Non-equidistant Multivariable New Information Optimizing MGRM(1,n) Based on Improving Background Value and Accumulated Generating Operation of Reciprocal Number

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Abstract: The background value is an important factor affecting the precision of non-equidistant multivariable MGM(1,n) model. Based on index characteristic of grey model GM(1,1), the characteristic of integral, improving the constructing method of background value, the function with non-homogeneous exponential law was used to fit the accumulated sequence via three points to obtain the background value of non-equidistant multivariable MGM(1,n) model, taking the mean relative error as objective function, taking the modified values of response function initial value, a new non-equidistant multivariable new information optimizing MGRM(1,n) model based on accumulated generating operation of reciprocal number and improving background value was put forward which was taken the mth component as the initialization. The proposed MGRM(1,n) model can be used in non-equal interval and equal interval time series and has the characteristic of high precision as well as high adaptability. Example validates the practicability and reliability of the proposed model.

Key words: Multivariable, background value, accumulated generating operation of reciprocal number, non-equidistance sequence, new information, optimizing, non-equidistance MGM(1,n) model, least square method

INTRODUCTION

The Grey predictive model is one important part in the grey theory and the GM(1,1) model is the basic predictive model in the Grey model, which is characteristic with a little samples and poor information and have been applied in many fields. Until now with researching GM (1,1) model deeper and extensive it has been to most accepted model in the grey theory. MGM(1,n) model is the expanded form of GM(1,1) model in the n variables and the parameters can reflect the affection and relationship between every variables. However the research about the MGM(1,N) model is not as common as GM(1,1) model. Zhai et al. (1997) established MGM(1,N) model. Literature Luo and Li (2009) took condition and establish optimized MGM(1,N) model. Based on the principles of the prior new information He and Luo (2009) took the nth vector as the initial conditions and established the multi-variables new information MGM(1,N) model. Luo and Xiao (2009) took the nth vector of x(0) as the initial conditions and establish the multi-variables new information MGM(1,N) model, in which the initial values and backgrounds values are optimized. But these model is the equal interval model. Wang (2007) adopted homogeneous index function to fit background values and establish the unequal interval MGM(1,N) model but the non-homogeneous index functions are more common and the above establishment mechanism of the model has some defects. Pingping et al. (2011) established the multi-variables unequal MGM(1,N) model and the background values are got by middle values which make the accuracy need to be improved further. Xiong et al. (2010) adopted the non-homogenous index functions to fit background values and establish the unequal multi-variables grey MGM(1,N) model and the accuracy is improved greatly. Luo and He (2009), Dai and Li (2005), Wang et al. (2008), Luo and He (2009) provided many different forms of background values and establish a few non-equal GM(1,1) model. In the grey theory how to expand GM(1,1) model to MGM(1,N) and establishment an unequal MGM(1,N) have great meaning. For the grey model accumulated operation is the key and reciprocal is the supplement. For the non-negative discrete data x(0) the data after AGO process is monotonic increasing. It is reasonable that the curve is monotonic increasing which is used to fit x(0). But if x(0) is monotonic decreasing that the AGO operation determines x(0) is monotonic increasing. So the fitting x(0) model is monotonic increasing. By the

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The non-equidistant multivariable new information optimizing MGM(1,n) based on improving background value and accumulated generating operation of reciprocal number

**Definition 1:** Given sequence \( X^{(o)} = [X_1^{(o)}(t_1), \ldots, X_n^{(o)}(t_n)] \). If \( \Delta t_i = t_i-t_{1,i} \) const, \( i = 1, 2, \ldots, n \), \( x_j \) is variables, \( m \) is the number of variables then \( x_j^{(o)} \) is the unequal sequence. Given:

\[
x_j^{(o)}(t_i) = \frac{1}{\Delta t_i}(j = 1, 2, \ldots, m)
\]

then \( x_j^{(o)}(t_i) = x_j^{(o)}(t_1), \ldots, x_j^{(o)}(t_n) \) is the reciprocal sequence of \( X^{(o)} \).

**Definition 2:** Given sequence \( X^{(s)} = \{X_1^{(s)}(t_1), X_1^{(s)}(t_2), \ldots, X_1^{(s)}(t_n), \ldots, X_n^{(s)}(t_1), \ldots, X_n^{(s)}(t_n) \} \). If \( x_j^{(s)}(t_i) = x_j^{(s)}(t_1), x_j^{(s)}(t_2) = x_j^{(s)}(t_2) + x_j^{(s)}(t_1) \).

\[ \Delta t_i, j = 2, \ldots, m, i = 1, 2, \ldots, n, \Delta t_i = t_{i+1} - t_i \], then \( X^{(o)} \) is the one order accumulated operation \( (1\text{-AGO}) \) of the unequal sequence \( X^{(o)} \).

If the original matrix of the multi-variables is:

\[
X^{(o)} = \begin{bmatrix}
X_1^{(o)}(t_1) & X_1^{(o)}(t_2) & \cdots & X_1^{(o)}(t_m)
\end{bmatrix}^T
\]

\[ \vdots \]

\[
\begin{bmatrix}
X_n^{(o)}(t_1) & X_n^{(o)}(t_2) & \cdots & X_n^{(o)}(t_m)
\end{bmatrix}
\]

where, \( x_j^{(o)}(t_i) = [x_j^{(o)}(t_1), x_j^{(o)}(t_2), \ldots, x_j^{(o)}(t_n)] \) is the objective values at \( t_i \) for \( X^{(o)}(t) \) \( (j = 1, 2, \ldots, m) \). Sequence \( \{x_j^{(o)}(t_1), x_j^{(o)}(t_2), \ldots, x_j^{(o)}(t_i), x_j^{(o)}(t_n)\} \) \( (i = 1, 2, \ldots, n, j = 1, 2, \ldots, m) \) is unequal which means \( t_i-t_{i-1} \) is not const.

In order to establish the model we accumulated the original data and formed the new matrix:

\[
X^{(o)} = \begin{bmatrix}
x_1^{(o)}(t_1) & x_1^{(o)}(t_2) & \cdots & x_1^{(o)}(t_m)
\end{bmatrix}^T
\]

\[ \vdots \]

\[
\begin{bmatrix}
x_n^{(o)}(t_1) & x_n^{(o)}(t_2) & \cdots & x_n^{(o)}(t_m)
\end{bmatrix}
\]

where, \( x_j^{(o)}(t_j) \) \( (i = 1, 2, \ldots, n) \) satisfied the definition 2, namely:

\[
x_j^{(o)}(t_j) = \sum_{k=1}^{i} x_j^{(o)}(t_k)(t_j - t_{k-1}) \quad (k = 2, \ldots, m)
\]

The \( n \)-variables first order differential equations of unequal MGM(1,n) model based on reciprocal accumulated operation is:

\[
\begin{align*}
\frac{dx_1^{(o)}}{dt} &= a_{11}x_1^{(o)} + a_{12}x_2^{(o)} + \cdots + a_{1n}x_n^{(o)} + b_1 \\
\frac{dx_2^{(o)}}{dt} &= a_{21}x_1^{(o)} + a_{22}x_2^{(o)} + \cdots + a_{2n}x_n^{(o)} + b_2 \\
& \quad \vdots \\
\frac{dx_n^{(o)}}{dt} &= a_{n1}x_1^{(o)} + a_{n2}x_2^{(o)} + \cdots + a_{nn}x_n^{(o)} + b_n
\end{align*}
\]

Given \( A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}, \quad B = \begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_n
\end{bmatrix}\]

and the Eq. 4 can be described as:
\[
\frac{dX^{(0)}(t)}{dt} = AX^{(0)}(t) + B
\]

(5)

Based on the prior new information principles of grey theory, if the first vector of \(X^{(0)}(t)\) is taken as the initial conditions we do not make full use of the new information. So taken the mth vector of \(X^{(0)}(t_m)\) as the initial conditions we can make full use of the newest information. The continuous time response equation in Eq. 5 is:

\[
X^{(0)}(t) = e^{At}X^{(0)}(t_m) + \sum_{i=1}^{m} A^i (e^{At} - I)B
\]

(6)

We take the m vector in the original sequence as the initial values in grey differential equations and fix it, which substitute \(X^{(0)}(t_m) + \beta\) into \(X^{(0)}(t_m)\) where \(\beta\) and \(X^{(0)}(t_m)\) have the same column, namely, \(\beta = [\beta_1, \beta_2, ..., \beta_m]^T\). After restoring the original data we can get:

\[
\tilde{X}(t) = \begin{cases} 
\lim_{\Delta t \to 0} \frac{X^{(0)}(t_j) - X^{(0)}(t_{j+1})}{\Delta t}, & j = 1 \\
(\tilde{X}^{(j)}(t_j) - \tilde{X}^{(j)}(t_{j+1}))/t_{j+1} - t_j, & j = 2, 3, ..., m 
\end{cases}
\]

(7)

Where:

\[
e^{At} = I + \sum_{k=1}^{\infty} \frac{A^k}{k!}I
\]

I+ is the unit matrix.

In order to identify A and B, we integrate the formula (4) between \([t_j, t_{j+1}]\) and get:

\[
x^{(0)}(t_j)\Delta t_j = \sum_{i=1}^{m} a_{ij} \int_{t_j}^{t_{j+1}} x^{(i)}(t)dt + b_{ij} \Delta t_j, i = 1, 2, ..., n; j = 2, 3, ..., m
\]

(8)

Given:

\[
z^{(i)}(t_j) = \int_{t_j}^{t_{j+1}} x^{(i)}(t)dt
\]

The traditional background values is equal to \(x^{(0)}(t)\Delta t\) of trapezoid area. When the interval is little and the change is not large the construction of the background values is appropriate. When the sequence change large the error of the background values is large. So the traditional construction of the background values in new information model is un-appropriate. So taken:

\[
z^{(i)}(t_j) = \int_{t_j}^{t_{j+1}} x^{(i)}(t)dt
\]

as the background values to get the estimated parameters will be more fit for the whitening Eq. 4. According to the index law of grey predictive model and the principle of Luo and He (2009) of the unequal interval GM(1,1) model, we supposed \(x^{(0)}(t) = A_i e^{B t} + C_i\) where \(A_i, B_i, C_i\) is undefined coefficient.

Given that \(x^{(0)}(t) = A_i e^{B t} + C_i\) is through \((t_j, x^{(i)}(t_j)), (t_{j+1}, x^{(i)}(t_{j+1}))\) then:

\[
x^{(i)}(t_j) = A_i e^{B t_j} + C_i, x^{(i)}(t_{j+1}) = A_i e^{B t_{j+1}} + C_i, x^{(i)}(t_{j+1}) = A_i e^{B t_{j+1}} + C_i
\]

(9)

From the Eq. 9 the unknown parameters \(A_i, B_i, C_i\) are:

\[
B_i = \ln\left(\frac{x^{(i)}(t_{j+1}) - x^{(i)}(t_j)}{x^{(i)}(t_{j+1}) - x^{(i)}(t_j)}\right), j = 3, 4, ..., m
\]

(10)

\[
A_i = \frac{x^{(i)}(t_{j+1}) - x^{(i)}(t_j)}{e^{B_i t_{j+1}} - e^{B_i t_j}}
\]

(11)

\[
C_i = x^{(i)}(t_j) - A_i e^{B_i t_j}
\]

(12)

Put, Eq. 10-12 into the background equation:

\[
\int_{t_j}^{t_{j+1}} x^{(i)}(t)dt
\]

and get:

\[
x^{(i)}(t_{j+1}) - x^{(i)}(t_j) = \int_{t_j}^{t_{j+1}} (A_i e^{B t} + C_i) dt + \int_{t_j}^{t_{j+1}} (A_i e^{B t} + C_i) dt + \int_{t_j}^{t_{j+1}} (A_i e^{B t} + C_i) dt
\]

(13)

Given \(a_i = [a_{i1}, a_{i2}, ..., a_{in}, b]_T\) (\(i = 1, 2, ..., n\), from the least square method we can get the estimated \(\hat{a}_i\):

\[
\hat{a}_i = [\hat{a}_{i1}, \hat{a}_{i2}, ..., \hat{a}_{in}, \hat{b}]_T = (L^T L)^{-1} L^T Y, i = 1, 2, ..., n
\]

(14)

Where:

\[
L = \begin{bmatrix}
z^{(i)}(t_2) & z^{(i)}(t_3) & ... & z^{(i)}(t_n) & A_i t_2 \\
z^{(i)}(t_3) & z^{(i)}(t_4) & ... & z^{(i)}(t_n) & A_i t_3 \\
... & ... & ... & ... & ... \\
z^{(i)}(t_m) & z^{(i)}(t_{m+1}) & ... & z^{(i)}(t_n) & A_i t_m
\end{bmatrix}
\]

(15)

\[
Y_i = [x^{(i)}(t_j) A_i t_2, x^{(i)}(t_j) A_i t_3, ..., x^{(i)}(t_j) A_i t_m]^T
\]

(16)

and get the discrimination value of \(A, B, C\):

\[
\hat{\lambda} = \begin{bmatrix}
\hat{a}_1 & \hat{a}_2 & \cdots & \hat{a}_n \\
\hat{a}_2 & \hat{a}_2 & \cdots & \hat{a}_n \\
\vdots & \vdots & \ddots & \vdots \\
\hat{a}_n & \hat{a}_2 & \cdots & \hat{a}_n \\
\end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}
\]

The equation of MGRM(1,n) model is:

\[
\hat{X}_{i}^{(t)}(t_j) = e^{(h_{ij}-t_j)}X_{i}^{(0)}(t_m) + \hat{\lambda}^{-1}(e^{(h_{ij}-t_j)} - 1)\hat{b}, j = 1, 2, \ldots, m
\]

(18)

The fitting values of the original sequence after restoring:

\[
\hat{X}(t_j) = \begin{cases}
\frac{\lim_{\Delta t \to 0}}{\Delta t} \frac{X_{i}^{(0)}(t_j) - X_{i}^{(0)}(t_j - \Delta t)}{X_{i}^{(0)}(t_j - \Delta t)}, & j = 1, 2, \ldots, m \\
\frac{\hat{X}_{i}^{(0)}(t_j) - X_{i}^{(0)}(t_j - \Delta t)}{X_{i}^{(0)}(t_j - \Delta t)}, & j = 2, 3, \ldots, m
\end{cases}
\]

(19)

Using the definition 1 the values of the original sequence in the model \(x_{i}^{(0)}(t)(j = 1, 2, \ldots, m)\) can be got.

The absolute error of the ith variable is: \(e_{i}(t) = x_{i}^{(0)}(t) - x_{i}^{(0)}(t)\). The relative error (%) of the ith variable is:

\[
e_{i}(t) = \left| \frac{x_{i}^{(0)}(t) - x_{i}^{(0)}(t)}{x_{i}^{(0)}(t)} \right| \times 100
\]

The average value of the relative error (%) of the ith variable is:

\[
\frac{1}{m} \sum_{j=1}^{m} e_{i}(t_j)
\]

The average error values of all data are:

\[
f = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \left| e_{i}(t_j) \right|
\]

(20)

Taken the average error \(f\) as the objective function and \(\hat{\beta}\) is the design variables, we use the MATLAB 7.5 to optimize the Fmincon or other method to solve it.

**EXAMPLES**

**Example 1:** In the calculation for contact strength, curvature function \(F(\rho)\) and the parameters \(m_a, m_b\) of the long, short radius, \(a, b\) in the point intact elliptical are got by the Table 1. The data is from Han and Dong (2008a).

\(T\) is the coefficient \(m_a\) of the short radius in the ellipse and \(x_i\) is the curvature function \(F(\rho)\), \(x_i\) is the coefficient \(m_b\) of the long radius in the ellipse. According to the reciprocal accumulated operation and improved background values we establish the new information optimized MGRM(1,2) model and the parameters are follows:

\[
A = \begin{bmatrix}
-0.1576 \\
0.1605 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
2.6534 \\
5.3928 \\
\end{bmatrix}, \quad \beta = \begin{bmatrix}
1.0488 \\
-0.3329 \\
\end{bmatrix}, \quad r = \begin{bmatrix}
0.15357 \\
-0.11719 \\
\end{bmatrix}
\]

• **The Fitting value of the curvature function \(F(\rho)\):**

\(\hat{F}(\rho) = [0.99822, 0.99833, 0.99805, 0.99715, 0.99612, 0.99505, 0.99399, 0.9929, 0.99182, 0.99079, 0.98977, 0.98876, 0.98778, 0.98686, 0.986, 0.98499, 0.98393, 0.98307, 0.98218, 0.98113, 0.98018, 0.9792, 0.97818, 0.97728]^{T}\)

• **The absolute error of curvature function \(f(\rho)\):**

\[q = [-0.0012758, -0.00066797, 0.0643e-005, 0.00014889, 0.00012112, 5.4785e-005, -6.4826e-006, -9.8755e-005, -0.00017895, -0.00020648, -0.00023364, -0.00024143, -0.0002157, -0.00013924, 4.94e-007, -1.2435e-005, -7.1179e-005, 7.1857e-005, 0.0001836, 0.00013227, 0.00018026, 0.00019553, 0.0002083, 0.00028944]^{T}\]

• **The Relative error (%) of curvature function \(F(\rho)\):**

\[e = [-0.12765, -0.066864, 0.050744, 0.014934, 0.012161, 0.0059506, -0.00065217, -0.0099451, -0.0180399, -0.020835, -0.0236, -0.0244212, -0.0218322, -0.014107, 5.0102e-005, -0.0012624, -0.0072336, 0.00731, 0.018697, 0.013484, 0.018394, 0.019973, 0.01849, 0.029165]^{T}\]

• **Mean relative error of \(x(1,:)\) (%) = 0.020819:**

The mean value of the relative error in curvature function \(F(\rho)\) is 0.020819% and the mean value of the relative error in the model is 0.7747%. The accuracy of the model is high. However in the un-optimized model the mean values of the relative error in curvature function

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F(\rho))</td>
<td>(0.9905, 0.9990, 0.9980, 0.9970, 0.9960, 0.9950, 0.9940, 0.9930, 0.9920, 0.9910, 0.9900, 0.9880)</td>
</tr>
<tr>
<td>(m_a)</td>
<td>(23.95, 18.53, 14.25, 12.26, 11.02, 10.15, 9.46, 8.92, 8.47, 8.10, 7.78, 7.49)</td>
</tr>
<tr>
<td>(m_b)</td>
<td>(0.163, 0.185, 0.212, 0.228, 0.241, 0.251, 0.260, 0.268, 0.275, 0.281, 0.287, 0.292)</td>
</tr>
<tr>
<td>(F(\rho))</td>
<td>(0.9880, 0.9870, 0.9860, 0.9850, 0.9840, 0.9830, 0.9820, 0.9810, 0.9800, 0.9790, 0.9780, 0.9770)</td>
</tr>
<tr>
<td>(m_a)</td>
<td>(7.25, 7.62, 6.84, 6.64, 6.47, 6.33, 6.19, 6.06, 5.95, 5.83, 5.72, 5.63)</td>
</tr>
<tr>
<td>(m_b)</td>
<td>(0.207, 0.301, 0.205, 0.310, 0.314, 0.317, 0.321, 0.325, 0.328, 0.332, 0.335, 0.338)</td>
</tr>
</tbody>
</table>
Table 2: The experimental data of the cutting force when \( f = 0.02 \text{ mm/rev} \)

<table>
<thead>
<tr>
<th>( a, \text{mm} )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_1, \text{N} )</td>
<td>838.98</td>
<td>1060.45</td>
<td>1261.79</td>
<td>1483.25</td>
<td>1704.72</td>
</tr>
<tr>
<td>( F_2, \text{N} )</td>
<td>255.10</td>
<td>290.16</td>
<td>355.22</td>
<td>420.28</td>
<td>469.08</td>
</tr>
</tbody>
</table>

\( F(p) \) is 20.4851\% and the mean values of the relative error in the model is 49.4419\%, which is different to the practice and can not be used.

**Example 2:** Using the Kentanium cutter YT14 in CA6140 engine lathe to cut excircle, at the condition of the given geometry parameters and cutting speed, the data of the cutting force changed with the cutting depth is as Table 2 Han and Dong (2008b).

The cutting deep \( a \), \( t \), and the main cutting force \( F_{\text{fs}}, F_{\text{g}} \), is \( x, y_1 \). According to the reciprocal accumulated operation we established the unequal MGRM(1,2) model and the parameters of the model are:

\[
A = \begin{bmatrix}
-9.4338 & 2.4827 \\
-81.4489 & 22.3382 
\end{bmatrix}, B = \begin{bmatrix}
0.0024 \\
0.0125 
\end{bmatrix}, \beta = \begin{bmatrix}
0.001347 \\
0.0044558 
\end{bmatrix}
\]

The fitting values of the main force is:

\[
\hat{F}_{\text{fs}} = [991.11727, 1072.0261, 1254.8004, 1468.2213, 1697.3949]
\]

The absolute error of the main force is:

\[
q = [152.1373, 11.57605, -6.89573, -15.02871, -7.325062]
\]

The relative error (%0f the main force is:

\[
e = [18.1336, 1.09162, -0.55394, -1.01323, -1.429693]
\]

The average value of the relative error is 4.2444\% and the average values of the model is 3.6903\%. The main error is the first form and if getting rid of it the average relative error is 0.7716\% and the accuracy of the model is high.

**CONCLUSION**

The construction of the background values is the key point that affect the predictive accuracy and adaptive in the grey theory. In the system of multi-variables non-equal sequence which the multi-variables are affected and constrained by each other and based on index characteristic of grey model, improving background value in non-equlistant multivariable new information optimization MGRM(1,n) was researched and the discrete function with non-homogeneous exponential law of three point fitting accumulated operation data was used to fit the accumulated sequence. The formula of improving background value was given. Taking the mean relative error as objective function and the modified values as the design variables as design variables, based on accumulated generating operation of reciprocal number, a non-equidistant multivariable new information optimization MGRM(1,n) model was put forward which was taken the nth component as the initialization. The new information optimization MGRM(1,n) model can be used in non-equal interval & equal interval time series and has the characteristic of high precision as well as high adaptability. Example validates the practicability and reliability of the proposed model.

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