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Development of Quadratic Plate Bending Macro-Elements

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Abstract: Macro-elements play a big role in reducing number of equations to be solved in finite-element analysis. This is because a single macro-element will represents many finite elements. Two types of quadratic quadrilateral plate bending macro-elements were developed in this paper. These macro-elements are based on equivalent energy theory. Implementation of these macro-elements in the analysis showed reduction in number of equations and computer time and excellent results were achieved. These new developed macro-elements were tested and the results were compared with the results of conventional plate bending finite element solutions and with closed form solutions if available.

Keywords: Quadraic Plate Bending, Macro-Elements

Introduction

The analysis of large structural systems using the conventional finite element method is impractical. This is because of the necessity to use relatively fine mesh to obtain an accurate model. This will lead to a large number of equations to be solved. Therefore, it is advantageous to seek for approaches that reduce the total number of degrees of freedom (d.o.f) needed to successfully model large systems. One of these methods is to use macro-elements.

In this paper two types of quadratic plate bending macro-elements were developed.

These macro-elements are based on transformation of many structural finite elements into single equivalent macro-element.

This is done by preserving the same potential energies of the structure modeled by finite elements and the same structure modeled by macro-elements. The first macro-element is based on the quadratic serendipity (Q-8) finite element. The second macro-element is based on the quadratic Lagrangian (Q-9) finite-element.

Plate Bending Finite Elments Used In The Formulations Of The Macro-Elements: What follows are brief information about the plate bending finite elements studied and used in the formulations of the macro-elements.

The (Q8) Quadratic Serendipity Finite Element

This element has eight nodes with three d.o.f per node [Rock and Hinton, 1976] see fig (2). It is Mindlen type plate bending finite element.

The displacement vector is

$\lfloor q_i \rfloor = [\, W_i \quad W_{i,y} \quad -W_{i,x} \,]$

Where i = 1, 2, ... 8

The (Q9) Quadratic Langrangian Finite Element:

This element is a Mindlen type plate bending element with nine nodes and three d.o.f per node [pugh etal. 1978] as shown in fig (3).

The displacement vector is:

$\lfloor q_i \rfloor = [\, W_i \quad W_{i,y} \quad -W_{i,x} \,]$

Where i = 1, 2, ... 9

Formulation Of The Macro-Elements: The stiffness matrix of a macro-element is formulated by equating the strain energy of the original structure modeled by finite-elements and that of the equivalent macro-element model as follows:

$$U_o = U_m \qquad (1)$$

Where:

U_o: The strain energy of the original structure modeled by many finite elements that constitute one macro-element.

U_m: The strain energy of the macro-element.

$$\frac{1}{2} \lfloor q_o \rfloor [Sk_o] \{ q_o \} = \frac{1}{2} \lfloor q_m \rfloor [K_m] \{ q_m \} \qquad (2)$$

Where:

q_o: Displacement vector of the structure modeled by many finite elements that constitute one macro-element.

q_m: Displacement vector of one macro-element.

$[SK_o]$: The assembled stiffness matrix of all stiffness matrices of the finite elements constituting one macro-element.

$[K_m]$: The stiffness matrix of the macro-element.

Let the displacement vector of the original structure, (which constitute one macro-element) $\{q_o\}$ be related to that of the macro-element $\{q_m\}$ as:

$$\{q_o\} = [T] \{q_m\} \qquad (3)$$

Where: [T] is the transformation matrix for the macro-element. Substituting Eq. (3) into Eq. (2) gives:

$$\lfloor q_m \rfloor [T]^T [SKo] [T] \{q_m\} = \lfloor q_m \rfloor [K_m] \{q_m\}$$

$$[T]^T [SK_o] [T] = [K_m] \qquad (4)$$

In the solution, matrix $[SK_o]$ is not needed, only $[K_o]$, the stiffness matrix of a single finite element bounded by the macro-element is needed. To explain this let.

n: The number of finite elements comprising the macro-element.

$[T_e]$: The finite-element transformation matrix.

Every time $[T_e]$ carries a partition of the transformation matrix [T] that corresponds to the degrees of freedom of the finite-element under consideration. The transformed stiffness matrix for each finite-element is placed in its proper place in the structural stiffness matrix of the equivalent model, which is the place of $[K_m]$, as:

$$\sum_{e=1}^{n} [T_e]^T [K_o] [T_e] = [K_m] \qquad (5)$$

The transformation matrix $[T_e]$ is simply the evaluation of the shape functions of the macro-element at the nodes of the finite-element. This evaluation is based on *local coordinates for the nodal points of the finite-*elements with respect to the macro-element nodes.

To form a general transformation matrix [Ti] corresponding to an arbitrary nodal point i of a certain finite element within a certain macro-element, consider the notation N_{ki} which means that shape function k of node i of this macro-element is evaluated at point i using its local coordinates within the macro-element. The transformation matrix will depend on the macro-element type as follows:

a. **The (Q8) Quadratic Serendipity Finite-Element:**

The displacement functions over this finite element are expressed as follow:

$$W = \sum_{i=1}^{n} N_i \, W_i \; ; \; \theta_x = \sum_{i=1}^{n} N_i \, \theta_{xi} \quad \& \quad \theta_y = \sum_{i=1}^{n} N_i \, \theta_{yi}$$

Where the shape functions (Ni) are the same in the above equations.

To construct [Te] of a certain finite element consider Fig- (6). The transformation matrix [Te] of the finite element [k,L,m,n,o,p,q,r] which is inside the macro-element (1,2,3,4,5,6,7,8) will be as follows:

$$[Te] = \begin{bmatrix} T_{K1} & T_{K2} & T_{K3} & T_{K4} & T_{K5} & T_{K6} & T_{K7} & T_{K8} \\ T_{L1} & T_{L2} & T_{L3} & T_{L4} & T_{L5} & T_{L6} & T_{L7} & T_{L8} \\ T_{m1} & T_{m2} & T_{m3} & T_{m4} & T_{m5} & T_{m6} & T_{m7} & T_{m8} \\ T_{n1} & T_{n2} & T_{n3} & T_{n4} & T_{n5} & T_{n6} & T_{n7} & T_{n8} \\ T_{o1} & T_{o2} & T_{o3} & T_{o4} & T_{o5} & T_{o6} & T_{o7} & T_{o8} \\ T_{p1} & T_{p2} & T_{p3} & T_{p4} & T_{p5} & T_{p6} & T_{p7} & T_{p8} \\ T_{q1} & T_{q2} & T_{q3} & T_{q4} & T_{q5} & T_{q6} & T_{q7} & T_{q8} \\ T_{r1} & T_{r2} & T_{r3} & T_{r4} & T_{r5} & T_{r6} & T_{r7} & T_{r8} \end{bmatrix}$$

Where:

$$[T_{K1}] = \begin{bmatrix} N_1 & 0 & 0 \\ 0 & N_1 & 0 \\ 0 & 0 & N_1 \end{bmatrix}$$

i.e. the participation of node [k] of the finite element that corresponds to node (1) of the macro-element under consideration.

In general:

$$[T_{ij}] = \begin{bmatrix} N_j & 0 & 0 \\ 0 & N_j & 0 \\ 0 & 0 & N_j \end{bmatrix}$$

Where: i = k , L , m , ...q , r the nodes of the finite element.

j = 1, 2 , 3 , ... 7 , 8 the nodes of the macro element.

Then :

$$\sum_{e=1}^{n} [T_e]^T [K_0] [T_e] = [K_m]$$

b. **The (Q9) Quadratic Lagrangian Finite Element:**

Here, there are nine nodes with three d.o.f. of type w , θ_x & θ_y.

Then:

$[T_e]$ is 27 * 27 and:

$$[T_{ij}] = \begin{bmatrix} N_j & 0 & 0 \\ 0 & N_j & 0 \\ 0 & 0 & N_j \end{bmatrix}$$

Where: i = k, L, m, n, o, p, q, r, s
j = 1, 2, 3, 4, 5, 6, 7, 8, 9

And: $$\sum_{e=1}^{n} [T_e]^T [K_0] [T_e] = [K_m]$$

Macro-Element load vector: The externals loading are applied at known nodes of the finite-element model. However, these nodes may not necessarily coincide with the macro-elements nodes. It is required to calculate the equivalent consistent nodal load vector of each macro-element.

In general, all forms of loading other than concentrated loads subjected to the original structure nodes must be first reduced to equivalent nodal forces acting on the original structure, as with the conventional finite element method. The nodal load vector of the original structure can then be transformed to equivalent macro-element structural load vector by equating the external work done on the original structure modeled by finite-elements and that of the macro-element model as follows:

$W_o = W_m$ (7)

Where:

W_o: The external work done on the original structure that constitute one macro-element.

W_m: The external work done on the macro-element.

$\lfloor q_o \rfloor \{F_o\} = \lfloor q_m \rfloor \{F_m\}$ (8)

Where:

$\{F_o\}$: The assembled nodal load vector of the finite-elements constituting one macro-element.

$\{F_m\}$: The equivalent nodal load vector of the macro-element. Substituting Eq. (3) into Eq. (8) gives:

$\lfloor q_m \rfloor [T]^T \{F_o\} = \lfloor q_m \rfloor \{F_m\}$

$[T]^T \{F_o\} = \{F_m\}$ (9)

Where [T] is the same transformation matrix used in deriving $[K_m]$.

The Assembly of all the macro-element stiffness matrices into a structural stiffness matrix and also the construction of the macro-element structural load vector and solution of the structure equation are the same as that of conventional finite element method.

Applications: Various problems of plate bending analysis are solved and presented below in order to demonstrate the efficiency of the macro-elements developed. The accuracy of the equivalent energy macro-elements are checked by using the conventional finite elements menthod and, if available, the exact solution. The moments and stresses are generally calculated at the Gauss points of the macro-elements in the problems presented below unless it is stated differently.

Problem No (1): The analysis of thin, square, simply supported isotropic plate under a uniformly distributed load, as shown in Fig. (7).

The following data are given for this problem:

L = 10 .... in units of length.

T = 0.1 .... in units of length.

$E = 10.92 * 10^7$ .... in units of force/area.

$G_{xy} = G_{xz} = G_{yz} = 4.2 * 10^7$.............. in units of force/area.

Nu = 0.3 ....

$Q_z = 1.0$ .... In units of force/area.

The results may be expressed in a normalized form as follows:

Deflection = $C * Q_z * L^4 * 10^{-2} / D$

Rotations (in x or y) = $C * Q_z * L^3 * 10^{-1} / D$

M_x, M_y or $M_{xy} = C * Q_z * L^2 * 10^{-4}$

(for Nu = 0.3)

where: $C * 10^{-2}$ represents the value of the function for the normalized data given above.

Due to symmetry only one quarter of plate is analyzed. The analysis is done using the (Q8) elements, as shown in Fig. (7).

The original finite element mesh has (65) nodes and (195) d.o.f. The equivalent energy model has (21) nodes and (63) d.o.f. The total reduction in d.o.f. is 67.7%.

The results for deflections and rotations are shown in Figs. (8 & 9). The maximum errors are (7.9%) & (4.2%) respectively.

Table (1) shows a comparative study for the execution time (CPU), the band width-solution operation count ($Ne * HBW^2$), central deflections and their corresponding errors. The analysis is done using (Q8) conventional F.E. and (Q8) equivalent M.E. meshes. The bandwidth-solutions operation count (Armanios & Negm, 1983) is a useful measure of the computer time required to solve banded equations. The errors in central deflections are measured from those of the original F.E. meshes.

Problem No (2): The analysis of thin, circular, clamped, isotropic plate with slopped bottom surface, under triangular loading, as shown in Fig. (10). The following data are given for this problem:

R= 3.0 m.

t = variable from 0.1m at center of plate to 0.25 m at edges of the plate, i.e. bottom slope is 5%.

$E = 25 * 10^6$ kN. / m^2

$G_{xy} = 10.5932 * 10^6$ kN. / m^2, $G_{xz} = G_{yz} = 10.5932 * 10^6$ kN. / m^2.

Nu = 0.18

Loading is triangular and varying from zero at center of plate to 25.958 kN./ m^2 at edges of plate.

Due to symmetry, only one quarter of plate is analyzed. The analysis is done using the (Q8) isoparametric elements.

Table (2) shows the details of the original F.E. model and the equivalent M.E. models for the problem.

The results for deflections, rotations (in x direction) and moments M_x along the x-axis are shown in Figs (13, 14 & 15).

When using the equivalent energy theory, the moments are calculated at the Gauss points of the individual F.E. inside the M.E., see Fig. (14).

The maximum errors for deflections when using the equivalent M.E. are at the center of the plate and having the values (0.28%) & (14.58%) for M.E. meshes (12 Q8) & (3 Q8) respectively. These errors are measured from the (48 Q8) conventional F.E. analysis for the problem.

Problem No (3): The analysis of thin, isotropic, annular plate with different boundary conditions, under a uniformly distributed load, as shown in Figs. (15 & 18). The following data are given for this problem:

R = 2.0 m.

t = 0.02 m.

$E = 200 * 10^6$ kN / m^2

$G_{xy} = G_{xz} = G_{yz} = 76.923 * 10^6$ kN / m^2

Nu = 0.3

$Q_z = 9.158$ kN / m^2

Due to symmetry, only on quarter of plate is analyzed. The analysis is done using the (Q9) isoparametric element.

The plate is first considered, as having clamped outer boundary and free inner boundary, as shown is Fig. (15).

Table (3) shows that the total number of d.o.f. is reduced by (44.4%) i.e. one macro-element contains two finite elements.

The results for deflections and moments (M_x) along the x-axis are shown in Figs (16 & 17).

The plate is then considered as having clamped outer boundary and guided inner boundary, as shown in Fig. (18). The same discretizations are used here as for the previous case, see table (3). The results for deflections and moments (M_x) along the x-axis are shown in Figs. (19 & 20). The maximum error in M_x is 8.3%.

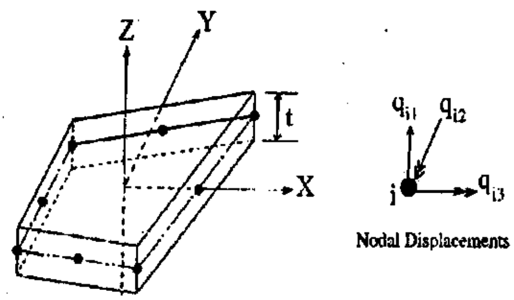


FIG.1. A General Quadrilateral Isoparametric Finite Element.

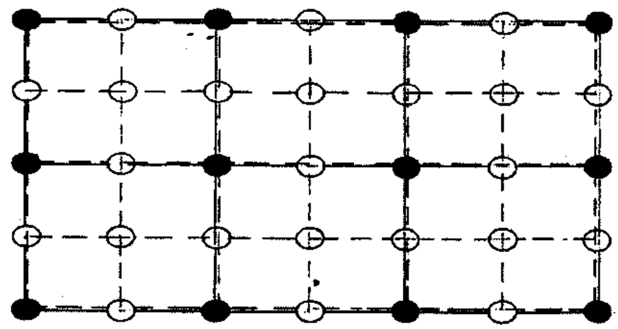


Fig 4 A General Macro - Element Discretization

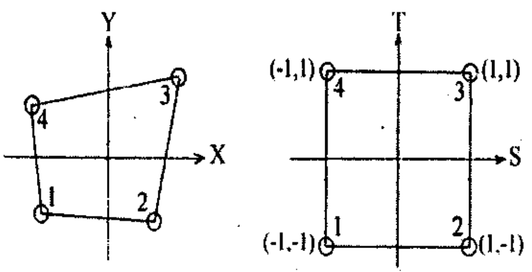


FIG.2. The Bilinear (L4) Quadrilateral Isoparametric Finite Element.

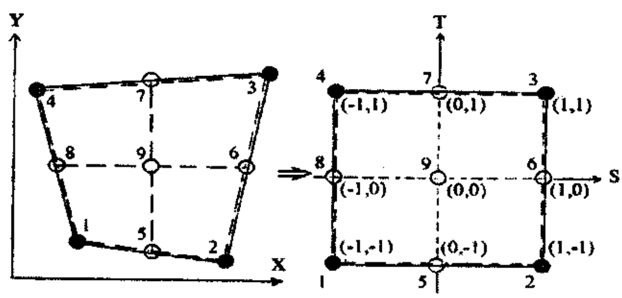


Fig.5 A General Description Of The Local Coordinates (S,T) Within A M.E.

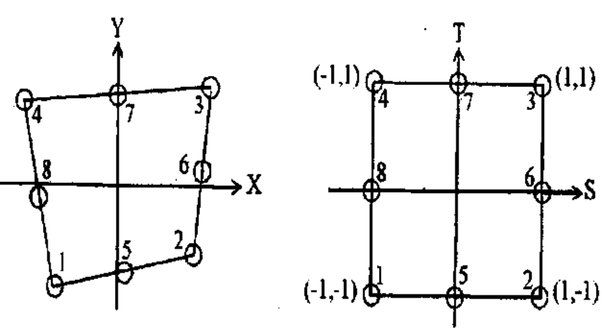


FIG.3. The Quadratic Serendipity (Q8) Quadrilateral Isoparametric Finite Element.

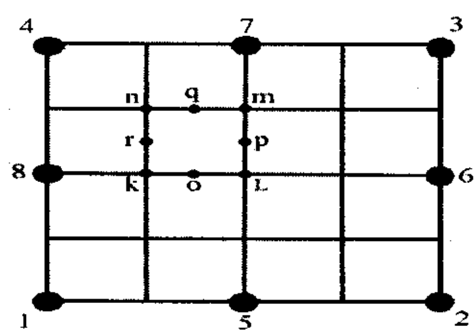


Fig.6 The Correspondance Between The Finite Element DOF And The Macro Element DOF

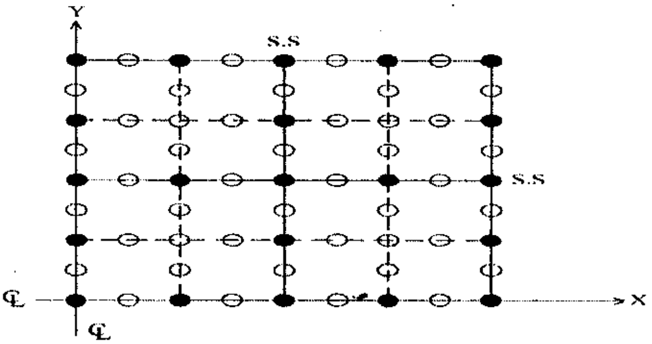


Fig 7 Quarter Of Plate For Problem No.1 Analyzed With the (Q8) Elements

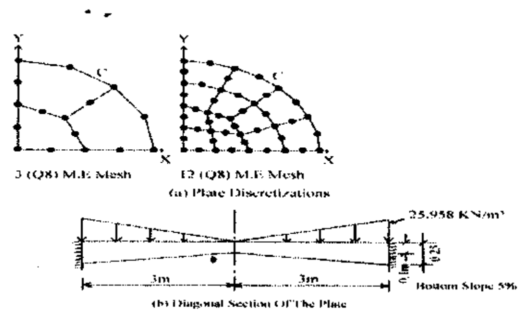


Fig. 10: The Plate For Problem No.2
(a) Plate Discretizations, (b)Diagonal Section of The Plate

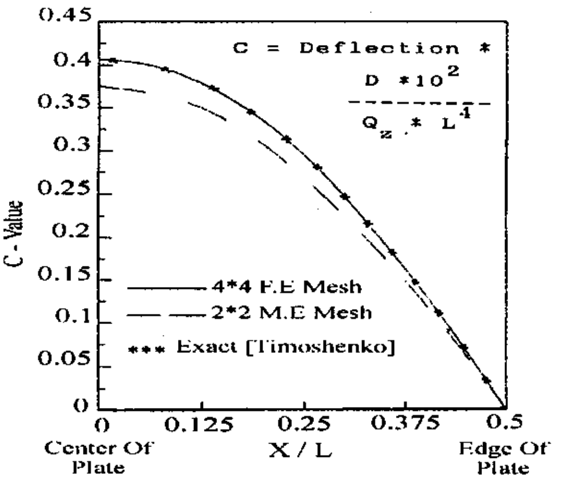


Fig.8 X-Axis Deflection For Problem No.1

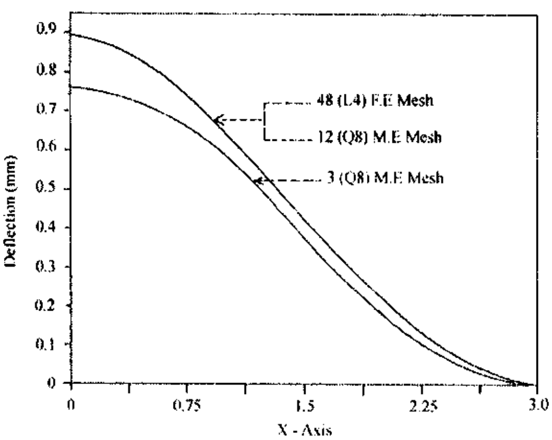


Fig. 11: X-Axis Deflection For Problem No.2

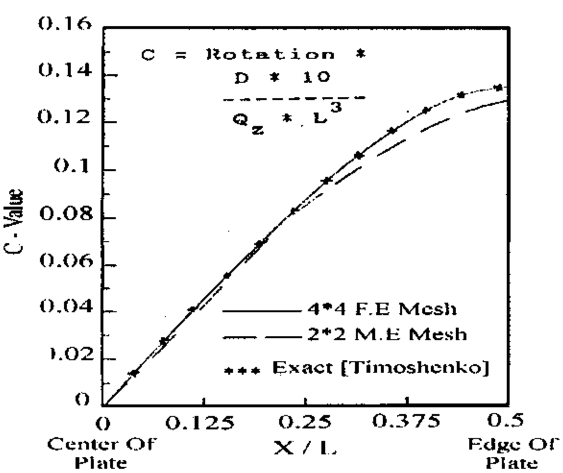


Fig.9 X-Axis Rotation For Problem No.1

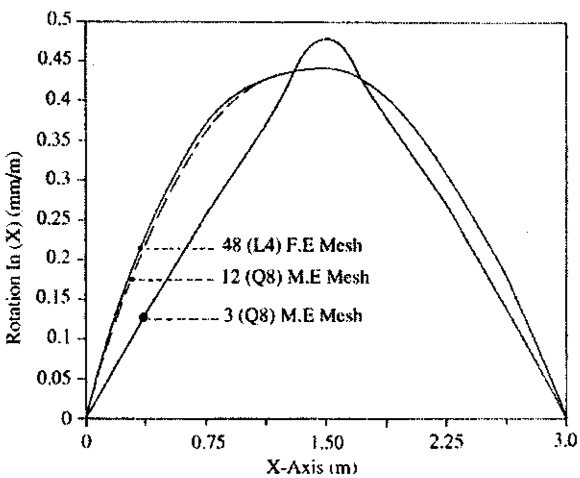


Fig. 12: X-Axis Rotation In (X) For Problem No.2

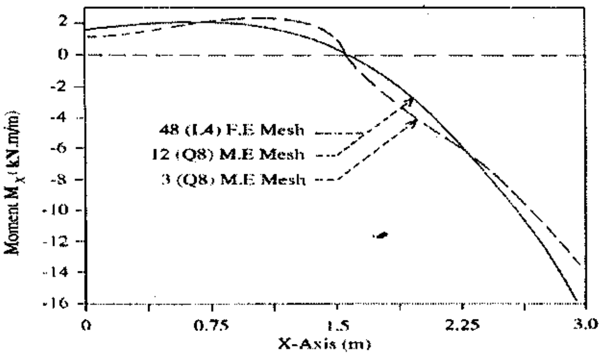


Fig. 13: Moment M_x Along X-Axis For Problem No.2

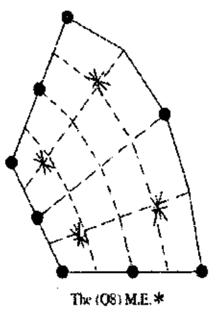


Fig.14: Gauss Points Locations In XY-Coordinate System For The (Q8) M.E. In Problem No.2

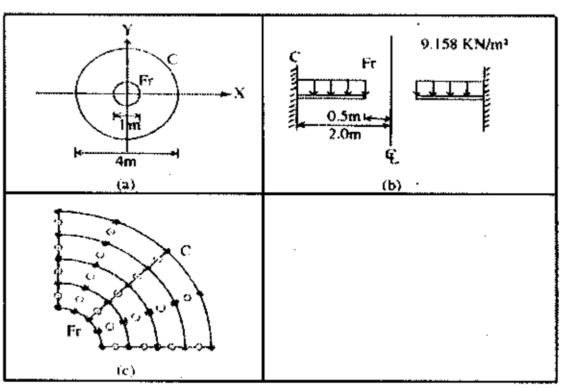
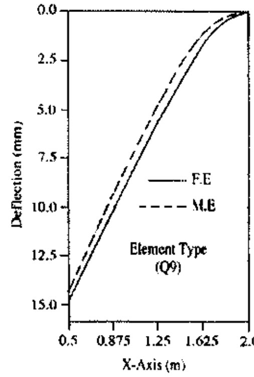
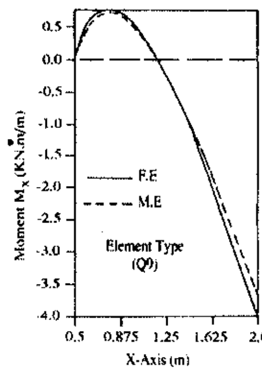


Fig. 15: Details Of The Clamped-Guided Annular Plate For Problem No.3 Analyzed With The (Q9) Elements
(a) Plan Of Plate, (b) Diagonal Section Of Plate
(c) Discretization Of Quarter Of Plate



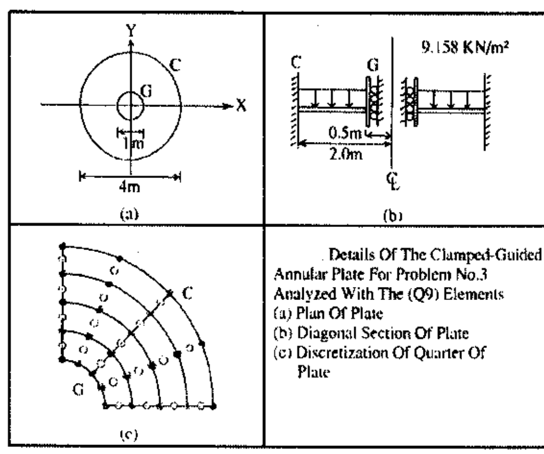
X-Axis Deflection For The Clamped-Free Plate Of Problem No.3

Fig. 16



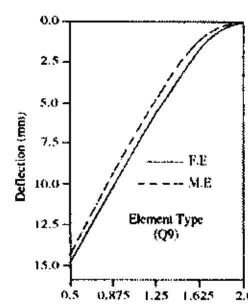
Moment M_x Along X-Axis For The Clamped-Free Plate Of Problem No.3

Fig. 17

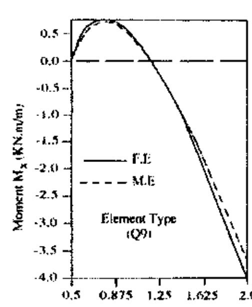


Details Of The Clamped-Guided Annular Plate For Problem No.3 Analyzed With The (Q9) Elements
(a) Plan Of Plate
(b) Diagonal Section Of Plate
(c) Discretization Of Quarter Of Plate

Fig. 18



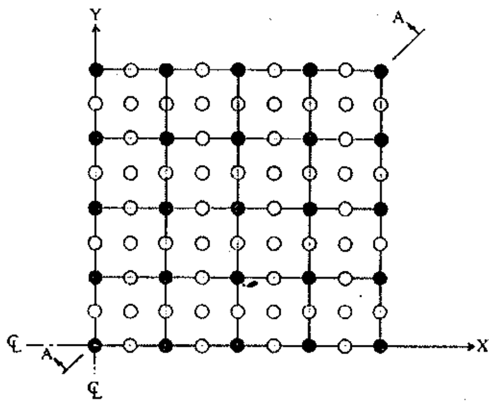
X-Axis Deflection For The Clamped-Free Plate Of Problem No.3



Moment M_x Along X-Axis For The Clamped-Free Plate Of Problem No.3

Fig. 19

Fig. 20



Quarter Of Plate For Problem No.4 Analyzed With (Q9) Elements

Fig. 21

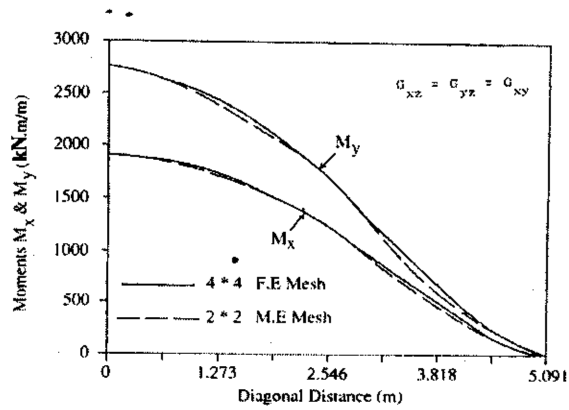
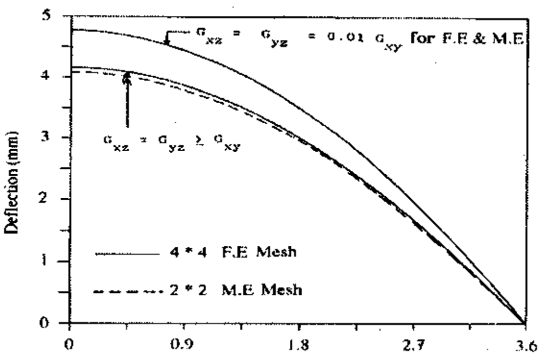


Fig. 24 Moments M_x & M_y Along Sec.A-A For The Thick Plate Of Problem No.4 (t/L=0.15)



X-Axis Deflection For The Thin Plate Of Problem No.4 (t/L=0.02)

Fig.22

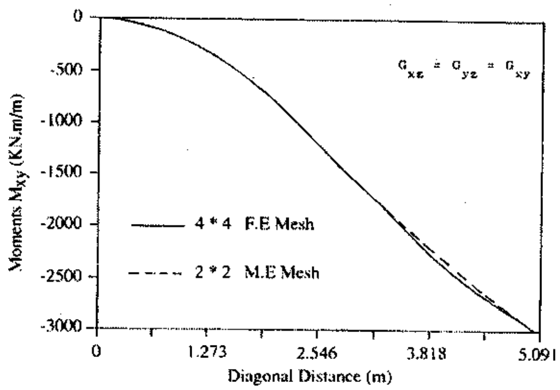
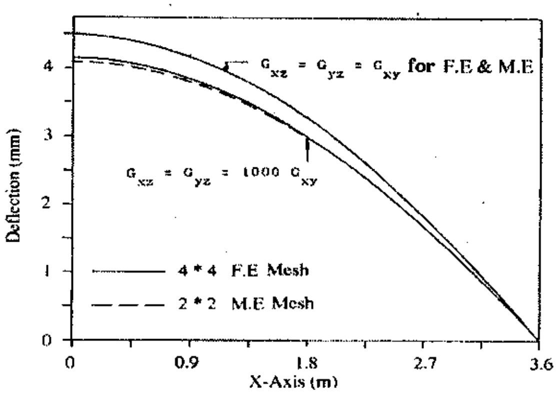


Fig. 25 Moment M_{xy} Along Sec.A-A For The Thick Plate Of Problem No.4



X-Axis Deflection For The Thick Plate Of Problem No.4 (t/L=0.15)

Fig. 23

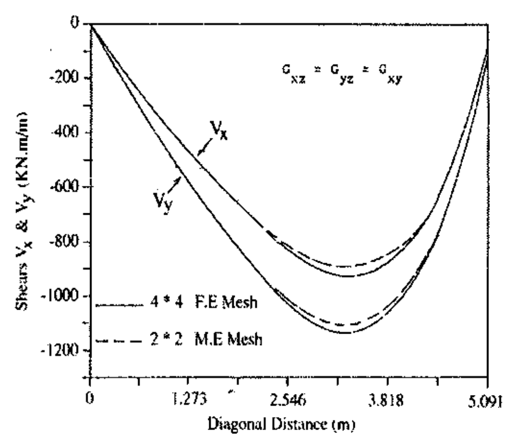


Fig. 26 Shears V_x & V_y Along Sec.A-A For The Thick Plate Of Problem No.4

Table 1: A Comparative Study of Different (Q8) Meshes for Problem No.1

Original F.E Mesh	M.E Mesh	M.E size (F.E*F.E)	CPU (seconds)	Ne*HBW^2	C=central def $\frac{\bar{D} * 10^2}{Qz * L^4}$	Error % in Def1.
	Conventional F.E. analysis		858.5	$1443*123^2$= 21831147	0.4064452	-
	6*6 (864 d.o.f)	2*2	492.1	$399*69^2$ = 1899639	0.4057476	0.17
12*12 Q8 (3456 .o.f)	4*4 (384 d.o.f)	3*3	438.7	$195*51^2$ = 507195	0.4043378	0.52
	3*3 (216 d.o.f)	4*4	428.3	$120*42^2$ = 211680	0.4009996	1.34
	2*2 (96 d.o.f)	6*6	421.7	$63*33^2$ = 68607	0.3731404	8.19
	Conventional F.E. analysis		206.9	$675*87^2$ = 5109075	0.4064444	-
8*8-Q8 (1536 d.o.f)	4*4 (384 d.o.f)	2*2	203.7	507195	0.4043634	0.51
	2*2 (96 d.o.f)	4*4	189.7	68607	0.3732003	8.18

ble 2: Details for Problem No. (2)

Mesh	No. of nodes	Total d.o.f	% Reduction in d.o.f
48(Q8) conventional F.E	61	183	-
Equivalent M.E based on F.E. of type (Q8)			
12 Q8 M.E	49	147	19.7%
3 Q8 M.E	16	48	73.8%

ble 3: Details for Problem No. (3)

Mesh(1)	No. of nodes (2)	Total d.o.f (3)
8 (Q9) conventional F.E	45	135
4 (Q9) M.E based on F.E. of type (Q9)	25	75

le 4: Details for Problem No. (4)

Mesh (1)	No. of nodes (2)	Total d.o.f (3)
4 * 4 conventional F.E	81	243
2 * 2 equivalent M.E.	25	75

Table 5: A Comparative Study for Different (Q9) Meshes for the Thin Plate of Problem No. (4)

Mesh	8 * 8 Conventional F.E.	4 * 4 equivalent M.E.	2 * 2 equivalent M.E.
M.E. size (F.E.*F.E.)	-	2 * 2	4 * 4
Total d.o.f	1728	432	108
CPU (Seconds)	414.6	291.1	261.8
Ne* HBW^2	867 * 111^2 = 10682307	243 * 63^2 = 964467	75 * 39^2 = 114075
Central deflection (mm)	4.160454	4.148805	4.099109
Error % in central deflection	-	0.28	1.47
Central moment M_x (kN-m/m)	4.5046936	4.5382051	4.6681530
Error in M_x	-	- 0.74	- 3.63
Central moment M_y (kN-m/m)	6.4825045	6.5213435	6.6506472
Error % in M_y	-	- 0.60	- 2.59

Problem No (4): The analysis of thin and thick, square, simply supported, orthotropic plate under a uniformly distributed load, as shown in Fig. (21). The following data are given for this problem:

L = 7.2 m.

Thickness t:

Case A: t = 0.114m (t/L = 0.02 , i.e. thin plate).

Case B: t = 1.080m (t/L = 0.15 , i.e. thick plate).

$E_x = 20 * 10^6$ kN / m^2.

$E_y = 30 * 10^6$ kN / m^2

$G_{xy} = 15 * 10^6$ kN / m^2

$G_{xz} = G_{yz}$: Variable and as defined on graphs

$Nu_{xy} = 0.15$

$Nu_{yx} = Nu_{xy} * E_y / E_x = 0.225$

Loading Q_z:

Case A: Q_z = 2.875 kN / m^2 (for thin plate)

Case B: Q_z = 1212.807 kN / m^2 (for thick plate)

Due to symmetry, only on quarter of plate is analyzed. The analysis is done using the (Q9) isoparametric elements.

The plate is first considered as a thin plate, i.e. case A, then considered as thick plate, i.e. case B. The same discretizations are used for both cases, which are shown in table (4). Table (4) shows that the total number of d.o.f is reduced by 69.1% with the equivalent models.

The analysis is done using the technique of reduced integration when the plate is thin, and using full numerical integration when the plate is thick.

The results for deflections for both cases along x-axis are shown in figures (22& 23). Also, the results for moments (M_x, M_y & M_{xy}) and shears (V_x & V_y) for the thick plate along sec. A-A are shown in figures (24,25 & 26).

Table (5) shows a comparative study for the execution time (CPU), the band width-solution operation count (Ne * HBW^2), central deflections, central moments (M_x & M_y) and their corresponding errors. The analysis is done on the thin plate using (8 * 8 Q9) original F.E mesh and:

$G_{xz} = G_{yz} = G_{xz}$. The errors in central deflection and moments are measured from the (8 * 8) original F.E. mesh analysis.

The central moment values in table (5) are obtained by extrapolating the moment values at the Gauss points using a technique known as "Local Stress smoothing", which is simply a bilinear extrapolation of the (2*2) Gauss point stress values within an element (Cook, 1981).

Results and Discussion

The four solved problems showed that using the macro-elements in the analysis reduced the number of equations to be solved. When the size

of the macro-element used is of moderate, excellent results are achieved with good amount of reduction in d.o.f and computer time. But when the size of the macro-element is large still acceptable results are chieved with substantial reductions in d.o.f and computer time as shown in tables (1) and table (5).

Conclusion

New quadratic plate bending macro-elements based on two types of finite elements were developed.

The solved examples demonstrated that using these macro-elements in the analysis largely reduced the total number of d.o.f required to model a certain structure. This in turn reduced the total number of equations to be solved.

Reduction in total number of equations reduced computer time and memory space for storage. This will allow personal computers to analyze relatively large structures.

At the same time these M.E. provided accurate results. In addition, finite elements of different sizes, thicknesses and material properties can easily be used inside the macro-elements if required in the analysis.

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