The Periodicity and the Horseshoes of Continuous Self-maps

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Abstract: The continuous self-maps on completely densely ordered linear ordered topological spaces (shortly, CDOLOTS) are discussed. This study is the generalization of the ones on the real line. It is pointed that n-chain of intervals implies periodic points with period-n. Then, the following two results are obtained: (1) Period-3 implies arbitrary integer period. (2) Horseshoes implies arbitrary integer period. At last, Sufficient conditions of the existence for periodic points of odd period are obtained.

Key words: Periodic points, horseshoes, continuous self-maps, CDOLOTS

INTRODUCTION

In engineering applications such as digital communication principles, environmental science, biological science, encryption and energy, the periodicity of a system is an important subject for study. The properties of periodic points or periodic solutions are intensively discussed recently (Buica and Ortega, 2012, Kurkev et al., 2011; Graff et al., 2010; Abbas and Rhoades, 2009). This study considers the periodicity of self-maps on CDOLOTSs.

The order relationship of a linear ordered set X is denoted by a < b. Notation a < b denotes “a < b or a = b”. Definitions and notations of interval in X are similar to the real line. Infinite intervals are denoted by ({a, -}, [b, -]) and (-, -). If there exists an element m e X such that x < m for all x ∈ X, m will be called for largest element of X and denoted by End {X}. If there exists an element m e X such that m < x for all x ∈ X, m will be called for smallest element of X and denoted by Begin {X}. If X is a topological space with the subbase {({a, -}, (-, a), a e X)}, then X is called a linear ordered topological space. X is densely ordered if whenever a, b ∈ X, a < b there is an element c e X such that a < c < b. X is complete if every nonempty set with upper bound has a least upper bound in X. X is called a completely densely ordered linear ordered topological spaces if it is a linear ordered topological space with complete and densely order relation.

Let X is a CDOLOTS, f: X → X a continuous map. If Δ_i, Δ_j, ..., Δ_n are closed subintervals of X such that:

\[ Δ_i = f(Δ_j) \] for 2 ≤ i ≤ n

then Δ_i, Δ_j, ..., Δ_n is called a n-chain of intervals for f. If Δ_i, Δ_j, ..., Δ_n are closed subintervals of X with pairwise disjoint interiors such that:

\[ Δ_i ∪ Δ_j ∪ ... ∪ Δ_n = f(Δ) \]

for every integer i from 1 to n, then Δ_i, Δ_j, ..., Δ_n is called a n-horseshoe, or simply a horseshoe if n = 2. Block and Coppel (1986) called a map with a horseshoe turbulent. Other concepts and terms (e.g., separation, connected) are from (Todorcevic, 1984; Nagata, 1985). If X is a CDOLOTS, then X and integers in X is connected.

THE EXISTENCE OF PERIODIC POINTS

This section introduces the relations between chain of intervals and periodic points, horseshoes and periodic points. In this section, X is a CDOLOTS and f: X → X a continuous map.

Lemma: 1 If [a, b] and [c, d] are subintervals of X satisfied f([a, b]) = [c, d], then there exists a closed interval Δ in [a, b] such that:

\[ f(Δ) = [c, d] \] and \[ f(Δ') = [c, d] \]

for arbitrary proper subinteval Δ' of Δ.

Proof:

U = {x ∈ [a, b]: f(x) = d} ∩ φ

V = {x ∈ [a, b]: f(x) = c} ∩ φ

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Without loss of generality, suppose \( u \in U, v \in V, v < u \).

Put:

\[
\begin{align*}
    u^* &= \mathbb{A}\{x : x \in U \cap [v, b]\} \\
    v^* &= \mathbb{A}\{x : x \in V \cap [v, u^*]\}
\end{align*}
\]

For \( \forall x \in (v^*, u^*) \), one has \( f(x) \cdot c, f(x) \cdot d \) and \( \Delta = [v^*, u^*] \).  

Since \( f(v^*) = c, f(u^*) = d \), one has \( f([v^*, u^*]) = [c, d] \).

On the other hand, hypothesis there exists a \( x \) in \((v^*, u^*)\) such that \( f(x) \in [c, d] \). Suppose \( d < d_1, f(x) \in [d_1, d] \) is similar. Since \( d_2 < c, d_2 \), there exists a point \( x' \in (v^*, u^*) \) such that \( f(x') \). This contradicts \( f(x) \cdot d \) for every \( x \in (v^*, u^*) \). So:

\[
\begin{align*}
    f([v^*, u^*]) &= [c, d]
\end{align*}
\]

Thus, \( f(\Delta) \). By the definitions of \( u^* \) and \( v^* \), one has \( f(\Delta^*) \cdot [c, d] \) for arbitrary proper subinterval \( \Delta^* \) of \( \Delta \).

**Theorem 1:** If \( \Delta_0, \Delta_1, \ldots, \Delta_m \) is a chain of intervals for \( f \) and \( f(\Delta_n) = \Delta_n \), then there exists a point \( x_0 \in \Delta_0 \) such that \( f(x_0) = \Delta_0 \) and \( f(x_0) \in \Delta_0 \), for every integer \( i \) from 1 to \( n-1 \).

**Proof:** Since \( f(\Delta_n) = \Delta_n \), there exists a closed subinterval \( \Delta_0^* \) of \( \Delta_0 \) such that \( f(\Delta_0^*) \). And because:

\[
\begin{align*}
    f(\Delta_0^*) &= \Delta_1 \supseteq \Delta_0^*
\end{align*}
\]

so there exists a closed subinterval \( \Delta_{01}^* \) of \( \Delta_{01} \) such that \( f(\Delta_{01}^*) = \Delta_{01}^* \).

Similarly, there are closed subintervals:

\[
\begin{align*}
    \Delta_{01}^* &= \Delta_i (i = 0, 1, \ldots, n-1)
\end{align*}
\]

such that:

\[
\begin{align*}
    f(\Delta_{01}^*) &= \Delta_i (i = 0, 1, \ldots, n-2)
\end{align*}
\]

and \( f(\Delta_{01}^*) = \Delta_0 \). Thus \( f(\Delta_{01}^*) = \Delta_i \supseteq \Delta_0^* \).

Then, there exists a point \( x_0 \) in \( \Delta_i^* \) such that \( f(x_0) = x_0 \). Since \( f(\Delta_i^*) = \Delta_i^* \), one has:

\[
\begin{align*}
    f(x_0) \in \Delta_i (i = 0, 1, \ldots, n-1)
\end{align*}
\]

Therefore, \( x_0 \in \Delta_0 \) such that \( f(x_0) = x_0 \) and \( f(x_0) \in \Delta_0 \) (\( \forall i \in \{1, \ldots, n-1\} \)).

**Corollary 1:** If \( f([a, b]) \supseteq [a, b] \) or \( f([a, b]) \supseteq [a, b] \), then \( f \) has a fixed point.

**Theorem 2:** For each positive integer \( n, f \) has a periodic point with period \( n \) if and only if \( f \) has a periodic point with period 3.

**Proof:** The necessity is manifested.

Now, let \( \{x_0, x_1, x_2\} \) is a periodic orbit with period 3 and \( x_0 < x_1 < x_2 \). Suppose \( f(x_0) = x_0 \), then \( f(x_1) = x_1 \). Let \( \Delta_0 = [x_0, x_1], \Delta_1 = [x_1, x_2] \) for \( \forall n \in \mathbb{N}, n+3 \).

Let:

\[
\begin{align*}
    \Delta_0 = \Delta_1 = \ldots = \Delta_{n-1} = \Delta_n = \Delta_0
\end{align*}
\]

then:

\[
\begin{align*}
    f(\Delta_k) &= \Delta_{k+1} (k = 0, 1, \ldots, n-2)
    f(\Delta_n) &= \Delta_0
\end{align*}
\]

By Theorem 1, there exists a point \( x^* \) in \( \Delta_0 \) such that \( f(x^*) = x^* \), \( f(x^*) \in \Delta_0 \), \( f(x^*) \in \Delta_{k+1} \) and \( f(x^*) \in \Delta_{k+1} \). It can be claimed that \( x^* \) is a periodic point with period 3. Otherwise there exists an integer \( k \in \{0, 1, \ldots, n-2\} \) such that \( f(x^*) = f(x^*) \in \Delta_0 \). Then \( f(x^*) \in \Delta_0 \). That is:

\[
\begin{align*}
    f(x^*) \in \Delta_0 \cup \Delta_1 = \{x_0, x_1\}
\end{align*}
\]

So:

\[
\begin{align*}
    x^* &= f(x^*) = f(x) = x_0
\end{align*}
\]

Hence \( f(x_0) = f(x^*) = x^* = x_0 \).

Since \( x_0 \) is a periodic point with period 3, one has \( 3|n \). And because \( n+3 \), then \( n \geq 6 \). Thus:

\[
\begin{align*}
    x_2 &= f(x_1) = f(x^*) \in \Delta_0 = \Delta_0 = [x_0, x_1]
\end{align*}
\]

This contradicts to \( x_2 \notin [x_0, x_1] \).

Therefore, \( f \) has a periodic point with period \( n \) for each positive integer \( n \).

According to Theorem 1 and Theorem 2, the following Corollary 2 is immediately checked.

**Corollary 2:** If there exist two disjoint closed subintervals \( \Delta_i \) and \( \Delta_j \) on \( X \) such that \( f(\Delta_i) \supseteq \Delta_i \), \( f(\Delta_j) \supseteq \Delta_j \), \( f(\Delta_i) \supseteq \Delta_j \), then for each integer \( n \), \( f \) has a periodic point with period \( n \).

**THE HORSESHOES AND PERIODIC POINTS**

Let \( L \subset X \) a closed interval and \( f : L \rightarrow L \) a continuous map.
Theorem 3: If \( f \) has a horseshoe then it admits periodic points of all periods.

Proof: Let \((\delta_1, \delta_2)\) be a horseshoe for \(f\).

Case 1: If \(\delta_1\) and \(\delta_2\) are disjoint. Let \(n \geq 1\). Applying Theorem 1 to the chain of intervals \((\Delta_0, \Delta_1, ..., \Delta_n)\) with \(\Delta_0 = \delta_1\) for \(1 \leq i \leq n-1\) and \(\Delta_n = \delta_2\), there exists a point \(x\) in \(\delta_1\) such that \(f^i(x) \in \delta_1\) for \(1 \leq k \leq n-1\). Thus, the period of \(x\) is exactly \(n\) because \(\delta_1\) and \(\delta_2\) are disjoint.

Case 2: \(\delta_1\) and \(\delta_2\) have a common endpoint \(b\). Write \(\delta_1 = [a, b]\) and \(\delta_2 = [b, c]\).

If \(b\) is a fixed point, put:

\[
d = Ah[x; b \prec x, f(x) \in (a,c)]
\]

Obviously, \(b \prec d\) then:

\[
\alpha \prec f([b,d]) \text{ and } c \prec f([b,d])
\]

Since \((\delta_1, \delta_2)\) is a horseshoe for \(f\), then \([a,c] \prec f([b,d])\), so \((a,c) \prec f([d,e])\). Put \(\delta_1^* = [d,e]\), one has \((\delta_1, \delta_2)\) is a horseshoe composed of disjoint intervals and the first point of the proof applies.

If \(b\) is not a fixed point, applying Theorem 1 to the chain of intervals \((\delta_1, \delta_2, ..., \delta_n, A)\), one can take a point \(x\) in \(\delta_1\) such that \(f^p(x) \in \delta_1\) for \(1 \leq k \leq n-1\), for some integer \(p\). Let \(p = 1\) then \(x \in \delta_1 \text{ and } f(x) \in \delta_1\). This is impossible. Therefore, \(p = 3\). By theorem 2, \(f\) has periodic points of all periods.

Theorem 4: Let \(x \in \Delta_0\), write \(x = f(x)\) for all \(n \geq 0\). If \(f\) has no horseshoe and \(x \prec x_n\), \(x_{n+1} \prec x\), then \(x_n \prec x_m\) and there exists a fixed point \(x \in [x_n, x_m]\).

Proof: Suppose \(x_n \prec x_m\) and assume that \(m > n\), the case \(m < n\) being symmetric. First, one has \(f([x_m, x_n]) \cap [x_m, x_n]\). By Corollary 1, there exists a point \(y\) in \([x_m, x_n]\) such that \(f(y) = y\). Let \(z \in [x_m, x_n]\) be the largest fixed point of \(f\), then \(y \prec f(y) = \forall (x, y)\). And \(x_n \prec z\) (Otherwise \(x_n = f^m(x_n - x_m)\)).

Put \(k = \min\{i; x_i \leq z, n+1 \leq i\}\).

Inductively, \(x_n \prec x_k\) for all integer \(i\) from \(n\) to \(k-1\). Since \(f(x_k) = z \prec x_{k+1}\) (by the definition of \(k\)) then \(x_k \prec f(z, x_k)\). One has \(x_n \prec x_k\). Suppose \(x_n \prec x_{k+1}\) \((n < k-1)\), then \(x_n \prec x_{k+1}\). Otherwise \([z, x_k], [x_k, x_{k+1}]\) form a horseshoe of \(f\) which is a contradiction. For \(i = k-1\), this gives \(x_{k-1} \prec x_n\) which is absurd.

So one has \(x_n \prec x_m\) and \(f([x_m, x_n]) \cap [x_m, x_n]\). It can be concluded that there exists a point \(z\) in \([x_m, x_n]\) such that \(f(z) = z\).

Theorem 5: Let \(x \in \Delta_0\), \(p\) an odd integer. If \(p(x) = x \prec f(x)\), then \(f\) has a periodic point of odd period \(q\) with \(1 < q < p\).

Proof: If \(f(x) = x\) then \(x\) is a periodic point of odd period \(q = p\). And because \(f(x) = x\), then \(f(x) = x\). Thus it can be assumed that \(f(x) = x\).

If \(f\) has horseshoes, by theorem 3, \(f\) has periodic points of all periods. One must only assume that \(f\) has no horseshoe.

Write \(x_n = f(x)\) for all \(n \geq 0\). Since \(f(x) = x \prec f(x)\), then there exist \(j, j \in \mathbb{N}, 0 \leq j < p\) such that \(x_{n+j} \prec x_n\).

Let:

\[
a = \text{End} \{x_n; x_n \prec x_{n+1}, 0 \leq n < p\}
\]

\[
b = \text{End} \{x_n; x_{n+1} \prec x_n, 0 \leq n < p\}
\]

Then \(x_n \prec f(a), f(b) \prec x = a\) and \(x_n \prec x = b\) (by theorem 4). Thus \(f(a, b) = [a, b]\). So there exists a fixed point \(z \in [a, b]\). If there exist \(1 \leq i \leq p\) such that \(x_i = z\) then:

\[
x_i \prec a \prec z = x_i
\]

which is a contradiction. Hence, \(x_i \neq z\) \((0 \leq i \leq p)\) and \(a \prec z \prec b\).

One can claims that there exist \(0 \leq k \leq p\) such that \(x_{k+1} \prec x_{k+i} \prec z\) or \(z \prec x_{k+i}\) \((0 \leq i \leq p)\).

Otherwise one gets from place to place that \(x_{k+i} \prec x_{k+i+j}\), \(x_{k+i+j} \prec x_{k+i+j+1}\), \(x_{k+i+j+1} \prec x_{k+i+j+2}\) \((i \leq j \leq p)\) (p odd) which is impossible because \(x_{k+i+j} = x_{k+i+j+1}\).

Suppose for example that \(x_{k+i+j} = x_{k+i+j+1}\). Define \(\Delta_i = [x_i, x_{i+1}]\) and \(\Delta_i = [x_i, z]\) for all \(0 \leq i \leq p, 1 \leq k\). Then:

\[
f(\Delta_k) \supset [x_{k+i}, b] \supset [x_{k+i}, z]
\]

and:

\[
f(\Delta_k) \supset [x_{k+i}, z] \text{ for } 0 \leq i \leq p, 1 \leq k
\]

Thus \((\Delta_0, ..., \Delta_p)\) is a chain of intervals. And because \(\Delta_i \supset \Delta_{i+1}\) by Theorem 1, there exists a point \(y \in \Delta_0\) such that \(f(y) = y\). Let \(q \) be the period of \(y\) then \(q \) divides \(p\). Thus \(q \) is odd and \(p < q\). If \(q = 1\) then \(y \in \Delta_0 \supset \Delta_i = [x_i, z]\) \((z, x)\) and thus \(y = z\). But one also has \(y \prec z\) because \(y \in \Delta_i = [x_i, a]\). A contradiction. Hence \(p < q\).

To prove Theorem 6, the order relation of odd periodic orbits (Luo and Zhu, 2010) is given first.

Lemma 2: Let \(X\) be a CDOLOTS and \(f: X \times X\) a continuous map. If \(f\) has a periodic orbit with period \(2n+1\) but has no periodic orbit with period \(2n-1\). Let \(x_i\) be the median point of the orbit. Then one of the following orders holds:

1453
Theorem 6: If \( f \) has a periodic point of odd period different from 1, then there exist two disjoint closed intervals \( \Delta_1 \) and \( \Delta_2 \) that do not contain the endpoints of and such that \( (\Delta_1, \Delta_2) \) is a horseshoe for \( f \).

Proof: Put:

\[
p = \min \{ \text{is an odd integer, } n>1, \ f \text{ has a periodic point of period } n \},
\]

and let \( x \) be a periodic point of period \( p \). According to the order relation of odd periodic orbits, the points of the orbit of \( x \) can be ordered in the following order:

\[
x_p, x_{p-1}, \ldots, x_2, x_0, x_1, x_2, \ldots, x_{p-1}
\]
or in the reverse order. Suppose the order above holds.

Since, \( f([x_0, x_1])=[x_0, x_1] \), then there exists a point \( d \) with \( x_d < x \), such that \( f(d) = x_0 \). This implies \( d < f(d) = x_1 \).

And because \( f^k([x_0, x_1])=[x_0, x_1] \), there exists a point \( a \) such that \( d < f(a) \). One has \( f([a, x_0])=[x_0, d] \), thus there exists a point \( b \) such that \( f(b) < a \). In the same way, since \( f([x_0, d])=[x_0, d] \), one can choose a point \( c \) in \( (x_{p-2}, d) \) such that \( f(c) < a \).

Let \( \Delta_1 = [a, b] \) and \( \Delta_2 = [c, d] \), then it can be easily checked that:

\[
f(\Delta_1) \cap f(\Delta_2) = \Delta_1 \cup \Delta_2
\]

Since, \( x_p, a \) and \( d < x_0 \), then \( \Delta_1 \) and \( \Delta_2 \) do not meet the endpoints of \( L \).

Remark: The results in this paper are correct for continuous maps of compact intervals. This study makes it clear that topological structure of spaces have no impact on some properties of periodic points. These properties of periodic points are only related to the order relations of spaces.

CONCLUSION

The following results are obtained:

- The relation between the existence of periodic points and the horseshoes
- Sufficient conditions of the existence for fixed points and periodic points of odd period

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