A New Achievable Rate Region for the Relay Channel with Forwarding Own Messages

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Abstract: The traditional relay channel as an important component of wireless networks has received considerable attention in recent years. In order to fully utilizing the wireless resources, a novel relay channel model, the Relay Channel with Forwarding Own Messages (RC-FOM), is proposed. In such study, the relay has an own message for the destination in addition to the traditional communication from the source to the destination with the help of the relay. Upper bounds as well as an achievable rate region are derived based on the Decode-and-Forward (DF) strategy for the discrete memoryless RC-FOM. For the Additive White Gaussian Noise (AWGN) channels, the corresponding achievable rate region is proved theoretically and validated numerically. This new achievable scheme can be flexibly reduced to the classic relay channel and thus seen as a generalization of the relay channel.

Key words: Wireless networks, relay channel, achievable rate region, decode-and-forward (DF)

INTRODUCTION

User cooperation in wireless networks is an effective technique that enables users to cooperate with each other in their transmissions, thus increasing the transmission efficiency. Relay channels are a special class of cooperative transmission networks, where a source wishes to transmit a message to a destination with the help of relays. The information theoretic study of relay channels have been broadly investigated (Cover and Gamal, 1979; Kramer et al., 2005; Hrost-Madsen and Zhang, 2005; Reznik et al., 2004; Paulraj et al., 2003; Wang et al., 2005) and practical relaying protocols have also been developed (Laneman et al., 2004).

Van Der Meulen (1971) was the first one who introduced the relay channel. This model was then studied deeply by Cover and Gamal (1979). Up to now, the studies on relay channels have been expanded to large-scale hybrid networks, for example (1) The wireless sensor networks (Youn and Kang, 2008; Shaik and Wang, 2008; Anvuthnan and Dhulipala, 2013), (2) The multi-way relay networks (Ong et al., 2011, 2012; Timo et al., 2013) and (3) The LTE-advanced networks (Sasikala and Srivatsa, 2006, Abed et al., 2011). In these relay modes, the relay nodes only assist the source with relaying messages to the destination without sending new messages of their own. When the traditional relay channel is applied to a practical large network, in order to make full use of the channel capacity, one has to consider the capacity limits if a relay node must play a role both in relaying the source messages, as well as its own messages. A representative channel model is proposed as illustrated in Fig. 1 which is referred to as Relay Channel with Forwarding Own Messages (RC-FOM). This RC-FOM consists of a point-to-point communication channel between the relay and the destination and relayed channels in comparison with the traditional relay channel.

In this study, the new channel model is established and the impact of the relay with forwarding own messages on capacity limits on upper bounds is studied. By proposing the random coding method with jointly typical sequences, achievable rates for both the discrete memoryless and Gaussian RC-FOM are derived. Finally, the achievable rate regions under different channel conditions are compared and validated by simulations.

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Fig. 1: Information flow of relay channel with forwarding own messages
MATERIALS AND METHODS

Experimental system model: The relay channel is a three-terminal discrete memoryless channel consisting of finite sets denoted by \(X_i, X_o, Y_i, Y_o\) (i.e., cardinalities of random variables \(X_i, X_o, Y_i, \) and \(Y_o\)) and a transition probability distribution \(p(y_i, y_o|x_i, x_o)\) where \(x_i \in X_i, x_o \in X_o, y_i \in Y_i,\) and \(y_o \in Y_o\). In this channel, \(X_i\) and \(X_o\) are the channel inputs from the source and the relay, respectively while \(Y_i\) and \(Y_o\) are the channel outputs at the relay and destination, respectively. This channel was described by Van Der Meulen (1971) as the first work on the achievable rate regions for the relay channel. Different cooperation schemes such as Decode-and-Forwarding (DF) and Compress-and-Forwarding (CF) were introduced. However, the relay can also send independent and new messages in addition to relaying messages that help the destination in decoding the source’s message. This novel system model is named as the relay channel with forwarding own messages in this study. Correspondingly, the relay uses DF scheme as the transmission strategy.

A \((2^m, 2^n)\) code for the RC-FOM consists of the following:

- Two sets of integers, the source message \(W_i \in W_i = \{0, \ldots, 2^m\}\) and the relay message \(W_o \in W_o = \{0, \ldots, 2^n\}\)
- An encoder at the transmitter, \(X_i: W_i \rightarrow X_i^n\) where, \(X_i^n(X_i^n, X_{i-1}^n, X_i)\)
- A set of relay functions \(f_{1:n}^{1:n}\) at the relay such that:

\[
x_{ij} = f(y_{i:j}, x_{i-1:j}, w_{ij}), \quad 1 \leq i \leq n
\]  

where, \(x_{ij}\) is the i-th component of the codeword \(x_j = (x_{i-1:j}, x_{i})\) that is only dependent of the past received information \((y_{i-1:j}, y_{i-1:j})\) and the information \(w_{ij}\)

- A decoder at the destination, \(\phi: Y^n \rightarrow W_i \times W_o\)

The average error probability of decoding \(W_i\) and \(W_o\) is defined as:

\[
P_e = \sum_{x_i \in X_i} \sum_{x_o \in X_o} \frac{1}{2^{m+n}} \Pr\{\hat{y}(y) \neq (w_i, w_o) | (w_i, w_o) \text{ were sent}\}
\]  

(2)

Figure 2 illustrates the encoding and decoding structure of the two messages. \(W_i\) and \(W_o\) denote the estimates of \(W_i\) and \(W_o\). The source and the relay generate their messages \((W_i, W_o)\) independently and uniformly over \(W_i \times W_o\).

A rate pair \((R_i, R_o)\) is said to be achievable for the RC-FOM if there exist a sequence of codes \((2^m, 2^n)\) such that for any \(\delta > 0\), the average probability of error:

\[
P_e \leq \delta
\]  

(3)
as \(n \rightarrow \infty\).

A relay channel with forwarding own messages is said to be degraded if its transition probability satisfies:

\[
p(y_i, y_o|x_i, x_o) = p(y_i|x_i, x_o) p(y_o|x_i, x_o)
\]  

(4)

That is to say, \(Y_i\) is independent of \(X_i\) on the condition of having \(Y_i\) and \(X_o\). Note that the definition of degradation precisely describes the notation that one channel is worse than other one in the RC-FOM.

General method: In this subsection, the result of the max-flow-min-cut theorem by Cover and Thomas (1991) is used as a method to establish upper rate bounds on the capacity region of the RC-FOM.

Given any \((2^m, 2^n)\) code for RC-FOM, the joint probability distribution satisfies:

\[
p(w_i, w_o, x_i, y_i, y_o) = p(w_o)p(w_i)p(x_i | w_i)\prod_{i=1}^{n} p(x_{i-1:j}, x_{i-1:j}) p(y_{i:j}, y_{i:j} | x_{i:j}, x_{i:j})
\]  

(5)

According to Fano’s inequality, the information entropy \(H(W_i, W_o | Y_i)\) can be bounded as:

\[
H(W_i, W_o | Y_j) \leq n (R_i + R_o) P_e + 1 \leq n \delta_e
\]  

(6)

Fig. 2: The encoding and decoding process for the relay channel with forwarding own messages
As the message can be decoded from $Y_2$, $P_\alpha \rightarrow 0$ and $\delta_1$ defined as $(R_1+R_2)P_\alpha+1/n$, also goes to zero as $n \rightarrow \infty$.

In the following, $R_1$ is upper bounded as:

$$nR_1 = H(W_1) = I(W_1; Y_1) + H(W_1 | Y_1)$$

$$\leq I(W_1; Y_1) + H(W_1, W_2, Y_1)$$

$$\leq I(W_1; Y_1) + n\delta_1$$

$$= \sum_{i=1}^{n} I(W_1; Y_{1,i} | Y_{1,i}^-) + n\delta_1$$

$$= \sum_{i=1}^{n} [H(Y_{1,i} | Y_{1,i}^-) - H(Y_{1,i} | W_1, Y_{1,i}^-)] + n\delta_1$$

$$\leq \sum_{i=1}^{n} [H(Y_{1,i} | W_1, Y_{1,i}^-)] + n\delta_1$$

$$= \sum_{i=1}^{n} [I(W_1, Y_{1,i}^-; Y_{1,i}; Y_{2,i}) + n\delta_1]$$

(7)

where, Eq. 8-9 follow as conditioning reduces entropy. Now, let $Q$ be a random variable uniformly distributed over $\{1, \ldots, n\}$, set $V_1 = QY_{1|Q} W_1$, $V_2 = QY_{2|Q} Y_1 = Y_{2,1}$ $Y_2 = Y_{1,2}$

Hence:

$$R_1 = \frac{1}{n} H(W_1) \leq I(V_1; V_2; Y_2)$$

(10)

Also:

$$nR_1 = H(W_1)$$

$$\leq \sum_{i=1}^{n} I(W_1; Y_{1,i} | Y_{1,i}^-) + n\delta_1$$

$$\leq \sum_{i=1}^{n} [I(W_1, Y_{1,i}^-; Y_{1,i}; Y_{2,i}) + n\delta_1]$$

$$= \sum_{i=1}^{n} [H(Y_{1,i} | Y_{1,i}^-) - H(Y_{1,i} | W_1, Y_{1,i}^-)] + n\delta_1$$

$$\leq \sum_{i=1}^{n} [H(Y_{1,i} | W_1, Y_{1,i}^-)] + n\delta_1$$

$$= \sum_{i=1}^{n} I(W_1, Y_{1,i}^-; Y_{1,i}; Y_{2,i}) + n\delta_1$$

(11)

Hence:

$$R_1 = \frac{1}{n} H(W_1) \leq I(V_1; Y_1; Y_2)$$

(12)

Then, for the relay-destination channel, $R_2$ can be bounded easily as below:

$$R_2 \leq I(V_1; Y_1 | V_2)$$

(13)

Based on the derivation of the upper bounds in Eq 10, 12 and 13 by taking $n \rightarrow \infty$, this subsection draws a conclusion. For any rate pair $(R_1, R_2)$ with $P_\alpha \rightarrow 0$ of the RC-FOM, there exist some random variables $U \rightarrow (V_1, V_2) \rightarrow (X_1, X_2) \rightarrow (Y_1, Y_2)$, such that $(R_1, R_2)$ satisfies the following conditions:

$$R_1 \leq \min\{I(V_1; Y_1, Y_2 | V_2), I(V_1, V_2; Y_2)\} + \delta$$

$$R_2 \leq I(V_1; Y_1 | V_2) + \delta$$

(14)

**PROPOSED APPROACH**

Here, proposes a random coding technique with jointly typical sequences to derive the achievable rate region for the RC-FOM based on the well-known DF strategy (Cover and Gamal, 1979). Here, the relay is assumed to be able to fully recover the source’s message forwarded to the destination. Then, it re-encodes the decoded source message $W_1$ together with its own message $W_2$ for the destination.

The coding schemes combine ideas from the relay channel (Cover and Gamal, 1979), the broadcast channel (Liang and Veeravalli, 2007) and the multiple access channel (Willems, 1982).

**Random codebook generation:**

- Randomly generated independently and identically distributed (i.i.d) $n$-sequence $u$ with probability:

$$p(u) = \prod_{i=1}^{n} p(u_i)$$

Label them as $u(w_i)$. Assume that all the nodes know $u$.

- For each $u(w_i)$ generate $2^m$ conditionally independent $n$-sequence $x_i$ at the source draw randomly according to:

$$p(x_i | u(w_i)) = \prod_{i=1}^{n} p(x_{i,1} | u_i(w_i))$$

index them as $x_i(w_i, w), w_i \in [1, 2^m]$.

- For each $u(w_i)$ generate at random $2^{m+1}$ i.i.d $n$-sequence $x_i$ at the relay, each with distribution:

$$p(x_i | u(w_i)) = \prod_{i=1}^{n} p(x_{i,1} | u_i(w_i))$$

index them as $x_i(w_i, w^2), w, w_i \in [1, 2^m]$.  


Assume a transmission period of B blocks, each consisting of n transmissions. A sequence of B-1 i.i.d source messages \( w_i(i) \in [1,2^m] \), \( i \in [1:B-1] \) and a sequence of B-1 i.i.d relay messages \( w_i(i) \in [1,2^m] \), \( i \in [1:B-1] \), are to be sent over the channel in nB transmissions, respectively. Note that the average rate pairs over B blocks are:

\[
\left( \frac{R_r}{R_s}, \frac{B-1}{B} \right)
\]

which can be made as close to \((R_r, R_s)\) as desired when \(B \to \infty\). The encoding and decoding are explained with the help of Table 1.

**Encoding:** At first consider block i, where \( i \neq 1, B \) which means it is not the first or the last block:

- The encoder at the source sends \( x_i(w_i(i), w_i(i-1)) \), where \( w_i(i-1) \) was denoted above as \( w_i \).
- Assuming that the relay already has a correct estimation of \( w_i(i-1) \) from the previous block, then it sends \( x_i(w_i(i), w_i(i-1)) \).

When \( i = 1 \), the source sends \( x_i(w_i(1), 1) \) and the relay sends \( x_i(w_i(1)) \), where \( w_i(0) = 1 \) by convention. When \( i = B \), the source sends \( x_i(1, w_i(B-1)) \) and the relay sends \( x_i(1, w_i(B-1)) \), where \( w_i(B) = w_i(B) = 1 \) by convention.

**Decoding:**

- Assuming the relay has decoded \( w_i(i-1) \) which was sent at block \( i=1 \), it can decode \( w_i(i) \) by looking for a unique \( \tilde{w}_i(i) \) such that:

\( u(w_i(i-1), x_i(\tilde{w}_i(i)), w_i(i-1), x_i(w_i(i), w_i(i-1)), y_i(i)) \)

are jointly typical. If:

\[
R_s < I(X_s; Y_i|X_s, U)
\]

then based on joint Asymptotic Equipartition Property (AEP), one has \( \tilde{w}_i(i) = w_i(i) \) with probability goes to 1.

- The destination decodes from the last block until all blocks are received. Assuming that it has decode \( w_i(i) \) in block \((i+1)\), then in block \( i \), it declare that \( \tilde{w}_i(i) \) is received, if \( (u(w_i(i-1)), x_i(w_i(i), \tilde{w}_i(i)), y_i(i)) \) are jointly typical. It is easy to see that if:

\[
R_i < I(U, X_i; Y_i)
\]

\( \tilde{w}_i(i-1) = w_i(i-1) \) with probability goes to 1, as \( n \) increases.

- Having \( w_i(i-1) \), the destination decodes \( w_i(i) \) by looking for a unique \( \tilde{w}_i(i) \) such that \((u(w_i(i-1)), x_i(\tilde{w}_i(i)), w_i(i-1), x_i(w_i(i), \tilde{w}_i(i)), y_i(i)) \) are jointly typical. If there does not exist such unique sequences, the destination occurs an error. Then based on AEP, the error probability will go to zero if:

\[
R_s < I(X_s; Y_i|X_s, U)
\]

By combining Eq. 15-17, the rate pairs in the closure of the convex hull of all \((R_r, R_s)\) for the RC-FOM satisfying:

\[
R_i < \min\{I(X_i; Y_i|X_i, U), I(U, X_i; Y_i)\}
\]

\[ R_s < I(X_s; Y_i|X_s, U) \]

for some joint distribution:

\[
p(u, v_i, v_j, x_s, y_s) = p(u) p(v_i|u) p(v_j|u) p(x_s|v_i, v_j) p(y_s|x_s, v_s)
\]

are achievable using DF scheme.

Note that if \( U = X_s \), Eq. 18 reduces to one of the main results by Cover and Gamal (1979).

**NUMERICAL RESULTS AND ANALYSIS**

Here, the achievability results in Eq. 18-19 are extended to the Gaussian RC-FOM because it approximately presents realistic wireless networks. The signals received, respectively at the relay and the destination node are given by:

Table 1: The encoding and decoding process for the RC-FOM

<table>
<thead>
<tr>
<th>Time block</th>
<th>Source Encoding</th>
<th>Relay Decoding</th>
<th>Encoding</th>
<th>Decoding</th>
<th>Destination Decoding</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x_i(w_i(1), 1) )</td>
<td>( \tilde{w}_i(1) )</td>
<td>( x_i(w_i(1), 1) )</td>
<td>( \tilde{w}_i(1) )</td>
<td>( \tilde{w}_i(1) )</td>
</tr>
<tr>
<td>2</td>
<td>( x_i(w_i(2), w_i(1)) )</td>
<td>( \tilde{w}_i(2) )</td>
<td>( x_i(w_i(2), \tilde{w}_i(1)) )</td>
<td>( \tilde{w}_i(2) )</td>
<td>( \tilde{w}_i(2) )</td>
</tr>
<tr>
<td>3</td>
<td>( x_i(w_i(3), w_i(2)) )</td>
<td>( \tilde{w}_i(3) )</td>
<td>( x_i(w_i(3), \tilde{w}_i(2)) )</td>
<td>( \tilde{w}_i(3) )</td>
<td>( \tilde{w}_i(3) )</td>
</tr>
<tr>
<td>B-1</td>
<td>( x_i(w_i(B-1), w_i(B-2)) )</td>
<td>( \tilde{w}_i(B-1) )</td>
<td>( x_i(w_i(B-1), w_i(B-2)) )</td>
<td>( \tilde{w}_i(B-2) )</td>
<td>( \tilde{w}_i(B-2) )</td>
</tr>
<tr>
<td>B</td>
<td>( x_i(1, w_i(B-1)) )</td>
<td>( \tilde{w}_i(1) )</td>
<td>( x_i(1, w_i(B-1)) )</td>
<td>( \tilde{w}_i(1) )</td>
<td>( \tilde{w}_i(1) )</td>
</tr>
</tbody>
</table>
$$Y_1 = X_1 + Z_1$$  \tag{20}$$

$$Y_2 = X_1 + X_2 + Z_2$$  \tag{21}$$

where, $Z_1 \sim \mathcal{N}(0, N_1)$ and $Z_2 \sim \mathcal{N}(0, N_2)$ are i.i.d. Gaussian noise. The source and the relay have average power constraints:

$$\frac{1}{n} \sum_{i=1}^{n} X_i^2 \leq P_s$$

and:

$$\frac{1}{n} \sum_{i=1}^{n} X_i^2, (Y_{12}, Y_{13}, \ldots, Y_{1n}, W_x) \leq P_r$$

If an AWGN RC-FOM is said to be degraded, then the relay has all the knowledge that the destination knows, i.e., the channel input-output relations in Eq. 20 and 21 are changed to:

$$Y_1 = X_1 + Z_1$$  \tag{22}$$

$$Y_2 = X_1 + X_2 + Z_1 + Z'$$  \tag{23}$$

where, $Z_1 \sim \mathcal{N}(0, N_1)$ and $Z' \sim \mathcal{N}(0, N_2 N_1)$ are i.i.d. Gaussian noise for $N_1 < N_2$.

Let:

$$C(x) = \frac{1}{2} \log_2 (1 + x)$$

denote the capacity function and let:

$$f(\alpha, \beta, P_s, P_r) = P_s + (1 - \beta) P_r + 2\sqrt{(1 - \alpha)P_s (1 - \beta)P_r}$$

in the rest of this section. All achievable rate values are expressed in bps Hz$^{-1}$.

Let $U \sim \mathcal{N}(0, P)$, $X' \sim \mathcal{N}(0, \alpha P_r)$ and $X' \sim \mathcal{N}(0, \beta P_r)$. Also, let:

$$X_1 = \sqrt{(1 - \alpha)P_s - U + X'_1}$$  \tag{24}$$

$$X_2 = \sqrt{(1 - \beta)P_r - U + X'_2}$$  \tag{25}$$

In the above expressions the parameter $\beta$ shows the relay's participation level in the help of relaying source's message which controls the trade-off between the relayed messages and relay's own messages. Although, for the traditional degraded relay channel it is always optimal to relay source's message with full power, this may not be the case when the relay sends its own messages.

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**Fig. 3:** The achievable rate regions for the degraded Gaussian relay channel with forwarding own messages.

Therefore, the rate pairs in the closure of the convex hull of all $(R_s, R_r)$ for the Gaussian relay channel with forwarding own messages satisfying:

$$R_s < \min \left\{ C \left( \frac{f(\alpha, \beta, P_s, P_r)}{BP_s + N_1} \right), C \left( \frac{\alpha P_r}{N_2} \right) \right\}$$  \tag{26}$$

$$R_r < C \left( \frac{\beta P_r}{N_2} \right)$$  \tag{27}$$

are achievable using DF scheme for some $\alpha, \beta \in [0, 1]$.

Note that, set $\beta = 0$ and Eq. 26 will reduce to the capacity of the degraded Gaussian relay channel as another main result by Cover and Gamal (1979).

Figure 3 illustrates the achievable rate regions to demonstrate the relationship between the source rate ($R_s$) and the rate of relaying own messages ($R_r$) for two scenarios for comparison:

- **Case 1:** The degraded gaussian RC-FOM with $P_s = P_r = 10$ dB, $N_1 = 2$ dB and $N_2 = 4$ dB
- **Case 2:** The non-degraded gaussian RC-FOM with $P_s = P_r = 10$ dB, $N_1 = 5$ dB and $N_2 = 4$ dB

In the case 2, using the relay performs worse than the case 1 because the system no longer benefits from using the relay sufficiently to achieve the best achievable rate when the relay has more noise. Therefore, it is profitable to use the relay whenever it is more helpful in forwarding the source message to the destination, i.e., the source-destination channel is the degraded version of the source-relay channel.
CONCLUSION

In this study, the relay channel with forwarding own messages was studied. In particular, the cooperative DF scheme was proposed and the coding strategy was developed for this channel. Furthermore, upper bounds on the capacity region and the corresponding achievable performance bounds for this channel were obtained in discrete memoryless channel and Gaussian channel, respectively. Finally, numerical results established the critical role of relay’s own message in the capacity region and shed light on the generalization of this channel.

This study can be further extended. First, achievable rate regions by using CF and partial decode-and-forward schemes can also be derived in RC-FOM. Then, RC-FOM in fading channels is a practical application in wireless networks. Finally, in order to get greater flexibility of power allocation, global power constraint instead of a per-node power constraint is worth studying.

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REFERENCES


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