Convergence of Pseudospectral Method for Solving Navier-Stokes Equations

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Abstract: In this paper a new kind of Pseudospectral scheme is constructed for the Unsteady Navier-Stokes equations. This method easily deal with nonlinear terms and saves computational time. The generalized stability of the scheme is analyzed and the convergence is proved. Numerical results are presented also.

Key Words: Navier-Stokes Equations, Pseudospectral Method, Generalized Stability, Convergence

Introduction
Consider the periodic problem of Navier Stokes equations as follows:

\[
\begin{align*}
\frac{du}{dt} + (u(x,t) \cdot \nabla) u(x,t) - \gamma u + \nabla p(x,t) &= f(x,t), \\
\nabla \cdot u(x,t) &= 0, \quad (x,t) \in \Omega \times (0,T), \\

u(x,0) &= u_0(x), \quad p(x,0) = P_0(x), \quad x \in \Omega.
\end{align*}
\]

where $\Omega = (0, 2\pi)^n$, $n = 2$, or $3$, $v \geq 0$, $u = (U_1, U_2, \ldots , U_n)$ is the velocity, $P$ is the ratio of pressure to density. The functions $u_0, P_0$ and $f$ are given with period $2\pi$ for all the space variables. We require in addition

\[
\int \rho (x,t) dx = 0 \quad \forall t \in (0, T).
\]

There is much literature concerning numerical solutions of Navier Stokes equations can be found in (Canuto et al., 1988; Cao Weiming and Guo Benyu, 1991; Guo Benyu, 1985; Temam, 1977). Specific algorithm in (Huang Wei and Guo Ben-Yu, 1992; Jing-Yu Hou and Ben-Yu Guo, 1999; Ben-Yu Guo, 1996; Guo Ben-yu and Cao Weiming, 1992; Li Jian and Guo Ben-yu, 1995) have been devoted to the semi-periodic cases, which describes channel flow, parallel boundary layers curved channel flow and cylindrical couette flow. This paper is devoted to the periodic pseudospectral method for two-dimensional unsteady Navier stokes equations with periodic boundary conditions. This method is performed easily and has the same high accuracy as spectral method.

Notations and Lemmas: Denote by (\cdot, \cdot) and $\| \cdot \|$ the inner product and norm of $L^2(\Omega)$ respectively. Let $\| \cdot \|_\infty$ be the norm of $L^\infty(\Omega)$. Define $C_p^\infty(\Omega) = \{ u | u \in C^\infty(\Omega), u \text{ has the period } 2\pi \}$ for the variable $x_i, 1 \leq q \leq n$ and let $H^\infty_p(\Omega)$ be the closure of $C_p^\infty(\Omega)$ in $H^\mu_p(\Omega)$.

Let $x$ be a Banach space. Define

\[
L^2(0,T;x) = \left\{ u | u : [0,T] \rightarrow X, \| u \|_{L^2(0,T;x)} = \left( \int_0^T \| u(t) \|_X^2 dt \right)^{1/2} < \infty \right\},
\]

\[
C(0,T;X) = \left\{ u | u : [0,T] \rightarrow X \text{ is strongly continuous}, \| u \|_{C(0,T;X)} = \max_{0 \leq s \leq T} \| u(s) \|_X \right\}
\]

Denote by $z$ the set of integers. For $k = (k_1, k_2, \ldots , k_n) \in \mathbb{Z}^n$, let $|k| = \max_{1 \leq q \leq n} |k_q|$, and

\[
|k| = \left( \sum_{q=1}^n k_q^2 \right)^{1/2}.
\]

For positive integer $N$, we define

\[
V_n = \text{span} \{ e^{i k_x} \mid k \in \mathbb{Z}^n, |k|_\infty \leq N \},
\]

\[
W_n = \text{span} \{ e^{i k_x} \mid k \in \mathbb{Z}^n, |k| \leq N \}.
\]

Let $P_n$ be the orthogonal projection operation form $L^2(\Omega)$ onto $W_n$. $P_c$ is the Lagrange interpolation operator from $C(\Omega)$ onto $V_n$ at points $x_j = \frac{2j}{2N+1}$, $j \in \mathbb{Z}^n$, $|j|_\infty \leq N$. Let $P_c = P_n P_c$. It is easy to see that

\[
(P_c u, v) = (P_c u, v), \quad \forall u, v \in V_n.
\]

Lemma 2.1: Rashid et al., (1994) If $u, v \in V_n$, then

1. $|u|^2 \leq n N^2 \| u \|^2$
2. $|\nabla u|^2 \leq n (2N + 1)^2 \left( \| u \|^2 + \| \nabla u \|^2 \right)$

Lemma 2.2: Canuto and Quartroni (1982) If $0 \leq \mu \leq \sigma$ and $u \in H^\sigma_p(\Omega)$ then

\[
\| P_n u - u \|_\mu \leq C N^{-\mu-\sigma} |u|_\sigma,
\]

and if $0 > \mu > n/2$, then

\[
\| P_c u - u \|_\mu \leq C N^{-\mu-\sigma} |u|_\sigma.
\]

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Lemma 2.3: Canuto and Quarneroni (1982) If $0 \leq \mu \leq \sigma$, and $u \in \mathbb{V}_h$, then

$$
\left\| u \right\|_{\infty} \leq CN^{\mu} \left\| u \right\|_{\infty}.
$$

Lemma 2.4: Song-nian and Cai ping (1999) Assume that the following conditions are fulfilled

(i) $E(t)$ is a non negative function defined on $R_+$,
(ii) $\rho$, $M_1$, and $M_2$ are non-negative constants;
(iii) for all $t \in R_+$,
(iv) $E(t) \leq \rho + M_1 \sum_{t=t}^{t=T}[E(t') + M_2 E^2(t')]$, respectively. Then the errors $\vec{u}(t)$, $\vec{p}(t)$ of $u(t)$ and $p(t)$ satisfy

\begin{align*}
\vec{u}_t(t) + d_c(u(t), \vec{u}(t)) + d_c(\vec{u}(t), u(t) + \vec{u}(t)) - \nu \nabla^2 \vec{u}(t) + \nabla \vec{p}(t) = P_c \vec{f}(t), & \\
\beta \vec{p}_t(t) + \nabla \cdot \vec{u}(t) = P_c \vec{g}(t).
\end{align*}

By taking the inner product of the first equation of (4.1) with $2 \vec{u}(t)$ we get

$$
\left[ \left\| \vec{u}(t) \right\|_1^2 \right]_t + 2 \nu \left\| \nabla \vec{u}(t) \right\|^2_1 + \sum_{j=1}^{2} F_j + 2 \left( \nabla \vec{p}(t), \vec{u}(t) \right),
$$

where

$$
F_1 = 2 \left( d_c(u(t), \vec{u}(t) + d_c(\vec{u}(t), u(t)), \vec{u}(t) \right),
F_2 = 2 \left( d_c(\vec{u}(t), \vec{u}(t)), \vec{u}(t) \right).
$$

Taking inner product of the second equation of (4.1) with $2 \nabla \vec{p}(t)$

$$
\beta \left[ \left\| \vec{p}(t) \right\|_1^2 \right] + 2 \left( \nabla \cdot \vec{u}(t), \nabla \vec{p}(t) \right) \leq \frac{\beta}{2} \left\| \vec{p} \right\|^2_1 + \frac{2}{\beta} \left\| \vec{g}(t) \right\|^2.
$$

(4.3)

Combing (4.2) and (4.3), Integration by parts and using Lemma 1, Lemma 2, we have

$$
\left[ \left\| \vec{u}(t) \right\|_1^2 \right]_t + 2 \nu \left\| \nabla \vec{u}(t) \right\|^2_1 + \sum_{j=1}^{2} F_j \leq \frac{1}{2} \left\| \vec{u}(t) \right\|^2_1 + \frac{\beta}{2} \left\| \vec{g}(t) \right\|^2 + \frac{2}{\beta} \left\| \vec{g}(t) \right\|^2.
$$

(4.4)

Now we are going to estimate $|F_1|$

$$
|F_1| \leq \frac{\nu}{2} \left\| \vec{u} \right\|_1^2 + \frac{c}{2} \left\| u \right\|_{\infty} \left\| \vec{u} \right\|^2_1,
$$

$$
|F_2| \leq \frac{\nu}{2} \left\| \vec{u} \right\|_1^2 + \frac{c M^2 N}{2} \left\| u \right\|^4.
$$

By substituting the above estimation in (4.5), we get

\begin{align*}
\left[ \vec{u}(t)^2 + \beta \vec{p}(t)^2 \right]_t + 2 \nu \vec{u}(t)^2 + \sum_{j=1}^{2} F_j & \leq \frac{1}{2} \vec{u}(t)^2 + \frac{\beta}{2} \vec{p}(t)^2 + \frac{c}{2} \vec{u}^2 \vec{u}^2 + \frac{c M^2 N}{2} \vec{u}^4
\end{align*}

(4.6)

let

$$
M_1 = c + \frac{c}{\nu} \left\| u \right\|_{\infty}^2,
M_2 = \frac{c M^2 N}{\nu},
$$

$$
E(t) = \left\| \vec{u}(t) \right\|^2 + \beta \left\| \vec{p}(t) \right\|^2 + 4 \nu \sum_{t=1}^{N} \left\| \vec{u}(t') \right\|^2.
$$

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\[ \rho(t) = 3\beta \| \bar{P}(0) \|^2 + \beta \| \bar{P}(0) \|^2 + 2 \beta \| \bar{P}(\tau) \|^2 + \beta \| \bar{P}(\tau) \|^2 + 8 \tau^2 \sum_{\tau=\tau} \left( \| \bar{P}(t') \|^2 + \frac{1}{\beta} \| \bar{G}(t') \|^2 \right) \]

Then by letting \( \tau \leq 1 \) and summing (4.6) for \( t \in \mathbb{R} \), we have from (2.1) that

\[ E(t) \leq \rho(t) + \tau \sum_{\tau=\tau} \left[ M_1 E(t') + M_2 E^2(t') \right] \]

We use the lemma (2.4) to obtain the following results

**Theorem 1:** There exist positive constant \( M_1 \) and \( M_2 \) depending only on \( \| \mathbf{u} \|_\infty \) and \( \nu \), such that, if for some \( t_1 \in \mathbb{R} \),

\[ \rho(t_1) e^{2M_1 t} \leq \frac{M_3}{M^2 N} \]

then for all \( t \in \mathbb{R} \), \( t \leq t_1 \),

\[ E(t) \leq \rho(t) e^{2M_1 t} \]

**Convergence:** Let \( U, P \) be the solution of (1.1) and \( U, P \) be the solution of (3.1). Define \( U^n = P^n U, P^n = P^n U, e = U^n - U, w = P^n - P \). We derive from (1.1) and (3.1) that

\[ \begin{align*}
& E_1(t) = \frac{\partial U}{\partial t} - U_N, \\
& E_2(t) = \left( U_N \cdot \nabla \right) U_N - d(U_N, U_N), \\
& E_3(t) = P_N - P_N, \\
& E_4(t) = (P_N - P_N) \cdot f, \\
& E_5(t) = P_N, \\
& E_6(t) = U_N - \hat{U}_N.
\end{align*} \]

where

\[ E_1(t) = \frac{\partial U}{\partial t} - U_N, \]

\[ E_2(t) = \left( U_N \cdot \nabla \right) U_N - d(U_N, U_N), \]

\[ E_3(t) = P_N - P_N, \]

\[ E_4(t) = (P_N - P_N) \cdot f, \]

\[ E_5(t) = P_N, \]

\[ E_6(t) = U_N - \hat{U}_N, \]

we have to estimate the right hand term in (5.1)

\[ \tau \sum_{\tau=\tau} \left[ E_1(t') \right]^2 \leq c \tau^4 \left\| P_N \right\|^2_{H^2(0,T;L^2(\Omega))}, \]

\[ \tau \sum_{\tau=\tau} \left[ E_2(t') \right]^2 \leq c \tau^4 \left\| \frac{\partial U}{\partial t} \right\|^2_{H^2(0,T;L^2(\Omega))}, \]

\[ \left\| U \right\|^2_{H^{2}} + cN^{-2} \left\| U \right\|^2_{H^{2}}, \]

\[ \tau \sum_{\tau=\tau} \left[ E_3(t') \right]^2 \leq c \tau^4 \left\| P_N \right\|^2_{H^2(0,T;L^2(\Omega))}, \]

\[ \tau \sum_{\tau=\tau} \left[ E_4(t') \right]^2 \leq c \tau^4 \left\| P_N \right\|^2_{H^2(0,T;L^2(\Omega))}, \]

\[ \tau \sum_{\tau=\tau} \left[ E_5(t') \right]^2 \leq c \tau^4 \left\| U \right\|^2_{H^2(0,T;L^2(\Omega))}, \]

\[ \tau \sum_{\tau=\tau} \left[ E_6(t') \right]^2 \leq c \tau^4 \left\| U \right\|^2_{H^2(0,T;L^2(\Omega))}. \]

Suppose that,

\[ c_0 t^2 \leq \beta \leq c_4 t^2 \]

where \( c_0 \) and \( c_4 \) are positive constant. If \( s > 2 \) then

\[ \beta^{-1} \left( \tau^4 + N^{-2s} \right) + \beta \leq \frac{c}{M^2 N}. \]

By an argument as in Theorem 1, we get the following result

**Theorem 2:** Assume \( \tau = O(N^{-s}) \), with \( s > 2, \mu > n/2 \) and that

\[ U \in H^2(0,T;L^2(\Omega)) \cap C(0,T;H^{2s}), \]

\[ P \in H^2(0,T;L^2(\Omega)) \cap \]

\[ H^1(0,T;H^{2s}(\Omega)) \cap C(0,T;H^{2s}), f \in L^2(0,T;H^2), \]

\[ \left\| U(t) - u(t) \right\|^2 \leq M_4 \left( \beta^{-1} \left( \tau^4 + N^{-2s} \right) \right) \]

where \( M_4 \) is positive constant depending only on \( \nu \) and the norm of \( U \) and \( P \) in the space mentioned in he above.

**The Numerical Results:** In this section, we examine the numerical performances. We choose the function \( f \) in such way that the solution of (1.1) is of the form

\[ \begin{align*}
U_1 &= \cos x \exp(\sin x + \sin x^2 + 0.1t), \\
U_2 &= -\cos x \exp(\sin x + \sin x^2 + 0.1t), \\
P &= -\cos(x_1 + \cos x_2) \exp(0.2t).
\end{align*} \]

The error for the speed and pressure are defined by

\[ \begin{align*}
E(U_i(t)) &= \left( \sum_{x \in \Omega_N} |U_i(t) - u_i(t)|^2 \right)^{1/2} \\
P(t) &= \left( \sum_{x \in \Omega_N} |P(t) - p(t)|^2 \right)^{1/2}
\end{align*} \]

where \( u_i \) and \( p \) are the solution of (3.1). We solve (1.1) by the scheme (3.1). The numerical results are tabulated in Table 1,2,3 and 4. The results are very accurate, if \( \beta \) is chosen suitably. Table 1 and 2. Table 3 and 4 show that \( \beta \) becomes less, the accuracy decreases. It agree with our theoretical analysis very well, since the nonlinear term are computed on the collocation points, this method is performed very simple.
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## Table 1: The Error of (3.1), \( \tau = 0.01, v = 0.001, \beta = 0.05 \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( E(U(t)) )</th>
<th>( E(U(t)) )</th>
<th>( E(P(t)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.000</td>
<td>0.27778E-05</td>
<td>0.26672E-05</td>
<td>0.12323E-05</td>
</tr>
<tr>
<td>2.000</td>
<td>0.99599E-05</td>
<td>0.99270E-05</td>
<td>0.70358E-05</td>
</tr>
<tr>
<td>3.000</td>
<td>0.32944E-04</td>
<td>0.34099E-04</td>
<td>0.25392E-04</td>
</tr>
<tr>
<td>4.000</td>
<td>0.12739E-03</td>
<td>0.14027E-03</td>
<td>0.10581E-03</td>
</tr>
<tr>
<td>5.000</td>
<td>0.71757E-03</td>
<td>0.78278E-03</td>
<td>0.53868E-03</td>
</tr>
</tbody>
</table>

## Table 2: The Error of (3.1), \( \tau = 0.005, v = 0.0001, \beta = 0.05 \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( E(U(t)) )</th>
<th>( E(U(t)) )</th>
<th>( E(P(t)) )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.46291E-05</td>
<td>0.15252E-05</td>
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<tr>
<td>2.000</td>
<td>0.17054E-04</td>
<td>0.17011E-04</td>
<td>0.10597E-04</td>
</tr>
<tr>
<td>3.000</td>
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<td>0.68518E-04</td>
<td>0.48212E-04</td>
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<tr>
<td>4.000</td>
<td>0.30433E-03</td>
<td>0.32366E-03</td>
<td>0.22345E-03</td>
</tr>
<tr>
<td>5.000</td>
<td>0.19929E-02</td>
<td>0.21263E-02</td>
<td>0.13431E-02</td>
</tr>
</tbody>
</table>

## Table 3: The Error of (3.1), \( \tau = 0.005, v = 0.001, \beta = 0.01 \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( E(U(t)) )</th>
<th>( E(U(t)) )</th>
<th>( E(P(t)) )</th>
</tr>
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<tbody>
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<tr>
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<tr>
<td>3.000</td>
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<td>0.27504E-02</td>
<td>0.22933E-02</td>
</tr>
<tr>
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<td>0.83360E-02</td>
<td>0.92043E-02</td>
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<tr>
<td>5.000</td>
<td>0.51157E-01</td>
<td>0.51165E-01</td>
<td>0.47461E-01</td>
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</table>

## Table 4: The Error of (3.1), \( \tau = 0.005, v = 0.001, \beta = 0.005 \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( E(U(t)) )</th>
<th>( E(U(t)) )</th>
<th>( E(P(t)) )</th>
</tr>
</thead>
<tbody>
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<td>0.37021E-03</td>
</tr>
<tr>
<td>2.000</td>
<td>0.30977E-02</td>
<td>0.30977E-02</td>
<td>0.20000E-02</td>
</tr>
<tr>
<td>3.000</td>
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<td>0.80258E-02</td>
<td>0.11171E-01</td>
</tr>
<tr>
<td>4.000</td>
<td>0.10826E+00</td>
<td>0.10826E+00</td>
<td>0.10842E+00</td>
</tr>
<tr>
<td>5.000</td>
<td>0.20355E+02</td>
<td>0.20437E+02</td>
<td>0.29057E+02</td>
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</tbody>
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## References


