Resolvent of Fourth Order Differential Equation in Half Axis

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Abstract: Let $H$ be a separable Hilbert space and $H_1 = L_2(0, \infty; H)$. The all functions are defined in range $[0, \infty)$, their values belong to space $H$, they are measurable in the meaning of Bochner and provides the condition of

$$\int_0^\infty |f(x)|^2 \, dx < \infty.$$ 

If the scalar product is defined in $H_1$ by the formula

$$(f, g)_{H_1} = \int_0^\infty f(x) \, g(x) \, dx,$$

$f(x), g(x) \in H_1$,

$H_1$ forms a separable Hilbert space. In this study, in space $H_1$, it is investigated that Green's function (resolvent) of the operator formed by the differential expression

$$(-1)^n y^{(2n)} + Q(x)y, \quad 0 \leq x < \infty,$$

and boundary conditions

$$y^{(j)}(0) - h_j y^{(j-n)}(0) = 0, \quad j = 1, 2, \ldots, 2n - 1,$$

where $Q(x)$ is a normal operator that has pure discrete spectrum for every $x \in [0, \infty)$ in $H$. Assumed that domain of $Q(x)$ is independent from $x$ and resolvent set of $Q(x)$ belongs to $|\arg \lambda - \pi| < \varepsilon$ ($0 < \varepsilon < \pi$) of complex plane $\lambda$. In addition assumed that the operator function $Q(x)$ satisfies the Titchmarsh-Levitan conditions. Here $s$ are arbitrary complex numbers. The obtained result has been applied to an example.

Key Words: Spectral analysis, Eigenvalues, Eigenfunctions, Operator

Introduction

Let $H$ be a separable Hilbert space. Assumed that $H_1 = L_2(0, \infty; H)$. The all functions are defined in range $[0, \infty)$ and their values belong to space $H$, they are measurable in the meaning of Bochner (Yosida, 1980) and provides the condition of

$$\int_0^\infty |f(x)|^2 \, dx < \infty.$$ 

If the scalar product is defined in $H_1$ by the equation

$$(f, g)_{H_1} = \int_0^\infty f(x) \, g(x) \, dx,$$

$f(x), g(x) \in H_1$,

$H_1$ forms a separable Hilbert space (Balakrishnan, 1976). Where $\|(., .)\|$ are norm and scalar product in $H$, respectively.

In this work, in space $H_1 = L_2(0, \infty; H)$, it is investigated that Green's function(resolvent) of operator $L$ formed by the differential expression

$$y^{(n)} + Q(x)y, \quad 0 \leq x < \infty$$

and the boundary conditions

$$y^{(0)}(0) - h_1 y^{(0)}(0) = 0,$$

$$y^{(n)}(0) - h_n y^{(n)}(0) = 0$$

where $Q(x)$ is a normal operator for every $x \in [0, \infty)$ in $H$ and its inverse is a compact operator, $h_1, h_2$ are arbitrary complex numbers.

In the case of $Q(x)$ is a normal operator and the boundary conditions are $y^{(j)}(0) = 0$ ($j=1, 2, \ldots, n$), resolvent of operator $L$ was investigated by Aliyev and Bayramoglu (1981). Green's function of Sturm-Liouville equation with infinity operator coefficient was first investigated by Levitan (1968).

In the space $L_2(\infty, \infty; H)$, Green's function and the asymptotic behaviour for the number of the eigenvalues of the operator formed by differential expression

$$\sum_{j=2}^{2n} Q_j(x)y(2n-j)$$

$$(-1)^n y^{(2n)} + \sum_{j=2}^{2n} Q_j(x)y(2n-j)$$

was obtained by Bayramoglu (1971), where $Q_j(x)$ ($j=2, \ldots, 2n$) are the self-adjoint operators in $H$. Later on many studies (Aslanov, 1994; Boymatov, 1973; Kleiman, 1977; Otelbayev, 1990 and Saito, 1975) were published in this subject. Wide reference of these studies is given in Kostyuchenko and Sargsyan, 1979 and Otelbayev, 1977). The main reference related to Green's function for the ordinary differential equation is Stakgold's book. (Stakgold, 1998).

Determination of the Problem: Resolvent of the operator formed by differential expression (1) and boundary conditions (2) in $H_1 = L_2(0, \infty; H)$ will be investigated.

$$y'' + Q(x)y + \mu y$$

$$y'(0) - h_1 y(0) = 0$$

$$y''(0) - h_2 y''(0) = 0$$

where $\mu > 0$ is a real number. It is assumed that $Q(x)$ is a normal operator mapping in $H$ for every $x \in [0, \infty)$ and satisfies the following specification:

- Let $Q(x)$ be a normal operator for each $x \in [0, \infty)$ in $H$, and $D = H$, where $D(Q(x)) = D$ independent from $x$. (Here $D$ shows the closure of $D$)
Let $Q^{-1}(x)$ be a compact operator for every $x$ ($Q^{-1}(x) \in \sigma_p$) and
\[
1 \leq |\alpha_1(x)| \leq |\alpha_2(x)| \leq \ldots \leq |\alpha_n(x)| \leq \ldots
\]
where $\alpha_1(x), \alpha_2(x), \ldots, \alpha_n(x), \ldots$ are the eigenvalues of $Q(x)$.

Assumed that resolvent set of $Q(x)$ belongs to $S_\epsilon = \{ \lambda : \pi - \epsilon < \arg \lambda < \pi + \epsilon, 0 < \epsilon < \pi \}$ of complex plane $\lambda$.

Suppose that function $F(x) = \sum_{i=1}^{\infty} \frac{1}{|\alpha_i(x)|^2}$ belongs to $L(0, \infty) = \int_0^\infty F(x)dx < \infty$

where $|x-s| \leq 1, c=constant$

Assumed that $Q^{-1}(x)[Q(s) - Q(x)] \leq c|x-s|$ while $|x-s| \leq 1, c=constant$ and $0 < a < \frac{5}{4}$ (c denotes different constants).

In (Kostyuchenko and Levitan, 1967) some examples are shown that these conditions satisfied. Let $B(H)$ be a Banach space whose elements are bounded operators mapping in $H$ (Kato, 1980) $G(x,s;\mu)$ function which belongs to $B(H)$ for $0 \leq x, s < \infty$ and satisfies the following conditions is called Green's function of problem (3)-(5).

The operator function $G(x,s;\mu)$ itself and its two partial derivative are continuous functions for variables $x$ and $s$ ($0 \leq x, s \leq \infty$).

When $s \neq x$ third derivative of $G(x,s;\mu)$ for $s$ is continuous.

$$
\frac{\partial^3 G}{\partial s^3}(x, x + 0, \mu) - \frac{\partial^3 G}{\partial s^3}(x, x - 0, \mu) = I
$$
(I is identity operator in $H$)

When $s \neq x$,

$$
\frac{\partial^4 G}{\partial s^4}(x, s; \mu) + G(x, s; \mu)Q(s) + \mu G(x, s; \mu) = 0
$$

$$
\frac{\partial G}{\partial s}(x, s; \mu) \bigg|_{s=0} - h_1 G(x, s; \mu) \bigg|_{s=0} = 0,
$$

$$
\frac{\partial G}{\partial s} G(x, s; \mu) \bigg|_{s=0} - h_2 \frac{\partial^2 G}{\partial s^2}(x, s; \mu) \bigg|_{s=0} = 0
$$

According to parametrics method, the operator function $G(x,s;\mu)$ will be found as a solution of integral equation given by

$$
G(x,s;\mu) = r(x-s)g(x,s;\mu) - \int_0^\infty \left[ \int_0^\infty \left( \int_0^\infty \frac{\partial^m}{\partial s^m} (x-\xi) g(x,\xi;\mu) + 4r''(x-\xi) g''(x,\xi;\mu) + 6r''(x-\xi) g''(x,\xi;\mu) + 4r''(x-\xi) g''(x,\xi;\mu) + r(x-\xi) g(x,\xi;\mu) \right) \right] G(\xi, s; \mu) \partial \xi
$$

where

$$
r(u) = \begin{cases} 1 & |u| \leq \rho \\ 0 & |u| \geq 2\rho, \quad 0 < \rho < \frac{1}{2} \end{cases}
$$

is any fixed sufficiently smooth function and

$$
g(x,s;\alpha) = \frac{\sqrt{2}}{8} \alpha^{-3} (1+i) e^{-\frac{\alpha^2}{2}|x-s|} - \frac{\sqrt{2}}{8} \alpha^{-3} (-1+i) e^{-\frac{\alpha^2}{2}|x-s|} - \frac{1}{4} (1+i) h_2 + \frac{\sqrt{2}}{4} \alpha^{-3} (-1+i) h_1 - \frac{\sqrt{2}}{4} \alpha^{-3} (1+i) + \frac{1}{4} (1+i) h_1 \alpha^2 i(-2h_1 h_2 - 2\alpha(h_1 h_2 - 2\alpha^2) e^{-\frac{\alpha^2}{2}|x-s|})
$$

$$
+ \frac{1}{4} (-1+i) h_1 + \frac{\sqrt{2}}{4} \alpha^{-3} (1+i) h_1 - \frac{\sqrt{2}}{4} \alpha^{-3} (1+i) + \frac{1}{4} (1+i) h_2 \alpha^2 i(-2h_1 h_2 - 2\alpha(h_1 h_2 - 2\alpha^2) e^{-\frac{\alpha^2}{2}|x-s|})
$$

where $c_i(x), e_1(x), e_2(x), \ldots, e_n(x), \ldots$ are respectively the orthonormalized eigenvectors corresponding to eigenvalues $\alpha_1(x), \alpha_2(x), \ldots, \alpha_n(x), \ldots$ of $Q(x)$.

$$
\frac{1}{4} \alpha_j(x) + \mu \text{'s are defined from}
$$

$$
- \pi + \epsilon < \arg(\alpha_n(x) + \mu) < \pi - \epsilon. \text{ Note that, } |\lambda + \mu| \geq |\lambda| \sin \epsilon, \quad |\lambda + \mu| \geq |\lambda| \sin \epsilon
$$

is satisfied for $\lambda \in S_\epsilon, \mu > 0$.\[423\]
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It will be shown that integral equation (6) has only one solutions and this solution is Green's fuction of the problem (3)-(5).

Equation (6) will be investigated in these spaces; $X_2$, $X_3^{(1)}$, $X_4^{(-1/4)}$, $X_5$. These are shown that they are Banach spaces and given by Levitan (1968).

Consider that integral Eq. (6) is in the space $X_2$. Let us show that this equation has one solution in $X_2$ for $\mu >> 0$ ($\mu$ is a big enough positive value) and the solution can be found by successive aproximation method. For this it is enough to show that $g(x,s;\mu) \in X_2$ and the operator of

$$NA = \int_{x-\xi}^{x+\xi} (x-\xi)g(x;\xi;\mu)[Q(x) - Q(\xi)]A(\xi,\eta)d\xi$$

is contraction operator in $X_2$ for $\mu >> 0$.

**Lemma 1:** If operator function $Q(x)$ satisfies the conditions 4) and 6) for $\mu >> 0$, operator $N$ is contraction operator in the space $X_2$.

**Proof:** If it is shown that the norm of operator $N$ for $\mu >> 0$ are small enough, it is demonstrated that $N$ is constriction operator for $\mu >> 0$.

$$NA = \sum_{j=1}^{5} N_j A$$

$$N_1 A = \int_{x-\xi}^{x+\xi} (x-\xi)g(x;\xi;\mu)[Q(x) - Q(\xi)]A(\xi,\eta)d\xi$$

$$N_2 A = \int_{x-\xi}^{x+\xi} 4r'(x-\xi)g''(x;\xi;\mu)A(\xi,\eta)d\xi$$

$$N_3 A = \int_{x-\xi}^{x+\xi} 6r''(x-\xi)g''(x;\xi;\mu)A(\xi,\eta)d\xi$$

$$N_4 A = \int_{x-\xi}^{x+\xi} 4r''(x-\xi)g''(x;\xi;\mu)A(\xi,\eta)d\xi$$

$$N_5 A = \int_{x-\xi}^{x+\xi} (x-\xi)g(x;\xi;\mu)A(\xi,\eta)d\xi$$

$$\|N\| \leq \|N_1\| + \|N_2\| + \|N_3\| + \|N_4\| + \|N_5\|$$

can be written from nature of norm.

Let's do the operations for operator $N_1 A$.

$$g(x,s;\mu) = g_1 + g_2 + g_3 + g_4 + g_5 + g_6$$

$$N_1 A = \int_{x-\xi}^{x+\xi} (x-\xi)g(x;\xi;\mu)[Q(x) - Q(\xi)]A(\xi,\eta)d\xi$$

is found. Hence

$$\|b_1\|_{X_2}^2 \leq c_1 \mu^{2q} \int_{x-\xi}^{x+\xi} (x-\xi)^{-q} \|A(\xi,\eta)\|_{X_2} d\xi$$

is obtained, or

$$\|b_1\|_{X_2} \leq c_1 \mu^{2q} \|A(x,\eta)\|_{X_2}$$

Therefore,

$$\|N_{11} A\|_{X_2} \leq c_1 \mu^{2q} \|A(x,\eta)\|_{X_2}$$

is derived. Here, $q < 0$ is constant. Thus, operator $N_{11}(x,\xi)$ is bounded with small enough norm of large $\mu > 0$. In a similar way, operators $N_{12} A, N_{13} A, N_{14} A, N_{15} A, N_{16} A$ are bounded with small enough norm of large $\mu > 0$. Then it is obtained that operator $NA$ is constriction operator in space $X_2$ for $\mu >> 0$. Thus Lemma 1 is proved.
If it can be shown that \( r(x-s)g(x,s;\mu) \) belongs to the space \( X_2 \), then, it is obtained that Eq. (6) has only one solution belonging to \( X_2 \) for \( \mu >> 0 \).

\[
\|g\|_2 \leq \|g_1\|_2 + \|g_2\|_2 + \|g_3\|_2 + \|g_4\|_2 + \|g_5\|_2 + \|g_6\|_2
\]

can be written.

Now let us show that \( rg \) belongs to space \( X_2 \) assuming that the condition 4) of function \( Q(x) \) is fulfilled. Let us perform the operation for any term included by \( rg \), for example the term \( r(x-s)g_1(x,s;\mu) \). That is, let us show that \( rg_1 \in X_2 \). In a same manner, it is indicated that other terms also belong to \( X_2 \). Since \( r(x-s)g_1(x,s;\mu) \) is a function of normal operator valued function \( Q(x) \), using the spectral expansion formula for normal operators (Yosida, 1980):

\[
\|g_1\|_2^2 = \sum_{j=1}^{\infty} \left( \frac{2}{\pi} \right)^{1/2} \left| \lambda_j \right| (1 + i \alpha_j(x) + \mu)^{-1/2} e^{-\alpha_j(x) + \mu} \right)^2 e^{\delta x}sdx = \frac{1}{(16\pi)^{3/2}} \sum_{j=1}^{\infty} \left( \frac{2}{\pi} \right)^{1/2} \left| \lambda_j \right| (1 + i \alpha_j(x) + \mu)^{-1/2} e^{-\alpha_j(x) + \mu} \right)^2 e^{\delta x}sdx
\]

d is implied. From the 4), property of \( Q(x) \)

\[
\int_{0}^{\infty} \|g_1\|_2^2 dsdx = c \sum_{j=1}^{\infty} \frac{dx}{(1 + i \alpha_j(x) + \mu)^{1/2}} < \infty,
\]

c is constant > 0

is obtained. Thus it is denoted that \( rg_1 \in X_2 \). Therefore the following theorem has been proved.

**Theorem 1:** If the conditions 4) and 6) of operator \( Q(x) \) are satisfied, then, for \( \mu >> 0 \), there exists a solution in the space \( X_2 \) for Eq. (6) and it is unique. This solution can be found by successive approximation method.

The following lemma can be proved.

**Lemma 2:** If operator function \( Q(x) \) satisfies the conditions in Lemma 1 then for \( \mu >> 0 \), operator \( N \) is a contraction operator in every spaces \( X_2, X_3, X_3^{(1)}, X_4^{(-1/4)} \) and \( X_3 \). In addition to the conditions 1) and 6), if operator function \( Q(x) \) satisfies the condition

\[
\left\| Q^{1/4}(x)Q^{-1/4}(s) \right\| \leq c,
\]

c constant = constant, then \( g \in X_4^{(-1/4)} \).

**Derivatives of Green's Function:** Let us try to show that operator function \( G(x,s;\mu) \) has the derivatives

\[
\frac{\partial^3 G(x,s;\mu)}{\partial s^j} \quad (j=1,2,3).
\]

If the derivatives of both sides of Eq. (6) is calculated according to \( s \)

\[
\begin{align*}
\frac{\partial^3 G(x,s+\varepsilon;x)}{\partial s^j} & = \frac{\partial^3 G(x,s;x)}{\partial s^j} + \varepsilon \left[ \frac{\partial^2 G(x,s;x)}{\partial s^{j-1}} \right] + \frac{\partial^3 G(x,s;x)}{\partial s^j} + \varepsilon^2 \left[ \frac{\partial^2 G(x,s;x)}{\partial s^{j-1}} \right] + \frac{\partial^3 G(x,s;x)}{\partial s^j} + \varepsilon^3 \left[ \frac{\partial G(x,s;x)}{\partial s^{j-1}} \right]
\end{align*}
\]

while \( |x-s| \leq 1 \). In this case;
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\[ \frac{\partial^4}{\partial s^4} \left[ r(x-s)g(x,s;\mu) \right] \in X_4^{-1/4} \]

that is,

\[
\sup_{0<s<\infty} \left\| \frac{\partial^4}{\partial s^4} \left[ r(x-s)g(x,s;\mu) \right] \right\| Q^{-1/4}(s) ds < \infty.
\]

**The Fourth Derivative of Green's Function:** In previous part it has been shown that the derivative \( \frac{\partial^4 G}{\partial s^4} \) of Green's function \( G(x,s;\mu) \) belongs to the space \( X_3 \) and it satisfies the continuity \( (x \neq s) \) for the variable \( s \) and the following expression

\[
\frac{\partial^4 G(x,s;\mu)}{\partial s^4} = \frac{\partial^4}{\partial s^4} \left[ r(x-s)g(x,s;\mu) \right] - \int_0^x P(x,\xi;\mu) \frac{\partial^3 G}{\partial s^3} \big|_{\xi} \, d\xi,
\]

where

\[
P(x,\xi;\mu) = r(x-\xi)g(x,\xi;\mu) + 4r(x-\xi)g(x,\xi;\mu) + 6r(x-\xi)g(x,\xi;\mu) + 4r'(x-\xi)g''(x,\xi;\mu) + r(x-\xi)g(x,\xi;\mu) \]

is obtained. Let us write Eq. (10) as follows

\[
L(x,s;\mu) = l(x,s;\mu) - \int_0^x P(x,\xi;\mu) \frac{\partial^3 G}{\partial s^3} \big|_{\xi} \, d\xi.
\]

Let us derive the Eq. (14) according to \( s \) as formal. From this

\[
\frac{\partial}{\partial s} \left[ L(x,s;\mu) \right] - \int_0^x P(x,\xi;\mu) \frac{\partial^3 G(x,\xi,s;\mu)}{\partial s^3} \, d\xi = \frac{\partial}{\partial s} \left[ L(x,s;\mu) \right] - \int_0^x P(x,\xi;\mu) \frac{\partial^3 r(x-s)g(x,s;\mu)}{\partial s^3} \, d\xi
\]

is obtained. If the expression

\[
\frac{\partial}{\partial s} \left[ r(x-(x+0))g(x,x+0;\mu) \right] - \frac{\partial}{\partial s} \left[ r(x-(x+0))g(x,x+0;\mu) \right] = 1
\]

is used and if we write as

\[
\frac{d}{dn} \left[ \int_0^x P(x,\xi;\mu) \frac{\partial^3 G(x,\xi,s;\mu)}{\partial s^3} \, d\xi \right] = \int_0^x P(x,\xi;\mu) \frac{\partial^3 G(x,\xi,s;\mu)}{\partial s^3} \, d\xi,
\]

\[
\frac{\partial}{\partial s} \left[ L(x,s;\mu) \right] = -P(x,s;\mu) - \int_0^x P(x,\xi;\mu) \frac{\partial^4 G(x,\xi,s;\mu)}{\partial s^4} \, d\xi
\]

is found. Let us say that

\[
\frac{\partial}{\partial s} \left[ L(x,s;\mu) \right] = 1 \left[ x, s; \mu \right] .
\]

If it can be shown that element \( L_1 \) belongs to \( X_4^{-1/4} \), according to Lemma 2, it is obtained that there exists a derivative of the function \( \frac{\partial^3 G}{\partial s^3} - \frac{\partial^3 (rg)}{\partial s^3} \) according to \( s \)

\[
\text{and } \frac{\partial}{\partial s} \left[ \frac{\partial^3 G}{\partial s^3} - \frac{\partial^3 (rg)}{\partial s^3} \right] = X_4^{-1/4}.
\]

From this point, according to Lemma 3, \( \frac{\partial^3 G}{\partial s^3} \in X_4^{-1/4} \) is obtained. It is found that the element \( L_1 \) belongs to \( X_4^{-1/4} \) by the studies (Levitan, 1968), (Bayramoglu, 1971).

**Satisfying Differential Equation of Green's Function:** Let us show that Green's function \( G(x,s;\mu) \) for \( x \neq s \) satisfies the equation

\[
\frac{\partial^4 G}{\partial s^4} + G(x,s;\mu) [Q(s) + \mu I] = 0,
\]

Let \( f \in D \). Then,

\[
\frac{\partial^4 G}{\partial s^4} (f) + [Q(s) + \mu I] f = -r [Q(s) - Q(x)] f - \int_0^x P(x,\xi;\mu) \frac{\partial^4 G(\xi,s;\mu)}{\partial s^4} \, d\xi,
\]

or

\[
\frac{\partial^4 G}{\partial s^4} (f) = -r [Q(s) + \mu I] f - \int_0^x P(x,\xi;\mu) \frac{\partial^4 G(\xi,s;\mu)}{\partial s^4} \, d\xi,
\]

is obtained. Let \( [Q(s) + \mu I] f = \alpha \). From this, Eq. (15) becomes as follows.

\[
\frac{\partial^4 G}{\partial s^4} \left[ \frac{[Q(s) + \mu I]^{-1}}{\alpha} \right] = G(x,s;\mu) \alpha
\]

is found. From the last expression fourth property is obtained as elements' set of \( \alpha \) for every constant \( s \geq 0 \) is dense everywhere in \( H \).

**Satisfying of Boundary Conditions:** Let us show that \( G(x,s;\mu) \) satisfies the conditions

\[
\frac{\partial G(x,s;\mu)}{\partial s} \bigg|_{s=0} = h_1 G(x,s;\mu) = 0,
\]

\[
\frac{\partial^3 G(x,s;\mu)}{\partial s^3} \bigg|_{s=0} = h_2 \frac{\partial^2 G(x,s;\mu)}{\partial s^2} = 0,
\]

that is Green's function fulfills the condition 5-).

\[
G(x,s;\mu) = r(x-s)g(x,s;\mu) - \int_0^x P(x,\xi;\mu) \frac{\partial G(\xi,s;\mu)}{\partial s} \, d\xi.
\]

From the equations (16) and (17),

\[
\frac{\partial}{\partial s} \left[ r(x-s)g(x,s;\mu) - \int_0^x P(x,\xi;\mu) \frac{\partial G(\xi,s;\mu)}{\partial s} \, d\xi \right] = 0,
\]

\[
\frac{\partial}{\partial s} \left[ r(x-s)g(x,s;\mu) - \int_0^x P(x,\xi;\mu) \frac{\partial G(\xi,s;\mu)}{\partial s} \, d\xi \right] = 0.
\]

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is obtained. Considering that,
\[
\frac{\partial (rg)}{\partial s}\bigg|_{s=0} - h_1 rg \bigg|_{s=0} = 0
\]
from Eq. (18);
\[
\int_0^\infty \int_0^\infty (Q(\xi) - Q(x)) \left[ -h_1 G(\xi, s; \mu) \right] \, d\xi \, ds = 0
\]
(19)
can be written. Homogen equation (19) can be written as below.
\[
N \left[ \frac{\partial G}{\partial s} - h_1 G \right]_{s=0} = 0
\]
Since operator N is constriction operator for \( \mu >> 0 \),
\[
\frac{\partial G(\xi, s; \mu)}{\partial s} - h_1 G(\xi, s; \mu) \bigg|_{s=0} = 0
\]
is obtained. Thus the first boundary conditions of 5-) is satisfied.
Now let us calculate the thirth derivation of \( G(x, s; \mu) \) according to s.
\[
\frac{\partial^3 (rg)}{\partial s^3}\bigg|_{s=0} - \frac{\partial^2 (rg)}{\partial s^2}\bigg|_{s=0} = \int_0^\infty \int_0^\infty (Q(\xi) - Q(x)) \left[ -h_1 G(\xi, s; \mu) \right] \, d\xi \, ds
\]
\[
\frac{\partial^3 (rg)}{\partial s^3}\bigg|_{s=0} - h_1 \frac{\partial^2 (rg)}{\partial s^2}\bigg|_{s=0} = \int_0^\infty \int_0^\infty (Q(\xi) - Q(x)) \left[ -h_1 G(\xi, s; \mu) \right] \, d\xi \, ds
\]
(20)
From the expression of \( g(y_s, \mu) \), considering that
\[
\frac{\partial^3 (rg)}{\partial s^3}\bigg|_{s=0} - h_1 \frac{\partial^2 (rg)}{\partial s^2}\bigg|_{s=0} = 0
\]
from the equation (20),
\[
\int_0^\infty \int_0^\infty (Q(\xi) - Q(x)) \left[ \frac{\partial G(\xi, s; \mu)}{\partial s} - h_1 \frac{\partial G(\xi, s; \mu)}{\partial s} \right] \, d\xi \, ds = 0
\]
is found. This homogen equation can be expressed by
\[
N \left[ \frac{\partial G}{\partial s} - h_1 \frac{\partial G}{\partial s} \right]_{s=0} = 0
\]
Since N is constriction operator for \( \mu >> 0 \),
\[
\frac{\partial^3 G(\xi, s; \mu)}{\partial s^3} - h_1 \frac{\partial^2 G(\xi, s; \mu)}{\partial s^2} \bigg|_{s=0} = 0
\]
is obtained. Thus the second boundary conditions of 5-) is also fulfilled.
Consequently, it is shown that operator function \( G(x, s; \mu) \) satisfies all properties of Green's function.
If integral operator
\[
Af = \int_0^\infty G(x, s; \mu)f(s) \, ds
\]
\(\mu > 0\)
is formed in \( H_1 \) by using Green's function obtained, it is seen that A is a Hilbert-Schmidt (H-S) type operator from the property proved
\[
\int_0^\infty \int_0^\infty (Q(x, s; \mu))_{s=0}^2 \, dx \, ds < \infty
\]
If \( Q(x) = Q^*(x) \), \( h_1 = h_2 \) are real numbers then \( G^*(x, s; \mu) = G(s, x; \mu) \) can be proved.

**Example:** Proof of existing the operator valued function \( Q(x) \) satisfied 1-) 6-) conditions.

Let \( H = L_2 \),
\[
Q(x) = \begin{pmatrix}
q_1(x) & 0 & 0 & \cdots & 0 \\
0 & q_1(x) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & q_1(x) & 0 \\
0 & 0 & \cdots & \cdots & 0
\end{pmatrix}
\]
Where \( q_j(x) = K_j x + d_j \), \( d_j = 0 \), any positif numbers satisfying \( \sum_j 1/K_j < \infty \) while
\( K_1 \leq K_2 \leq \ldots \leq K_n \leq \ldots \). Let show that all \( Q(x) \)'s conditions in section 2 were provided.
Let pay attention that it is enough to indicate \( Q(x) \) satisfies the conditions almost every where.
1. \( Q(x) \) is a normal operator for every \( x > 0 \) in \( L_2 \).
because of \( Q^*(x) = -iq(x) \).
2. \( Q^{-1}(x) = -i \begin{pmatrix}
q_1^{-1}(x) & 0 & 0 & \cdots & 0 \\
0 & q_2^{-1}(x) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & q_n^{-1}(x) & 0 \\
0 & 0 & \cdots & \cdots & 0
\end{pmatrix}
\)
\( q_j^{-1}(x) = (K_j x + d_j)^{-1} \rightarrow 0, (x \neq 0) \).
This also demonstrates \( Q^{-1}(x) \) is a compact operator in every \( x, (x \neq 0) \).
3. \( (Q(x) + \lambda)^{-1} = \begin{pmatrix}
(q_1(x) + \lambda) & 0 & 0 & \cdots & 0 \\
0 & (q_2(x) + \lambda) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & (q_n(x) + \lambda) & 0 \\
0 & 0 & \cdots & \cdots & 0
\end{pmatrix}
\)
This statement shows \( (Q(x) + \lambda)^{-1} \) is exist and bounded every point in complex plane exterior half axis \(( -\infty, 0] \). That is set \( s \) is a resolvent set of \( Q(x) \).
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\[ \int_0^\infty \frac{1}{q_j^{7/4}(x)} \, dx = \int_0^\infty \frac{1}{\sum_{j=1}^\infty (K_j x + d)^{7/4}} \, dx \]

\[ = \sum_{j=1}^\infty \frac{1}{K_j} \int_0^\infty \frac{d(K_j x)}{(K_j x + d)^{7/4}} \]

\[ = \sum_{j=1}^\infty \frac{1}{K_j} \int_0^\infty \frac{dt}{(t + d)^{7/4}} = \sum_{j=1}^\infty \frac{4}{3} K_j \leq \frac{4}{3} \sum_{j=1}^\infty \frac{1}{K_j} < \infty \]

5. \( q_j^{1/4}(x) q_j^{1/4}(\xi) = (K_j x + d)^{-1/4} (K_j \xi + d)^{1/4} \)

\[ (K_j x + d)^{-1/4} (K_j \xi + d)^{1/4} \leq (K_j x + d)^{-1/4} (K_j x + x + d)^{1/4} \]

\[ \left( 1 + \frac{d}{K_j x + d} \right)^{1/4} \leq \left( 1 + \frac{d}{d} \right)^{1/4} = \sqrt[4]{2} \]

while \( |x - \xi| \leq 1 \Rightarrow -\xi \leq x \leq 1 \Rightarrow \xi \leq x + 1 \).

6. \( |x - \xi| \leq 1 \); Let's take \( a = 1 \) satisfying condition \( 0 < a < \frac{5}{4} \). Assumed that \( x \geq 1 \) without changing generalization.

\[ |q_j^{-1}(x)[q_j(\xi) - q_j(x)]| = (K_j x + d)^{-1} |K_j \xi - K_j x| \]

\[ = |x - \xi| \frac{d}{K_j x + d}^{-1} = |x - \xi| \left( 1 + \frac{d}{K_j} \right)^{-1} \]

\[ \leq |x - \xi| \left( 1 + \frac{d}{K_j} \right)^{-1} \leq |x - \xi| \]

Thus \( |q_j^{-1}(x)[q_j(\xi) - q_j(x)]| \leq |x - \xi| \) while \( |x - \xi| \leq 1 \). Therefore it is showed that 1-) - 6-) conditions are satisfied.

References


