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Scalar Irreducible Patrs of Sixth Rank Tensor

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Abstract: Reduction procedure of decomposing a Cartesian tensor of rank n into its irreducible parts is given. The numbers of different irreducible parts of the sixth rank Cartesian tensor are calculated. The number of independent components of sixth rank Cartesian tensor is computed. The scalar normalized irreducible parts of sixth rank Cartesian tensor are presented. Examples of different symmetry of sixth rank tensors are given.

Key words: Six rank, Irreducible parts, Tensor

Introduction

In kinetic theory of gases and liquids and in other branches of molecular science one often encounters the necessity of evaluating integrals involving tensors of rank six. On integration, such integrals are expressed as a linear combination of irreducible isotropic tensors of the same rank. One typical example is the evaluation of stress tensors in the Burnett order in kinetic theory, Burnett (1935) and Ferziger and Kaper(1972), which requires integrals of Cartesian products of six momenta. In the course of investigation on nonlinear transport phenomena for dense fluids such a necessity also arises in a more general setting, Eu (1979).

The reduction of a Cartesian tensor $T_{i_1i_2\ldots i_n}$ results in a sum of irreducible tensors with some weights represented more than once. Hence we can write, Coope and Sniders (1970) and Jerphagnon *et al.*, (1978)

$$T_{i_1i_2\ldots\ldots i_n} = \sum_{j=0}^{n}\sum_{q=1}^{N_n^{(j)}} T^{(j;q)}_{i_1i_2\ldots\ldots i_n} \quad (1)$$

Where q is called the seniority index of the irreducible tensor $T^{(j;q)}_{i_1i_2\ldots i_n}$ and $N_n^{(j)}$ is the multiplicity of weight j in this reduction.

The natural projection of x^j onto the irreducible subspace H_j^j of traceless symmetric tensors of order j is denoted by $E^{(j)} = E^{(j)}_{k_1k_2\ldots k_j;l_1l_2\ldots l_j}$.

The principle element in the reduction procedure is the mappings $Q^{(0;q)}_{i_1i_2\ldots i_n;k_1k_2\ldots k_j}$ of minimal rank tensor subspace $H^j_{j,q}$ onto $H^n_{j,q}$. We will choose the mappings such that they are orthonormal and g_{pq} will be reduced to identity matrix where g_{pq} is a symmetric matrix, which was used and defined in Ferziger and Kaper (1972), through the relation,

$$g_{pq}E^{(j)}_{k_1k_2\ldots k_j;l_1l_2\ldots l_j} = Q^{(0;p)}_{i_1i_2\ldots i_n;k_1k_2\ldots k_j} Q^{(0;q)}_{i_1i_2\ldots i_n;l_1l_2\ldots l_j} \quad (2)$$

In this work equation (2) is reduced to,

$$\delta_{pq}E^{(j)}_{k_1k_2\ldots k_j;l_1l_2\ldots l_j} = Q^{(0;p)}_{i_1i_2\ldots i_n;k_1k_2\ldots k_j} Q^{(0;q)}_{i_1i_2\ldots i_n;l_1l_2\ldots l_j} \quad (3)$$

The mapping $\widetilde{Q}^{(0;p)}_{k_1k_2\ldots k_j;i_1i_2\ldots i_n}$ dual to $Q^{(0;p)}_{i_1i_2\ldots i_n;k_1k_2\ldots k_j}$ are defined by the relation,

$$\widetilde{Q}^{(0;p)}_{k_1k_2\ldots k_j;i_1i_2\ldots i_n} = Q^{(0;p)}_{i_1i_2\ldots i_n;k_1k_2\ldots k_j} \quad (4)$$

The dual mapping extract the natural forms $t^{(j;p)}_{\lambda_1\lambda_2\ldots\lambda_j}$ from the tensor $T_{i_1i_2\ldots i_n}$ as follows,

$$t^{(j;p)}_{\lambda_1\lambda_2\ldots\lambda_j} = \widetilde{Q}^{(0;p)}_{\lambda_1\lambda_2\ldots\lambda_j;i_1i_2\ldots i_n} T_{i_1i_2\ldots i_n} \quad (5)$$

These tensors can be embedded in the tensor space of order n through the mapping,

$$T^{(j;q)}_{i_1 i_2 \ldots\ldots i_n} = Q^{(0;q)}_{i_1 i_2 \ldots i_n ; k_1 k_2 \ldots k_j} \, t^{(j;p)}_{k_1 k_2 \ldots k_j} \quad (6)$$

$$T^{(j;q)}_{i_1 i_2 \ldots\ldots i_n} = Q^{(0;q)}_{i_1 i_2 \ldots i_n ; k_1 k_2 \ldots k_j} \, \widetilde{Q}^{(0;p)}_{i_1 i_2 \ldots i_n ; l_1 l_2 \ldots l_j} \, T_{l_1 l_2 \ldots\ldots l_n} \quad (7)$$

The Sixth Rank Tensor: For $n = 6$, equation (1), implies that,

$$T_{ijklmn} = \sum_{j=0}^{6} \sum_{q=1}^{N_6^{(j)}} T^{(j;q)}_{ijklmn} =$$

$$\sum_{q=1}^{N_6^{(0)}} T^{(0;q)}_{ijklmn} + \sum_{q=1}^{N_6^{(1)}} T^{(1;q)}_{ijklmn} + \sum_{q=1}^{N_6^{(2)}} T^{(2;q)}_{ijklmn} + \sum_{q=1}^{N_6^{(3)}} T^{(3;q)}_{ijklmn} + \sum_{q=1}^{N_6^{(4)}} T^{(4;q)}_{ijklmn} + \sum_{q=1}^{N_6^{(5)}} T^{(5;q)}_{ijklmn} + \sum_{q=1}^{N_6^{(6)}} T^{(6;q)}_{ijklmn} \quad (8)$$

Where,

$$N_n^{(j)} = \sum_k (-1)^k \binom{n}{k} \binom{2n-3k-j-2}{n-2} \quad (9)$$

and

$$0 \le k \le [(n-j)/3] \quad (10)$$

From relation's (9) and (10) we have the following table,

Table 1: Values of $N_n^{(j)}$ for $n=6$ and $j = 1,2,3,4,5,6$.

n	j	$N_n^{(j)}$
6	0	15
6	1	36
6	2	40
6	3	29
6	4	15
6	5	5
6	6	1

Each irreducible tensor has $(2j+1)$ independent components, so that the total number of components in the reduction form is:

$$\sum_{j=0}^{n} (2j+1) N_n^{(j)} = 3^n.$$

So for $n = 6$, we have,

$$\sum_{j=0}^{n} (2j+1) N_n^{(j)} =$$

$$N_6^{(0)} + 3N_6^{(1)} + 5N_6^{(2)} + 7N_6^{(3)} + 9N_6^{(4)} + 11N_6^{(5)} + 13N_6^{(6)}$$

$$= 1\times15 + 3\times36 + 5\times40 + 7\times29 + 9\times15 + 11\times5 + 13\times1$$

$$= 729 = 3^6$$

Equation (8) and table (1) imply that,

$$T_{ijklmn} = \sum_{j=0}^{6} \sum_{q=1}^{N_6^{(j)}} T^{(j;q)}_{ijklmn} =$$

$$\sum_{q=1}^{15} T^{(0;q)}_{ijklmn} + \sum_{q=1}^{36} T^{(1;q)}_{ijklmn} + \sum_{q=1}^{40} T^{(2;q)}_{ijklmn} + \sum_{q=1}^{29} T^{(3;q)}_{ijklmn} + \sum_{q=1}^{15} T^{(4;q)}_{ijklmn} + \sum_{q=1}^{5} T^{(5;q)}_{ijklmn} + T^{(6;1)}_{ijklmn} \quad (11)$$

Table 2: Natural Projections for Traceless Symmetric Tensors

Order j	$E^{(j)}_{k_1 k_2 \ldots k_j ; l_1 l_2 \ldots l_j}$
0	1
1	δ_{kl}
2	$(\delta_{kl})^2 - \frac{1}{2}\delta_{kk}\delta_{ll}$
3	$(\delta_{kl})^3 - \frac{3}{5}\delta_{kl}\delta_{kk}\delta_{ll}$
4	$(\delta_{kl})^4 - \frac{6}{7}(\delta_{kl})^2\delta_{kk}\delta_{ll} + \frac{3}{35}(\delta_{kk})^2(\delta_{ll})^2$
.	.
.	.
.	.
.	.
j	$(\delta_{kl})^j - \frac{j(j-1)}{2(2j-1)}(\delta_{kl})^{j-2}\delta_{kk}\delta_{ll} + \frac{(j-1)(j-2)}{2(2j-1)(2j-3)}(\delta_{kk})^2(\delta_{ll})^2 + \ldots\ldots$

Scalar Parts of Sixth Rank Tensor: Let us take $n = 6$ and $j = 0$. From equation (11) and table (1), the seniority index q will take the values $1,2,3,\ldots\ldots\ldots,15$:: For $q = 1$:

$Q^{(0;1)}_{ijklmn} = a\delta_{ij}\delta_{kl}\delta_{mn}$ Where a is the normalization constant, and the normalization

Condition is:

$$Q^{(0;1)}_{ijklmn} Q^{(0;1)}_{ijklmn} = a^2 \delta_{ij}\delta_{kl}\delta_{mn}\delta_{ij}\delta_{kl}\delta_{mn}$$

$$= 1, \rightarrow 27a^2 = 1, \rightarrow a = \frac{1}{\sqrt{}}$$

$$\Rightarrow Q^{(0;1)}_{ijklmn} = \frac{1}{3\sqrt{3}}\delta_{ij}\delta_{kl}\delta_{mn}, \Rightarrow \widetilde{Q}^{(0;1)}_{ijklmn} = \frac{1}{3\sqrt{3}}\delta_{ij}\delta_{kl}\delta_{mn}$$

Equation (7) implies that:

$$T^{(0;1)}_{ijklmn}=\frac{1}{27}\delta_{ij}\delta_{kl}\delta_{mn}T_{ppqqrr} \quad (12)$$

In the same way we can find the other irreducible parts:

$$T^{(0;2)}_{ijklmn}=\frac{1}{48}\left(\delta_{im}\delta_{jl}\delta_{kn}-\delta_{ik}\delta_{lm}\delta_{jn}\right)\left(T_{pqrqpr}-T_{pqprrq}\right) \quad (13)$$

$$T^{(0;3)}_{ijklmn}=\frac{1}{48}\left(\delta_{in}\delta_{jm}\delta_{kl}-\delta_{il}\delta_{jk}\delta_{mn}\right)\left(T_{pqrrqp}-T_{pqqprr}\right) \quad (14)$$

$$T^{(0;4)}_{ijklmn}=\frac{1}{48}\left(\delta_{kl}\delta_{in}\delta_{jm}-\delta_{mn}\delta_{ik}\delta_{jl}\right)\left(T_{pqrrqp}-T_{pqpqrr}\right) \quad (15)$$

$$T^{(0;5)}_{ijklmn}=\frac{1}{48}\left(\delta_{kl}\delta_{im}\delta_{jn}-\delta_{mn}\delta_{il}\delta_{jk}\right)\left(T_{pqrrpq}-T_{pqqprr}\right) \quad (16)$$

$$T^{(0;6)}_{ijklmn}=\frac{1}{48}\left(\delta_{in}\delta_{jk}\delta_{lm}-\delta_{ik}\delta_{ln}\delta_{jm}\right)\left(T_{pqqrrp}-T_{pqprqr}\right) \quad (17)$$

$$T^{(0;7)}_{ijklmn}=\frac{1}{36}\left(\delta_{ij}\delta_{km}\delta_{ln}-\delta_{ij}\delta_{kn}\delta_{lm}\right)\left(T_{ppqrqr}-T_{ppqrrq}\right) \quad (18)$$

$$T^{(0;8)}_{jklmn}=\frac{1}{36}\left(\delta_{kl}\delta_{im}\delta_{jn}-\delta_{kl}\delta_{in}\delta_{jm}\right)\left(T_{pqrrpq}-T_{pqrrqp}\right) \quad (19)$$

$$T^{(0;9)}_{jklmn}=\frac{1}{36}\left(\delta_{mn}\delta_{ik}\delta_{jl}-\delta_{mn}\delta_{il}\delta_{jk}\right)\left(T_{pqpqrr}-T_{pqqprr}\right) \quad (20)$$

$$T^{(0;10)}_{jklmn}=\frac{1}{216}\left(\delta_{ij}\delta_{kl}\delta_{mn}-3\delta_{il}\delta_{jk}\delta_{mn}\right)\left(T_{ppqqrr}-3T_{pqqprr}\right) \quad (21)$$

$$T^{(0;11)}_{ijklmn}=\frac{1}{144}\left(\delta_{in}\delta_{jm}\delta_{kl}+\delta_{il}\delta_{jk}\delta_{mn}-2\delta_{ij}\delta_{km}\delta_{ln}\right)\left(T_{pqrrqp}+T_{pqqprr}-2T_{ppqrqr}\right) \quad (22)$$

$$T^{(0;12)}_{ijklmn}=\frac{1}{144}\left(\delta_{kl}\delta_{im}\delta_{jn}+\delta_{mn}\delta_{ik}\delta_{jl}-2\delta_{ij}\delta_{km}\delta_{ln}\right)\left(T_{pqrrpq}+T_{pqpqrr}-2T_{ppqrqr}\right) \quad (23)$$

$$T^{(0;13)}_{ijklmn}=\frac{1}{3600}\left(\delta_{ij}\delta_{kl}\delta_{mn}+\delta_{in}\delta_{jm}\delta_{kl}-4\delta_{im}\delta_{jn}\delta_{kl}\right)\left(T_{ppqqrr}+T_{pqrrqp}-4T_{pqrrpq}\right) \quad (24)$$

$$T^{(0;14)}_{ijklmn}=\frac{1}{4752}\left(9\delta_{im}\delta_{jl}\delta_{kn}+9\delta_{ik}\delta_{lm}\delta_{jn}-2\delta_{ij}\delta_{kl}\delta_{mn}\right)\left(9T_{pqrqpr}+9T_{pqprrq}-2T_{ppqqrr}\right) \quad (25)$$

$$T^{(0;15)}_{ijklmn}=\frac{1}{6744}\left(3\delta_{ik}\delta_{ln}\delta_{jm}+\delta_{il}\delta_{jk}\delta_{mn}+11\delta_{ij}\delta_{kn}\delta_{lm}-13\delta_{in}\delta_{jm}\delta_{kl}\right)\left(3T_{pqprqr}+T_{pqqprr}+11T_{ppqrrq}-13T_{pqrrqp}\right) \quad (26)$$

Examples of Sixth Rank Tensors: 1. Sixth rank tensor **C** symmetric with respect to the first, second, third pairs of indices and to their permutation (Sitotin and Shaskolskaya, 1979), satisfies: $C_{ijklmn}=C_{jiklmn}=C_{ijlkmn}=C_{ijklnm}=C_{ijmnkl}=C_{klijmn}$.

The coefficients $C_{\lambda\mu\nu}=C_{ijklmn}$ $ij\leftrightarrow\lambda$, $kl\leftrightarrow\mu$, $mn\leftrightarrow\nu$, $(\lambda,\ \mu,\ \nu\ =1,\dots,6)$

2. Sixth rank tensor **q** symmetric with respect to the first, second, third pairs of indices and to the permutation of the second and third pairs, satisfies:

$q_{ijklmn}=q_{jiklmn}=q_{ijlkmn}=q_{ijklnm}=q_{ijmnkl}$.

The coefficients $q_{\lambda\mu\nu}=q_{ijklmn}$ $ij\leftrightarrow\lambda$, $kl\leftrightarrow\mu$, $mn\leftrightarrow\nu$, $(\lambda,\ \mu,\ \nu\ =1,\dots,6)$

Appendix:

Definitions:

1 - Isotropic Tensors: A tensor is called isotropic if its components retain the same values however the axes are rotated like δ_{ik}, ε_{iks} and $\varepsilon_{iks}\varepsilon_{mps}$, where the symbol δ_{ik} is called the identity matrix (Kronecker delta) defined by:

$$\delta_{ik}=\begin{cases}1 & if \quad i=k\\ 0 & if \quad i\neq k\end{cases}$$

and the symbol ε_{iks} is called Levi – Civita antisymmetric tensor defined by:

$$\varepsilon_{iks}=\begin{cases}1 & for \quad ikm= \quad 123,231,312\\ -1 & for \quad ikm= \quad 132,213,321\\ 0 & in \quad all \quad other\ cases.\end{cases}$$

and $\varepsilon_{iks}\varepsilon_{mps}=\delta_{im}\delta_{kp}-\delta_{ip}\delta_{km}$. There are no isotropic tensors of the first rank, $f^{(0)r}_{(m)}$ isotropic tensor of rank m, which are product of $m/2$ Kronecker deltas if m is even, and products of $(m-3)/2$ kronecker deltas, and Levi – Civita antisymmetric tensor if m is odd, the index r in $f^{(0)r}_{(m)}$, is used to differentiate the various index permutation of f, each of which contracts with a tensor of rank m to give one of rank j.

2 - Isotropy*:* A material is isotropic with respect to certain properties if these properties are the same in all directions.

3 - $trace(E_{ij})=tr(E_{ij})=E_{ii}=E_{11}+E_{22}+E_{33}$.

4 - Irreducible sets obtained by reducing sets of tensors serve to construct new tensors in various ways. Consider first the simultaneous reductions of the set of components of an ordinary tensor **T** and of the corresponding set of base unit tensors. This operation splits the tensor into a sum of tensors, each of which consists of an irreducible set of base tensors. Each of these tensors may accordingly be called an irreducible tensor. For example a second rank tensor represented by:

$$T = iiT_{xx} + ijT_{xy} + \dots\dots\dots + kkT_{zz},$$

revolves into three parts $T = T^{(0;1)} + T^{(1;1)} + T^{(2;1)}$, the form of this expression emphasizes the structure of the irreducible tensor $T^{(0;1)}$, $T^{(1;1)}$ and $T^{(2;1)}$ in terms of irreducible sets, where:

$$T^{(0;1)} = \frac{1}{3}\delta_{ij}T_{pp}$$

$$T^{(1;1)} = \frac{1}{2}\left(T_{ij} - T_{ji}\right)$$

$$T^{(2;1)} = \frac{1}{2}\left(T_{ij} + T_{ji}\right) - \frac{1}{3}\delta_{ij}T_{pp}.$$

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