**T$_{ir}$-interval-valued Fuzzy Rings and Their Homomorphism Properties**

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**Abstract:** In this study, the concept of T$_{ir}$-interval-valued fuzzy ring is first introduced based on the notion of fuzzy ring, then some meaningful properties of T$_{ir}$-interval-valued fuzzy ring are investigated. The results show that fuzzy ring and interval-valued fuzzy ring are special cases of T$_{ir}$-interval-valued fuzzy ring.

**Key words:** Interval-valued fuzzy set, T$_{ir}$-interval-valued fuzzy ring, homomorphism

**INTRODUCTION**

The research on interval-valued fuzzy set has attracted many authors’ attention. For example, Biswas (1994) first introduced the concept of interval-valued fuzzy set to discuss fuzzy algebra and studied interval-valued fuzzy group. Sun and Gu (1998) investigated the properties of fuzzy algebra based on interval-valued fuzzy set. Wang and Zhou (1997) and Li and Wang (2000) proposed the concept of T$_{ir}$-interval-valued fuzzy group and T$_{ir}$-interval-valued fuzzy group and studied their properties.

In this study, we first introduce the concept of T$_{ir}$-interval-valued fuzzy ring. Then, the corresponding properties of T$_{ir}$-interval-valued fuzzy ring are studied. Through the remarks below, one would find that the results of this study are an interesting and meaningful extension of fuzzy ring and interval-valued fuzzy ring.

**PRELIMINARIES**

Here, some basic notions and notations of fuzzy set are reviewed, the detailed descriptions could be found by Zadeh (1965), Mengers (1993) and Kumar et al. (1992).

**Definition 1:** Let $X$ be a crisp set. The mapping $A : X \rightarrow [I]$ is called an interval-valued fuzzy set based on $X$. We denote by $(X)$ the set of all the fuzzy set on $X$. For any $A \in$ IF($X$), if we define that $A(x) = [A^-(x), A^+(x)]$, $A^-(x) = A^-(x, X)$, $A^+(x) = A^+(x, X)$, then $A = A^-(X) = A^+(X)$ are called the lower fuzzy set and the upper fuzzy set of $X$, respectively.

For any $x \in X$, $A \in$ IF($X$), one can define that $(A \cup B)(x) = A(x) \vee B(x)$, $(A \cap B)(x) = A(x) \wedge B(x)$.

For any $A \in$ IF($X$), $[\lambda_b, \lambda_c]$, we have that $A_{[\lambda_b, \lambda_c]} = \{x \in X | A^-(x) \geq \lambda_b, A^+(x) \geq \lambda_c\}$, $A_{[\lambda_b, \lambda_c]}$ is called the $[\lambda_b, \lambda_c]$-cut sets of $A$. Obviously, we have that $A_{[\lambda_b, \lambda_c]} = A_{[\lambda_b, \infty]} \cap A_{[\lambda_c, \infty]}$, $A_{[0, \infty]} = X$.

For any $A, B \in$ IF($X$) and $[\lambda_b, \lambda_c], [\lambda_d, \lambda_e] \subseteq [I]$, the following two equations are satisfied:

$(A \cup B)_{[\lambda_b, \lambda_c]} = A_{[\lambda_b, \lambda_c]} \cup B_{[\lambda_d, \lambda_e]} \cup (A_{\lambda_b} \cap B_{\lambda_e})$

$(A \cap B)_{[\lambda_b, \lambda_c]} = A_{[\lambda_b, \lambda_c]} \cap B_{[\lambda_d, \lambda_e]}$

**Definition 2:** A mapping $T : I \times I \rightarrow I$ is said to be an idempotent norm, if the following conditions satisfy, where $a, b, c, d \in I$:

1. If $a \leq b \leq c$ then $T(a, b) \leq T(c, d)$
2. $T(a, b) = T(b, a)$
3. $T(T(a, b), c) = T(a, T(b, c))$
4. $T(a, 0) = 0, T(a, 1) = a$
5. $T(a, a) = a$

**Proposition 1:** If $T$ is an idempotent norm, then for any $a, b, c, d \in I$ we have:

$T(a, b) \leq T(c, d) \Rightarrow T(a \wedge c, b \wedge d)$

**Proof:** Noting that $a \leq a \wedge c, b \leq b \wedge d$ according to (1) in Definition 2, $T(a, b) \leq T(a \wedge c, b \wedge d)$ is satisfied. Analogously, $T(c, d) \leq T(a, b) \Rightarrow T(a, b) \leq T(a \wedge c, b \wedge d)$.

**Definition 3:** A mapping $T_{ir} : [I] \times [I] \rightarrow [I]$ is said to be an idempotent interval normal, if for any $a, b \in [I]$, $T_{ir} = [T(a, b)]$, $T(a, b) = [T(a^-, b^+), T(a^+, b^-)]$ is satisfied, where $T$ is an idempotent norm.

**Proposition 2:** For any $a, b, c, d \in [I]$, if $a \leq b, c \leq d$, then $T_{ir}(a \wedge b, c \wedge d)$.

**Proof:** If $a \leq b, c \leq d$ then we have that $a^{-} \leq c^{-}, a^{+} \leq c^{+}$, $b^{-} \leq d^{-}, b^{+} \leq d^{+}$. According to Definition 2, $T(a^{-}, b^{-}) \leq T(c^{-}, d^{-})$ and $T(a^{+}, b^{+}) \leq T(c^{+}, d^{+})$. Therefore:

Theorem 2: Let R be a ring. If A_i and A_j are two interval-valued fuzzy rings of R, then A_i ∩ A_j is a T^i_R interval-valued fuzzy ring of R.

Proof:
- For any a, b ∈ R, we have that: (A_i ∩ A_j)(a+b) = A_i(a+b) ∩ A_j(a+b).
- (A_i ∩ A_j)(ab) = A_i(ab) ∩ A_j(ab).

and (A_i ∩ A_j)(a) = A_i(a) ∩ A_j(a).

Thus, according to Definition 5, A_i ∩ A_j is a T^i_R interval-valued fuzzy ring of R.

**Homomorphism Properties**

Here, we will discuss the homomorphism properties about the T^i_R interval-valued fuzzy ring.

Definition 6: Let R_i and R_j be two rings and a mapping ϕ between R_i and R_j is given as ϕ: R_i → R_j. Suppose ϕ: IF(R_i) → IF(R_j) and ϕ^-1: IF(R_j) → IF(R_i), they are denoted as:

\[ \psi(A)(y) = \text{sup}_{x \in [0,1]} A(x) \psi^i(y) = \Phi, \quad y \in R \]

\[ \psi^i(B)(x) = B(\psi(x)), \quad \forall x \in R_i \]

where, A ∈ IF(R_i), B ∈ IF(R_j), ψ^i(y) = \{x ∈ R_i | ϕ(x) = y\}. Then ϕ and ϕ^-1 are respectively said to be interval-valued fuzzy transformation and interval-valued fuzzy inverse transformation generated by ϕ. From Definition 5, we can easily get the following two equations:

\[ \psi^i(A)(y) = \left[ \bigvee_{x \in [0,1]} A^+(x), \bigvee_{x \in [0,1]} A^-(x) \right] \]

\[ \psi^i(A)(y) = \left[ \bigwedge_{x \in [0,1]} A^+(x), \bigwedge_{x \in [0,1]} A^-(x) \right] \] for any y ∈ R_j.

\[ \psi(A(x)) = \begin{cases} \min(b^+(\psi(x)), b^+(\psi(x))) & \text{for any } x \in R_1 \\ \max(b^+(\psi(x)), b^+(\psi(x))) & \end{cases} \]

**Theorem 3:** Let \( R_1 \) and \( R_2 \) be two rings, \( \phi: R_1 \rightarrow R_2 \), a homomorphism mapping from \( R_1 \) to \( R_2 \), and \( \psi \) interval-valued fuzzy transformation generated by \( \phi \). If \( A_i \) is a \( T_{(1)} \)-interval-valued fuzzy ring of \( R_1 \), \( A_i = \phi(A_i) \), then \( A_i \) is a \( T_{(1)} \)-interval-valued fuzzy ring of \( R_2 \).

**Proof:**

- Firstly, if for any \( y \in R_1 \), there is \( \psi^{-1}(y) = \phi \), then we can conclude that \( \psi^{-1}(y) = \phi \). In fact, if there exists \( x \in \psi^{-1}(y) \), then there must have \( \psi(x) = y \). Since \( \psi \) is a homomorphism mapping, we have that \( \psi(x) = y \). Consequently, \( \psi^{-1}(y) = x \). This is contradicted with hypotheses. By Definition 4.1, it follows that:

\[ A_i(y) = \phi(A_i)(y) = A_i(y) \]

- Secondly, if \( \psi^{-1}(y) + \phi \), we have that:

\[ A_i(y) = \phi(A_i)(y) = \psi^{-1}(y) A_i(x) \]

Hence, for all \( y \in R_1 \), we have:

\[ A_i(x) = \phi(A_i)(x) = \psi^{-1}(y) A_i(x) \]

**Theorem 4:** Let \( R_1 \) and \( R_2 \) be two rings, \( \phi: R_1 \rightarrow R_2 \), a homomorphism mapping from \( R_1 \) to \( R_2 \), and \( \psi \) interval-valued fuzzy transformation generated by \( \phi \). If \( A_i \) is a \( T_{(1)} \)-interval-valued fuzzy ring of \( R_1 \), then \( A_i \) is a \( T_{(1)} \)-interval-valued fuzzy ring of \( R_2 \).

**Proof:** For any \( x \), we have:

\[ A_i(x) = \phi(A_i)(x) = A_i(x) = A_i(y) \]

Thus, we conclude that there exist \( x_i, x_j \in R_1 \) satisfying \( \psi(x_i) = y_i, \psi(x_j) = y_j \) and \( A_i(y_i+y_j) = A_i(y_i) \).

Noting that \( \psi \) is an epimorphism mapping, we have that \( \psi(x_i) = y_i, \psi(x_j) = y_j \) such that:

\[ A_i(y_i+y_j) = A_i(y_i) \]

Hence, \( A_i(x_i+y_j) = A_i(x_i) \).

This is contradicted with the fact that \( A_i \) is a \( T_{(1)} \)-interval-valued fuzzy ring of \( R_1 \). Therefore, for all \( y_i, y_j \in R_2 \), we have:

\[ A_i(y_i+y_j) = A_i(y_i) \]
Therefore, according to Definition 5, we conclude that $A_i$ is a $T_{ir}$-interval-valued fuzzy ring of $R_i$.

**CONCLUSION**

In this paper we have studied the problem of fuzzy ring. We have proposed the notion of $T_{ir}$-interval-valued fuzzy ring. And based on the notion, the corresponding homomorphism properties have been researched and some interesting results have been got. In the future we may involve in the investigation of the isomorphic properties of $T_{ir}$-interval-valued fuzzy ring.

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