Solutions of Initial Value Problems Using Fifth-Order Runge-Kutta Method Using Excel Spreadsheet

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Abstract: Fifth order Runge - Kutta method (Butcher’s method) is used in solving initial value problems of the form \( \frac{dy}{dx} = f(x, y) \) by the aid of Excel Spread Sheet for different values of step size \( h \), and the true relative percent error \( E_r \) is calculated for every case.

Keywords: Excel, Runge-Kutta, Initial Value Problem and the True Value

Introduction
Microsoft Excel is used throughout the paper, and gives a simple approach in solving initial value problems of the form \( \frac{dy}{dx} = f(x, y) \) by using fifth order Runge – Kutta method (Butcher’s method). A spreadsheet is made up of cells as follows:

\[
\begin{array}{cccccc}
A & B & C & D & E \\
1 & A1 & B1 & C1 & D1 & E1 \\
2 & A2 & B2 & C2 & D2 & E2 \\
3 & A3 & B3 & C3 & D3 & E3 \\
4 & A4 & B4 & C4 & D4 & E4 \\
5 & A5 & B5 & C5 & D5 & E5 \\
\end{array}
\]

In these boxes we type the data into, the data that can be typed into every cell, (A1, A2, ..., B1, B2, ..., E1, E2, ...) can be numeric or algebraic.

Fifth-Order Runge-Kutta Method: Butcher’s fifth-order Runge-Kutta method gives us very good accurate results, Buchanan et al. (1992), Mathews et al. (1999) and Chapra et al. (1998).  

\[
y_{i+1} = y_i + \frac{1}{90}(7k_1 + 32k_2 + 12k_3 + 32k_4 + 7k_5)h
\]

Where:

\[
k_1 = f(x_i, y_i) \\
k_2 = f\left(x_i + \frac{1}{4}h, y_i + \frac{1}{4}k_1h\right) \\
k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h + \frac{1}{8}k_2h\right) \\
k_4 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h + \frac{1}{2}k_2h + k_3h\right) \\
k_5 = f\left(x_i + \frac{1}{4}h, y_i + \frac{2}{16}k_1h + \frac{9}{16}k_2h\right) \\
k_6 = f\left(x_i + h, y_i + k_3h, k_5h + \frac{12}{7}k_2h + \frac{12}{7}k_3h - \frac{12}{7}k_4h + k_3h\right)
\]

In Chapra et al. (1998) is stated that higher-order Runge-Kutta formulas such as Butcher’s method are available, but in general, beyond fourth-order methods the gain in accuracy is offset by the added computational effort and complexity. But by using Excel this complexity is solved and we will obtain a very good results quickly and easily and also we can change the value of the step size \( h \) and we will have the results immediately as we see in the following example.

Example: Use fifth-order Runge-Kutta method by the aid of Excel to solve

\[
\frac{dy}{dx} = f(x, y) = 4e^{ax} - 0.5y
\]

with \( y(0) = 2 \) from \( x = 0 \) to \( x = 4 \) with various step sizes, compare the results with the true values by computing the true percent relative error \( |E_r| \).

Excel Spreadsheet can be used to solve this problem as follows:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x_i )</td>
<td>( y_i )</td>
<td>( f(x_i, y_i) )</td>
<td>( h )</td>
<td>( k_1 )</td>
<td>( x_i + \frac{1}{4}h )</td>
<td>( y_i + \frac{1}{4}k_1h )</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>( 4*\text{EXP}(0.8<em>A2) - 0.5</em>B2 )</td>
<td>0.5</td>
<td>0 = ( \text{C2} = \text{A2} + (1/4)*\text{D2} )</td>
<td>( \text{B2} + (1/4)<em>\text{E2}</em>\text{D2} = 4*\text{EXP}(0.8*\text{F2}) - 0.5*\text{G2} )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( \text{A2} + \text{D2} )</td>
<td>( = \text{U2} = 4*\text{EXP}(0.8*\text{A3}) - 0.5*B3 )</td>
<td>( \text{D2} = \text{C3} = \text{A3} + (1/4)*\text{D3} )</td>
<td>( \text{B3} + (1/4)<em>\text{E3}</em>\text{D3} = 4*\text{EXP}(0.8*\text{F3}) - 0.5*\text{G3} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>O</td>
<td>P</td>
<td>Q</td>
<td>R</td>
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<td></td>
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<td>---</td>
<td>---</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>$x_i + \frac{1}{4}h$</td>
<td>$y_i + \frac{1}{8}k_1h + \frac{1}{8}k_2h$</td>
<td>$k_3$</td>
<td>$x_i + \frac{1}{2}h$</td>
<td>$y_i - \frac{1}{2}k_2h + k_3h$</td>
<td></td>
</tr>
<tr>
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<td>$k_1$</td>
<td>$x_i + \frac{3}{4}h$</td>
<td>$y_i + \frac{3}{16}k_1h + \frac{1}{16}k_2h$</td>
<td>$k_4$</td>
<td>$x_i + h$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$k_1$</td>
<td>$x_i + \frac{3}{4}h$</td>
<td>$y_i + \frac{3}{16}k_1h + \frac{1}{16}k_2h$</td>
<td>$k_4$</td>
<td>$x_i + h$</td>
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</tr>
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</table>

The results are as follows:

<table>
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<tr>
<th>S</th>
<th>T</th>
<th>U</th>
<th>V</th>
<th>W</th>
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<tbody>
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<td>$y_{i+1}$</td>
<td>$k_6$</td>
</tr>
<tr>
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<td>$x_i + \frac{3}{4}h$</td>
<td>$y_{i+1}$</td>
<td>$k_6$</td>
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</tbody>
</table>

The results are as follows:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
</tr>
</thead>
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<td>$y_i$</td>
<td>$f(x_i, y_i)$</td>
<td>$h$</td>
<td>$k_1$</td>
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<td>$y_i + \frac{1}{4}h$</td>
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<td>3</td>
<td>0.125</td>
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<td>4.091538</td>
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<th>K</th>
<th>L</th>
<th>M</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>$x_i + \frac{1}{4}h$</td>
<td>$y_i + \frac{1}{4}k_1h + \frac{1}{4}k_2h$</td>
<td>$k_3$</td>
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</table>
Radwan: Solutions of Initial Value Problems Using Fifth-Order Runge-Kutta Method

\begin{tabular}{|c|c|c|c|c|c|}
\hline
N & O & P & Q & R & S \\
\hline
1 & \( k_1 \) & \( x_i + \frac{3}{4}h \) & \( y_i + \frac{9}{16}k_1h \) & \( y_i + \frac{9}{16}k_1h \) & \( \frac{2}{7}y_i + \frac{9}{16}k_1h \) \\
\hline
2 & 3.483285 & 0.375 & 3.260923856 & 3.768973 & 0.5 \\
3 & 4.857694 & 0.875 & 5.501330181 & 5.304346 & 1 \\
4 & 6.983004 & 1.375 & 8.702807524 & 7.66526 & 1.5 \\
5 & 10.21195 & 1.875 & 13.36918276 & 11.24216 & 2 \\
6 & 15.07442 & 2.375 & 20.32518726 & 16.62098 & 2.5 \\
7 & 22.36378 & 2.875 & 30.43646374 & 24.6785 & 3 \\
8 & 33.26578 & 3.375 & 45.58825832 & 36.7248 & 3.5 \\
9 & 49.55113 & 3.875 & 68.15173517 & 54.71594 & 4 \\
10 & 73.86273 & 4.375 & 101.7810669 & 81.57127 & 4.5 \\
\hline
\end{tabular}

Now if change \( h \) from 0.5 to 0.25 we will have the results immediately as follows:

\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
A & B & C & D & E & F & G \\
\hline
1 & \( x_i \) & \( y_i \) & \( f(x_i, y_i) \) & \( h \) & \( k_i \) & \( y_{i+1} + \frac{1}{4}k_i \) \\
\hline
2 & 0 & 2 & 3 & 0.25 & 3 & 0.0625 & 2.1875 & 3.111334 \\
3 & 0.25 & 2.807781 & 3.481721 & 0.25 & 3 & 0.3125 & 3.025389 & 3.623407 \\
4 & 0.5 & 3.751521 & 4.091538 & 0.25 & 3 & 0.5625 & 4.007242 & 4.269268 \\
5 & 0.75 & 4.866362 & 4.855294 & 0.25 & 3 & 0.8125 & 5.169818 & 5.077254 \\
6 & 1.0 & 6.194631 & 5.804848 & 0.25 & 3 & 1.0625 & 6.557434 & 6.07987 \\
7 & 1.25 & 7.787509 & 6.979373 & 0.25 & 3 & 1.3125 & 8.223722 & 7.318745 \\
8 & 1.5 & 9.707042 & 8.426947 & 0.25 & 3 & 1.5625 & 10.233737 & 8.844509 \\
9 & 1.75 & 12.082661 & 10.20649 & 0.25 & 3 & 1.8125 & 12.66652 & 10.7192 \\
10 & 2 & 14.84392 & 12.39017 & 0.25 & 3 & 2.0625 & 15.61831 & 13.01877 \\
11 & 2.25 & 18.26467 & 15.06625 & 0.25 & 3 & 2.3125 & 19.26032 & 15.83612 \\
12 & 2.5 & 22.42701 & 18.34272 & 0.25 & 3 & 2.5625 & 23.57343 & 19.28489 \\
13 & 2.75 & 27.49698 & 22.35156 & 0.25 & 3 & 2.8125 & 28.89396 & 23.5050 \\
14 & 3 & 33.67717 & 27.25412 & 0.25 & 3 & 3.0625 & 35.38055 & 28.66311 \\
15 & 3.25 & 41.21483 & 33.24754 & 0.25 & 3 & 3.3125 & 43.2928 & 34.96976 \\
16 & 3.5 & 50.41177 & 40.5727 & 0.25 & 3 & 3.5625 & 52.94757 & 42.67734 \\
17 & 3.75 & 61.6365 & 49.5239 & 0.25 & 3 & 3.8125 & 64.73174 & 52.09551 \\
18 & 4 & 75.33896 & 60.46064 & 0.25 & 3 & 4.0625 & 79.11775 & 63.60248 \\
\hline
\end{tabular}
Radwan: Solutions of Initial Value Problems Using Fifth-Order Runge-Kutta Method

<table>
<thead>
<tr>
<th>N</th>
<th>O</th>
<th>P</th>
<th>Q</th>
<th>R</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>k₄</td>
<td>xᵢ + 3₄h</td>
<td>yᵢ + 2₆k₄h + 9₄k₅h</td>
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<table>
<thead>
<tr>
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<th>T</th>
<th>U</th>
<th>V</th>
<th>W</th>
</tr>
</thead>
<tbody>
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<td>yᵢ₊₁</td>
<td>True Value</td>
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<td>92.06861</td>
</tr>
</tbody>
</table>

**Conclusion**
Using of Excel Spreadsheet makes the computation very simple and we obtained a very good results which are very close to the true values and also can use any value for the step size and will obtain the results quickly.

**References**


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