent \( i \) is designed as:

**Plant rule m:**

If \( \theta(t) \) is \( M_1^\mu \) and \( \ldots \), and \( \theta(t) \) is \( M_N^\mu \),

\[
\begin{align*}
\dot{x}(t) &= \sum_{k} \delta(t-t_k)B_{k}\sum_{\mu=1}^{N} h_{\mu}(\theta(t))\left( x_{\mu}(t)-x(t) \right), & k &\in \mathbb{N}, & i = 1, 2, \ldots, N, & m = 1, 2, \ldots, q
\end{align*}
\]

where, the discrete instants \( t_k \) satisfy \( 0 < t_0 < t_1 < \ldots < t_k < \ldots \) and:

\[
\lim_{k \to \infty} t_k = +\infty
\]

\( \delta(t) \) is the Dirac delta function, \( B_k \in \mathbb{R}^{n \times m} \) are constant matrices to be designed later, \( N \) is the set of neighbors of agent \( i \).
Using the same method as above, the overall control input of agent \(i\) has the following form:

\[
u^i(t) = \sum_{\tau = 1}^{\infty} h_{\tau}(t - \tau_0) \sum_{n=1}^{N} h_n(\theta(t))B_n^i \times \sum_{m=1}^{N} (x^i(t) - x^j(t)), m = 1, 2, \ldots, N,
\]

(6)

Adopting a similar approach (Zheng and Chen, 2009; Guan and Zhang, 2008), from Eq. 4 and 5 we have:

\[
x^i(t_{k+1}) = x^i(t_{k}) + \int_{t_k}^{t_{k+1}} \left( \sum_{n=1}^{N} h_n(\theta(s))A_n x^i(s) + u^i(s) \right) ds,
\]

where, \(\varepsilon > 0\) is sufficiently small. As \(\varepsilon \rightarrow 0^+\), this becomes to:

\[
\Delta x^i(t_k) = x^i(t_{k+1}) - x^i(t_k) = \sum_{n=1}^{N} h_n(\theta(t_k))B_n^i \sum_{m=1}^{N} (x^i(t_k) - x^j(t_k)).
\]

Where:

\[
x^i(t_{k+1}) = \lim_{t \rightarrow t_{k+1}} x^i(t)
\]

and:

\[
x^i(t_k) = \lim_{t \rightarrow t_k} x^i(t)
\]

Without loss of generality, we assume that:

\[
\lim_{t \rightarrow \infty} x^i(t) = x^i(t_k)
\]

which means that the solution \(x^i(t)\) is right continuous at time \(t_k\). This implies that the agent \(i\) will suddenly update its state variable according to the state variables of itself and its neighbors at the instants \(t_k\). Thus the control input \(u^i(t)\) is called a fuzzy impulsive control protocol.

Under the fuzzy impulsive control protocol Eq. 5, the dynamics of agent \(i\) satisfy the following equations:

\[
\begin{align*}
x^i(t) &= \sum_{\tau = 1}^{\infty} h_{\tau}(t) A^i x^i(t), t \neq t_k, \\
\Delta x^i(t_k) &= \sum_{n=1}^{N} h_n(\theta(t_k))B_n^i \sum_{m=1}^{N} (x^i(t_k) - x^j(t_k)), \\
i = 1, 2, \ldots, N, k \in N_n.
\end{align*}
\]

(7)

Definition 1: For the system Eq. 4, the exponential consensus is said to be achieved under the fuzzy impulsive control protocol Eq. 5 if there exist \(M, \gamma > 0\) such that:

\[
\|x^i(t) - x^j(t)\| \leq M e^{-\gamma t} \to 0, \\
t \to \infty, i, j = 1, 2, \ldots, N
\]

(8)

Consider the following fuzzy impulsive system:

\[
\begin{align*}
z(t) &= \sum_{n=1}^{N} h_n(\theta(t))C_n z(t), t \neq t_k, \\
\Delta z(t_k) &= \sum_{n=1}^{N} h_n(\theta(t_k))G_n^i z(t_k), k \in N_n.
\end{align*}
\]

(9)

**Lemma 2:** (Jiang et al., 2008) For a given positive constant \(\alpha\), if there exist \(\mu, \lambda \in \mathbb{R}^+\), positive constants

\[
0 < \mu \leq e^{\alpha(\mu - \lambda)}
\]

and symmetric and positive definite matrix \(X\), such that the following LMIs hold:

\[
XC_n^T + C_n X + 2\alpha X < 0
\]

(10)

\[
\begin{bmatrix}
-X & (X + G_n^T X) \\
X + G_n^T X & -X
\end{bmatrix} < 0
\]

(11)

then the system Eq. 9 is globally exponential stable. In this way, there exist \(M, \gamma > 0\) such that:

\[
\|z(t)\| \leq Me^{-\gamma t} \to 0, t \to \infty
\]

(12)

**NETWORK WITH FIXED TOPOLOGY**

In this section, we provide the convergence analysis of the consensus problem for network with fixed topology, i.e., \(G(t) = G\) for time \(t\).

Let \(x(t) = (x^1(t), x^2(t), \ldots, x^N(t))^T\), then the system Eq. 7 can be described as:

\[
\begin{align*}
x(t) &= (I_n \otimes \sum_{n=1}^{N} h_n(\theta(t))A_n) x(t), t \neq t_k, \\
\Delta x(t_k) &= (I_n \otimes \sum_{n=1}^{N} h_n(\theta(t_k))B_n^i \sum_{m=1}^{N} (x^i(t_k) - x^j(t_k)), k \in N_n.
\end{align*}
\]

(13)

Since \(L\) is symmetric, there is an orthogonal matrix \(Y \in \mathbb{R}^{N \times N}\) such that:

\[
YLY^{-T} = YLY^T = D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N),
\]

where \(\{\lambda_1, \lambda_2, \ldots, \lambda_N\}\) is the spectrum of \(L\).

Inspired by the method (Wang et al., 2008), let:

\[
x(t) = (Y \otimes I_N) x(t)
\]

(14)
Using the properties of Kronecker product, we have when $t \in [t_0, t_1)$, $k \in \mathbb{N}_+$:

$$
\dot{x}(t) = (Y \otimes I_{n})x(t) = (Y \otimes I_{n})I_{n} \otimes \left( \sum_{n=1}^{N} f_{n}(\theta(t)) A_{n} \right)x(t)
$$

$$
= (Y \otimes I_{n})I_{n} \otimes \left( \sum_{n=1}^{N} f_{n}(\theta(t)) A_{n} \right)(Y \otimes I_{n})^{-1}\dot{x}(t)
$$

$$
= (I_{n} \otimes \left( \sum_{n=1}^{N} f_{n}(\theta(t)) A_{n} \right))\dot{x}(t)
$$

and:

$$
\Delta \dot{x}(t_k) = (Y \otimes I_{n})\Delta x(t_k)
$$

$$
= (Y \otimes I_{n})(I_{n} \otimes \left( \sum_{n=1}^{N} f_{n}(\theta(t_k^n)) B_{n} \right))
$$

$$
\times (-L \otimes I_{n})(Y \otimes I_{n})^{-1}\dot{x}(t_k)
$$

$$
= (-D \otimes \left( \sum_{n=1}^{N} f_{n}(\theta(t_k^n)) B_{n} \right))\dot{x}(t_k)
$$

Thus Eq. 13 becomes to:

$$
\begin{cases}
\dot{x}(t) = (I_{n} \otimes \left( \sum_{n=1}^{N} f_{n}(\theta(t)) A_{n} \right))x(t), t \in (t_k, t_{k+1}) \\
\Delta \dot{x}(t_k) = (I_{n} \otimes \left( \sum_{n=1}^{N} f_{n}(\theta(t_k^n)) B_{n} \right))\dot{x}(t_k)
\end{cases}
$$

(15)

Therefore:

$$
\begin{cases}
\dot{x}(t) = \left( \sum_{n=1}^{N} f_{n}(\theta(t)) A_{n} \right)\dot{x}(t), t \in (t_k, t_{k+1}) \\
\Delta \dot{x}(t_k) = \lambda_{k} \left( \sum_{n=1}^{N} f_{n}(\theta(t_k^n)) B_{n} \right)\dot{x}(t_k)
\end{cases}
$$

(16)

**Theorem 1:** Consider the system Eq. 4. Assume that the graph $G$ of the network is connected. For a given positive constant $\alpha$, if there exist discrete instants $t_k$, positive constants $0 < \eta_k < e^{\alpha N}$, symmetric and positive definite matrix $X$, such that the following LMIs hold for $i = 2, \ldots, N, k \in \mathbb{N}_+$:

$$
X A_{n}^T + A_{n} X + 2\alpha X < 0
$$

(17)

$$
\begin{bmatrix}
-\eta_{X} & (X + \lambda_{k} B_{n}^T X)_{i}^{T} \\
(X + \lambda_{k} B_{n}^T X)_{i} & -X
\end{bmatrix} < 0
$$

(18)

then the exponential consensus is achieved under the fuzzy impulsive control protocol Eq. 5.

**Proof:** Since the graph $G$ is connected, by Lemma 1, $\lambda_i = 0$ is the algebraically simple eigenvalue of $L$. All the other eigenvalues of $L$ are positive. Then we have:

$$
0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n
$$

It can be verified that:

$$
\begin{cases}
(L \otimes I_{n})x(t) \\
= (Y \otimes I_{n})(L \otimes I_{n})(Y \otimes I_{n})^{-1}x(t)
\end{cases}
$$

$$
= (Y \otimes I_{n})(D \otimes I_{n})x(t)
$$

$$
= (Y \otimes I_{n})^{-1}\begin{bmatrix}
0 & \lambda_1 \hat{x}(t) & \cdots & \lambda_n \hat{x}(t)
\end{bmatrix}^{T}
$$

For a given positive constant $\alpha$, if there exist discrete instants $t_k$, impulsive matrices $B_{n}$, constants $0 < \eta_k < e^{\alpha N}$, symmetric and positive definite matrix $X$, such that the following LMIs hold for $i = 2, \ldots, N, i = 2, \ldots, n$ by Lemma 2 the system Eq. 16 is exponentially stable, i.e. $\|x(t)\| \leq M e^{-\alpha t} \rightarrow 0$, $t \rightarrow +\infty$, $i = 2, \ldots, n$, then:

$$
\|L \otimes I_{n}x(t)\| < \|Y \otimes I_{n}\|^{-1} \max \{M_1, \ldots, M_n\} e^{-\alpha N} \rightarrow 0, t \rightarrow \infty
$$

Since the graph $G$ is connected, by Lemma 1, $0$ is the eigenvalue of $L \otimes I_{n}$ with multiplicity $n$. The $n$ linearly independent eigenvectors associated with the eigenvalue $0$ of $L \otimes I_{n}$ are $1_{\eta} \xi_{i}$, $i = 1, 2, \ldots, n$. Therefore, $x \rightarrow 1_{\eta} \xi_{i}$, $t \rightarrow +\infty$, where:

$$
|s| = \sum_{i=1}^{n} \xi_{i} \xi_{i}^{T} \in \mathbb{R}^{n}
$$

$\xi \in \mathbb{R}^{n}$, $i = 1, 2, \ldots, n$. Thus the system Eq. 4 achieves the exponential consensus under the fuzzy impulsive control protocol Eq. 5. This completes the proof.

**SIMULATIONS**

Consider the following multi-agent fuzzy systems:

**Plant rule m:**

\[
\text{IF } x_{i}(t) \text{ is } M_{i}^{a} \text{ and } x_{j}(t) \text{ is } M_{j}^{b} \text{ THEN } \dot{x}(t) = A_{n}x(t) + u(t), t \geq t_{0} \geq 0, i, j = 1, 2, m = 1, 2, 3, 4,
\]

where, $x(t) = (x_{1}(t), x_{2}(t), x_{3}(t))^{T}$, $a = 10$, $b = 28$, $c = 8/3$, $\alpha = -20$, $\beta = 30$, $M_{1}^{10} = 1, 2, m = 1, 2, 3, 4$ are fuzzy sets which represent “Large” and “Small” in [-20, 30], respectively.

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The control input of agent $i$ is designed as:

**Plant rule m:**

$$u^i(t) = \sum_{k=1}^{N_m} \sum_{t_k} \delta(t-t_k) B_k^i (x^i(t) - x^i(t)).$$

Since the graph is a path on two vertices, we get $\lambda_2 = 2$. For simplicity, the impulsive matrices $B_k^i, k \in \mathbb{N}_a$, are chosen as $p^i_k I$, where, $0 < p = 0.2$. Choose the equidistant impulsive interval $t_k - t_{k-1} = -\Delta - 0.025$. By solving the linear matrix equalities, we get:

$$X = \begin{bmatrix} 0.2540 & -0.3992 & 0.1283 \\ -0.3992 & 0.4397 & 0.0541 \\ 0.1283 & 0.0541 & 0.7695 \end{bmatrix}$$

Thus the conditions of Theorem 1 are satisfied. Simulation results are shown in Fig. 1-3. The simulation results show that the fuzzy impulsive protocol is efficient to solve the exponential consensus problem.

**CONCLUSION**

In this study, we have introduced fuzzy impulsive control protocols for multi-agent fuzzy systems. Convergence analysis of the fuzzy impulsive control protocol is presented.

**ACKNOWLEDGMENTS**

This study is partially supported by the National Nature Science Foundation of China (Grant Nos. 11202180, 61273106, 11171290) and the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (Grant No. 10KJB510026).

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