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Norm Ratios and Anisotropy Degree

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Abstract: Decomposition of elastic constant tensor into irreducible parts is given. Elastic constant tensor norm and norm ratios for various anisotropic materials of the same or different symmetry are calculated. The norm of a tensor is used as a criterion for comparing the overall effect of the properties of anisotropic materials and the norm ratios are used as a criterion to represent the anisotropy degree of the properties of materials.

Key Words: Norm Ratios, Anisotropy Degree, Elastic Constant

Elastic constant tensor decomposition: The constitutive relation characterizing linear anisotropic solids is the generalized Hook's law Nye (1964):

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl},$$

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl}, \qquad (1)$$

Where σ_{ij} and ε_{kl} are the symmetric second rank stress and strain tensors, respectively C_{ijkl} is the fourth-rank elastic stiffness tensor (hereafter we call it elastic constant tensor) and S_{ijkl} is the elastic compliance tensor.

There are three index symmetry restrictions on these tensors. These conditions are:

$$C_{ijkl}=C_{jikl}, \quad C_{ijkl}=C_{ijlk}, \quad C_{ijkl}=C_{klij}, \qquad (2)$$

which the first equality comes from the symmetry of stress tensor, the second one from the symmetry of strain tensor, and the third one is due to the presence of a deformation potential. In general, a fourth-rank tensor has 81 elements. The index symmetry conditions (2) reduce this number to 21. Consequently, for most asymmetric materials (triclinic symmetry) the elastic constant tensor has 21 independent components.

Elastic compliance tensor S_{ijkl} possesses the same symmetry properties as the elastic constant tensor C_{ijkl} and their connection is given by Teodosio, (1982):

$$C_{ijkl}S_{klmn} = \frac{1}{2}\left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right), \qquad (3)$$

Where δ_{ij} is the kronecker delta. The Einstein summation convention over repeated indices is used and indices run from 1 to 3 unless otherwise stated. Schouten (1954) has shown that C_{ijkl} can be decomposed into two scalars, two deviators, and one-nonor parts. The same decomposition in terms of the irreducible representations of the three-dimensional rotation group has been given in Heine (1960) as:

$$2D_0 + 2D_2 + D_4, \qquad (4)$$

where the subscripts denote the weight of the representation. By applying the symmetry conditions (2) to the decomposition results obtained for a general fourth-rank tensor, the following reduction spectrum for the elastic constant tensor is obtained. It contains two scalars, two deviators, and one-nonor parts:

$$C_{ijkl}^{(0;2)} = \frac{1}{90}\left(3\delta_{ik}\delta_{jl} + 3\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}\right)\left(3C_{pqpq} - C_{ppqq}\right)$$

These parts are orthonormal to each other. Using Voigt's notation Nye, (1964) for C_{ijkl}, can be expressed in 6 by 6 reduced matrix notation, where the matrix coefficients $c_{\mu\lambda}$ are connected with the tensor components by the recalculation rules:

$$c_{\mu\lambda}=C_{ijkl}$$

$$(ij \leftrightarrow \mu = 1,\ldots, 6, kl \leftrightarrow \lambda = 1,\ldots, 6);$$

that is:

$$11 \leftrightarrow 1, 22 \leftrightarrow 2, 33 \leftrightarrow 3,$$
$$23 = 32 \leftrightarrow 4,$$
$$31 = 13 \leftrightarrow 5, 12 = 21 \leftrightarrow 6$$

The Norm Concept: Generalizing the concept of the modulus of a vector, norm of a Cartesian tensor (or the modulus of a tensor) is defined as the square root of the contracted product over all indices with itself:

$$C_{ijkl}^{(0;2)} = \frac{1}{90}\left(3\delta_{ik}\delta_{jl} + 3\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}\right)\left(3C_{pqpq} - C_{ppqq}\right)$$

Denoting rank-n Cartesian ,

$T_{ijkl......}$ by T_n,

the square of the norm is expressed as Jerphagnon *et al.* (1978):

This definition is consistent with the reduction of the tensor in tensor in Cartesian formulation when all the irreducible parts are embedded in the original rank-n tensor space.

Since the norm of a Cartesian tensor is an invariant quantity, we suggest the following:

Rule 1: The norm of a Cartesian tensor may be used as a criterion for representing and comparing the overall effect of a certain property of the same or different symmetry. The larger the norm value, the more effective the property is.

It is known that the anisotropy of the materials, i.e., the symmetry group of the material and the anisotropy of the measured property depicted in the same materials may be quite different. Obviously, the property, tensor must show, at least, the symmetry of the material. For example, a property, which is measured in a material, can almost be isotropic but the material symmetry group itself may have very few symmetry elements. We know that, for isotropic materials, the elastic constant tensor has two irreducible parts, i.e., two scalar parts, so the norm of the elastic constant tensor for isotropic materials depends only on the norm of the scalar parts, i.e., $N = N_s$. Hence, the ratio $\frac{N_s}{N} = 1$ for isotropic materials. For cubic symmetry materials the constant tensor has two scalar parts and one nonor part, so we define two ratios: $\frac{N_s}{N}$ for the scalar irreducible parts and $\frac{N_n}{N}$ for the nonor irreducible part. For more anisotropic materials, the elastic constant tensor additionally contains two deviator parts, so we can define $\frac{N_d}{N}$ for the deviator irreducible parts.

Generalizing this to irreducible tensors up to rank four, we can define the following norm ratios: $\frac{N_s}{N}$ for scalar parts, $\frac{N_v}{N}$ for vector parts, $\frac{N_d}{N}$ for deviator parts, $\frac{N_{sc}}{N}$ for septor parts, and $\frac{N_n}{N}$ for nonor parts. It is to be noted that we calculate norms for weights only, i.e., for values of j = 0,2,3,4. Although norm ratios of different irreducible parts represent the anisotropy of that particular irreducible part, they can also be used to asses the anisotropy degree of a material property as a whole, we suggest the following two more rules

Rule 2: When N_s is dominating among norms of irreducible parts: the closer the norm ratio $\frac{N_s}{N}$ is to one, the closer the material property is isotropic.

Rule3. When N_s is not dominating or not present, norms of the other irreducible parts can be used as a criterion. But in this case the situation is reverse; the larger the norm ratio value we have , the more anisotropic the material property is.

The square of the norm of the elastic constant tensor $C_{\mu\lambda}$, is:

$$\|N\|^2 = \sum_{\mu\lambda}\left(C_{\mu\lambda}^{(0;1)}\right)^2 + \sum_{\mu\lambda}\left(C_{\mu\lambda}^{(0;2)}\right)^2 + 2\sum_{\mu\lambda}\left(C_{\mu\lambda}^{(0;1)}.C_{\mu\lambda}^{(0;2)}\right) + \sum_{\mu\lambda}\left(C_{\mu\lambda}^{(2;1)}\right)^2 + \sum_{\mu\lambda}\left(C_{\mu\lambda}^{(2;2)}\right)^2 +$$

$$2\sum_{\mu\lambda}\left(C_{\mu\lambda}^{(2;1)}.C_{\mu\lambda}^{(2;2)}\right) + \sum_{\mu\lambda}\left(C_{\mu\lambda}^{(4;1)}\right)^2 \quad (11)$$

Let us consider the irreducible decompositions of the elastic constant tensor in the following materials:

Table 1: Elastic constants

Materials	C_{11}	C_{33}	C_{44}	C_{12}	C_{13}
Zinc	165.0	61.8	39.3	31.1	50.0
Cadmium Sulfide	87.0	94.1	14.9	-54.6	47.5
Cadmium	116.0	50.9	19.6	42.0	41.0

Both materials (tool steel and rocks) are listed with increasing anisotropy degrees, that is from smaller $\frac{N_s}{N}$ to larger values. Among these five materials, Normal Tool Steel is the elastically strongest and Slate is the elastically most anisotropic.

Both materials (rocks and wood) are listed with increasing anisotropy degrees, that is from smaller $\frac{N_s}{N}$ to larger values. Among these materials Dunite is the elastically strongest and Oak is the elastically most anisotropic.

Table 2: Norms and the anisotropy degrees (norm ratios)

Material	N_s	N_d	N_n	N	N_s/N	N_d/N	N_n/N
Cadmium Sulfide	165.14	129.48	10.19	210.10	0.786	0.616	0.048
Zinc	267.26	86.28	12.14	281.10	0.951	0.307	0.043
Cadmium	196.61	55.56	2.50	204.32	0.962	0.272	0.012

Among these materials, Zinc is the elastically strongest and Cadmium Sulfide is elastically most anisotropic.

Table 3: Elastic constants of tool steel and rocks, hexagonal system transversely isotropic

Material	C_{11}	C_{33}	C_{44}	C_{12}	C_{13}
TOOL STEEL					
Normal	289.0	284.0	84.5	116.0	117.0
Hardened	277.0	272.0	80.8	113.0	112.0
ROCKS					
Micaschist	165.0	64.8	39.6	31.1	50.0
Slate	87.0	94.1	14.9	54.6	47.5
Eclogite	116.0	50.0	19.6	42.0	41.0

Table 4: Norms and the anisotropy degrees (norm ratios)

Material	N_s	N_d	N_n	N	N_s/N	N_d/N	N_n/N
TOOL STEEL							
Normal	592.461	4.658	0.384	592.479	0.99996	0.0079	0.0006
Hardened	567.798	4.438	1.461	567.817	0.99996	0.0078	0.0026
ROCKS							
Micaschist	152.791	16.753	5.078	153.790	0.99350	0.1089	0.0330
Eclogite	260.413	27.829	10.385	362.102	0.99356	0.1062	0.0396
Slate	181.220	63.857	23.2171	93.539	0.93653	0.3299	0.1199

Table 5: Orthorhombic system (orthotropic symmetry) non-crystalline materials elastic constants

Material	C_{11}	C_{22}	C_{33}	C_{44}	C_{55}	C_{66}	C_{12}	C_{13}	C_{23}
ROCKS									
Dunite	263.0	194.0	213.0	70.0	78.0	71.0	95.0	74.0	67.0
Zoistic Prasinite	175.0	164.0	158.0	63.6	51.1	45.5	63.0	72.0	72.0
Enslat.	186.0	179.0	159.0	51.6	55.6	60.0	60.0	54.0	56.0
Olivin.	323.0	210.0	199.0	73.3	70.9	68.6	93.0	92.0	82.0
Marble	119.0	110.0	104.0	29.7	30.7	32.6	51.0	52.0	47.0
Hornb	144.0	125.0	130.4	38.0	42.5	52.0	49.0	52.0	52.3
WOOD									
Oak	1.034	6.76	2.98	1.29	0.39	0.76	1.01	1.01	1.47
Beech	1.66	15.4	3.30	1.61	0.46	1.06	1.43	1.28	2.15
Pine	1.24	17.1	1.79	1.18	0.079	0.91	0.74	0.76	0.94
Spruce	0.755	17.2	0.965	0.624	0.035	0.854	0.550	0.332	0.541

Table 6: Orthorhombic system non-crystalline materials norms and norm ratios

Material	N_s	N_d	N_n	N	N_s/N	N_d/N	N_n/N
ROCKS							
Dunite	450.668	56.528	6.945	454.252	0.9921	0.1244	0.0153
Olivinite	451.137	37.834	16.696	453.029	0.9958	0.0835	0.0368
Enslatite	312.494	22.575	6.638	313.379	0.9972	0.072	0.0212
Zoistic Prasinite	348.197	9.754	22.742	3490.075	0.9975	0.0279	0.0652
Hornb	272.307	14.958	6.352	272.791	0.9982	0.0548	0.0233
Marble	234.427	12.530	1.455	234.766	0.9986	0.0534	0.0062
WOOD							
Spruce	8.7391	12.2398	5.3786	15.9723	0.5471	0.7663	0.3368
Pine	9.9011	11.6131	4.8855	16.0238	0.6179	0.7247	0.3049
Beech	11.4242	10.0296	3.4586	15.5906	0.7328	0.6433	0.2218
Oak	7.1576	3.9101	0.9574	8.2119	0.8716	0.4762	0.1166

Table 7: Average elastic coefficients measured in units of Gpa, from fresh unembalmed human and canine femora

Material	C_{11}	C_{22}	C_{33}	C_{44}	C_{55}	C_{66}	C_{12}	C_{13}	C_{23}
Human	18.0	20.0	27.6	6.23	5.61	4.52	9.98	10.1	10.7
Canine	19.0	22.2	29.7	6.67	5.67	4.67	9.73	11.9	11.9

Table 8: Norms and the anisotropy degrees (norm ratios)

Material	N_s	N_d	N_n	N	N_s/N	N_d/N	N_n/N
Human	46.347	7.026	0.902	46.886	0.9885	0.1499	0.0192
Canine	49.969	7.966	1.075	50.611	0.9873	0.1574	0.0212

Considering the ratio $\frac{N_s}{N}$, we can say that Canine is more anisotropic than Human and elastically Canine is a little bit stronger than Human.

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