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On Finite Topological Permutation Groups

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Abstract: In this paper, we show that the permutation group S_n as well as its quotientgroup is a topological group. Furthermore, we show that S_n operates properly on itself and its quetient group.

Keywords: Topological group, Permutation group, Properly continuous

Introduction

Let f be a mapping of a topological space X into a topological space Y. f is a said to be proper if f is continuous and if the mapping $f \times I_Z : X \times Z \to Y \times Z$ is closed, for every topological space Z or equivalently, f is proper if and only if f is closed and $f^{-1}(y)$ is compact for each $y \in Y$ (Bourbaki, 1966).

Let X be a topological space and let G be a topological group. G is said to operate continuously on X if the following conditions are satisfied.

(1) X has G as a group of operators; in other words, X endowed with an external law of composition $(s,x) \to s.x$ for which G is the set of operators, and which is such that s (t.x)=(st). x and e.x=x for all $s,t \in G$ and all $x \in X$.

(2) The mapping $(s,x) \to s.x$ of $G \times X$ into X is continuous (Bourbaki, 1966).

Let G be a topological group operating continuously on a topological space X. Gis said to operate properly on X if the mapping $\theta : (s,x) \to (x, s.x)$ of $G \times X$ into $X \times X$ is proper(Bourbaki, 1966).

A semi- topological group is G endowed with a topology such that, for each $a \in G$, the translations $x \to a.x$ and $x \to x.a$ are continuous on G, and such that the symmetry $x \to x^{-1}$ is continuous on G. A paratopological group is a group G endowed with a topology such that the mapping (x,y) $\to$ xy of $G \times G$ into G is continuous. A semi-topological group G is quasi-topological if, for each $a \in G$, the mapping x $\to xax^{-1}$ of G into G is continuous. A filter on a set X is a set F of subset of X which has the following properties: (1) Every subset of X which contains aset of F belongs to F. (2) Every intersection of sets of F belongs to F. (3) The empty set is not in F. Let B(e) be the neighboorhood filter of the identy element e. For $a \in G$, define $aB(e)=\{aV : V \in B(e)\}$ where the set $aV=\{ab : b \in V\}$. Recall that if A and B are two subset of a group G, then $AB=\{xy : x \in A \text{ and } y \in B\}$. Note that $aB(e)=B(a)$.

Let B be a filter on G.

I Given any $U \in B$, there exists $V \in B$ such that $V.V \subset U$.

II Given any $U \in B$, we have $U^{-1} \in B$.

III For all $a \in G$ and all $V \in B$, we have $aVa^{-1} \in B$(Bourbaki, 1966).

Theorem1.1: Let G be group and let B be a filter on G satisfyıng the axioms I, II and III. Then there is a unique topology on G, compatible with the group structure of G, for which B is the neighbourhood filter of the identity e. For this topology the neighbourhood filter of any point $a \in G$ is the same as each as of the two filters aB and Ba (Bourbaki, 1966).

Let Xbe a topological space, $x \in X$ and $B(x)$ be the neighbourhood filter of x. Then $B(x)$ has the following properties: (i) Every subset of X which contains a set belonging to $B(x)$ itself belongs to $B(x)$. (ii) Every finite intersection of sets of $B(x)$ belongs to $B(x)$. (iii) The element x is in every set of $B(x)$. (iv) If V belongs to $B(x)$, then there is a set W belonging to $B(x)$ such that, for each $y \in W$, Vbelongs to $B(y)$(Bourbaki, 1966).

Theorem1.2: If to each element x of a set X there corresponds a set $B(x)$of a subset of X such that the properties (i), (ii), (iii) and (iv) are satisfied, then there is a unique topological structure on X such that, for each $x \in X$, $B(x)$ is the set of neighbourhoods of x in this topogy(Bourbaki, 1966).

Topology On S_n: Let Ω denote a finite set and S_n be the symmetric group on Ω. Let A_n be a group of even permutation on Ω and xA_n be a set of odd permutation of Ω, where $x \in S_n$(Hungerford, 1974).

Next, we define a topology τ^* on S_n and show that (S_n, τ^*) is a topological group. Let e be the identity element of S_n and $B'(e)=\{U \subset S_n : A_n \subset U\}$.

Lemma 2.1: (i) $B'(e)$ is a base for S_n.
(ii) $B'(e)$ is a neighbourhood filter of e.

Proof: (i) Note that $S_n \in B'(e)$. Let B_1, $B_2 \in B'(e)$ and $x \in B_1 \cap B_2$. It follows that $A_n \subset B_1 \cap B_2$. If $x \in A_n$ then let $B_3=A_n$ and consequently $B_3 \in B'(e)$. If $x \notin A_n$, then let $B_3=A_n \cup \{x\}$. Note that $x \in B_3 \in B'(e)$ and $B_3 \subset B_1 \cap B_2$. Hence $B'(e)$ is a base for S_n.

(ii) Let $U \in B'(e)$ and $U \subset V$. Since $U \in B'(e)$, it follows that $A_n \subset U \subset V$. Let $V \in B'(e)$. $U,V \in B'(e)$. Then $A_n \subset U \cap V$ and cosequently $U \cap V \in B'(e)$. Since $e \in A_n$, it follows that for all $U \in B'(e)$, $e \in U$. Let $V \in B'(e)$, $W=A_n \in B'(e)$ and $x \in W$. Since $A_n \in B'(e)$ and $A_n=W=xW \in B'(e)=B(x)$(by introduction) it follows that $V \in B(x)$. Note also that $B'(e)$ is a filter on S_n. Therefore $B'(e)$ is a neighbourhood filter of e.

Since $B'(e)$ is a neighbourhood system of e by theorem1.2 we get a unique topology

$\tau^* = \{U \subset S_n : \text{for all } x \in U,\ U \in xB'(e)=B(x)\}$

on S_n. It easy to see that $\tau^* \{\phi, S_n, A_n, xA_n, : x \in S_n\}$.

Hence we have:

Theorem 2.2: (S_n, τ^*) is a topological group.

Remark 2.3: τ^* is a topology induced by A_n.
Let n=4. Since V_4={e, (12)(34), (13)(24), (14)(23) } is a normal subgroup on S_4. We next try to find a topology on S_4 induced by V_4. Let $B_1(e)=\{U \subset S_4 : V_4 \subset U\}$. By theorem1.2 $B_1(e)$ is a neighbourhood system of e. $\tau_1^*=\{U \subset S_4 : \forall x \in U,\ U \in xB_1(e)=B_1(x)\}$.
By theorem1.1, (S_4, τ_1^*) is a topological group.

Corollary 2.4: (S_n, τ^*) is compact, para-topological, quasi-topological and semi-topological group.

Proof: since S_n is finite, S_n is compact. Since (S_n, τ^*) is a topological group, it follows that (S_n, τ^*) is a semi-topological group and para-topological group. Note that f: $(S_n, \tau^*) \to (S_n, \tau^*)$, $f(x)=axa^{-1}$, $a \in S_n$ is continuous (Bourbaki, 1966). Hence (S_n, τ^*) is a quasi-topological group.
Let $\varphi: S_n \to S_n/A_n$ be the conanical map $\varphi(x)=xA_n$, $x \in S_n$. $\tau_* =\{U \subset S_n/A_n: \varphi^{-1}(U) \in \tau^*\}$ is a topology on S_n/A_n (Bourbaki, 1966). Since $S_n/A_n=\{xA_n, A_n\}$, it follows easily that $\tau_* = \{\phi, S_n/A_n, \{xA_n\}, \{A_n\}\}$. Hence $(S_n/A_n, \tau_*)$ is a topological group.

Remark 2.5: (1) $(S_n/A_n, \tau_*)$ is compact Hausdorff topological group.
(2) $(S_4/A_4, \tau_{1*})$ is a compact, semi-topological group, quasi-topological group and para-topological group.
(3) $\tau^*_{A_n} =\{U \cap A_n : U \in \tau^*\}=\{\phi, A_n\}$. Note that $(A_n, \tau^*_{A_n})$ is a topological group.

Lemma 2.6: (1) Both $(A_n, \tau^*_{A_n})$ and (S_n, τ^*) operate properly on (S_n, τ^*).
(2) (S_n, τ^*)operates continuously on $(S_n/A_n, \tau_*)$.

Proof: (1) Since A_n is a subgroupof S_n, it follows easily S_n has A_n as a group of operators. We now show thatthe mapping $\theta: A_n \times S_n \to S_n$ is continuous. İ.e., if $U \in \tau^*$ then $\theta^{-1}(U) \in \sigma$, where $\sigma=\{\phi, A_n \times S_n, A_n \times A_n, A_n \times xA_n\}$ is the product topology on $A_n \times S_n$. It is easy to see that $\theta^{-1}(\phi)=\phi$, $\theta^{-1}(S_n)= A_n \times S_n$, $\theta^{-1}(A_n)= A_n \times A_n$ and $\theta^{-1}(xA_n)= A_n \times xA_n$. Hence θ is continuous and $(A_n, \tau^*_{A_n})$ operates continuously on (S_n, τ^*).
Next, we show that the mapping $\gamma: A_n \times S_n \to S_n \times S_n$, $\gamma(s,x)=(x, sx)$, $s \in A_n$, $x \in S_n$ is proper. To do this first we show that γ is closed. Note that the only closed subset of $A_n \times S_n$ are ϕ, $A_n \times S_n$, $A_n \times A_n$, $A_n \times xA_n$. It follows easily that $\gamma(\phi)=\phi$, $\gamma(A_n \times S_n)= S_n \times S_n$, $\gamma(A_n \times A_n)= A_n \times A_n$ and $\gamma(A_n \times xA_n)= xA_n \times A_n$. All of these are closed subsets of $S_n \times S_n$. Hence γ is closed map. Let $w \in S_n \times S_n$. Since $A_n \times S_n$ is finite, it follows that $\gamma^{-1}(w)$ is a finite subset of $A_n \times S_n$ and consequently $\gamma^{-1}(w)$ is compact. Therefore γ is proper and so $(A_n, \tau^*_{A_n})$ operates properly on (S_n, τ^*).

Next, we show that (S_n, τ^*) oprates properly on (S_n, τ^*). Note that S_n has S_n as a group of operators. Let B={ ϕ, $S_n \times S_n$, $S_n \times A_n$, $A_n \times S_n$, $S_n \times xA_n$, $A_n \times A_n$, $A_n \times xA_n$, $xA_n \times S_n$, $xA_n \times A_n$, $xA_n \times xA_n$ }. Let σ^* be the product topology on $S_n \times S_n$ induced by B. Since $\theta^{-1}(\phi)=\phi$, $\theta^{-1}(S_n)= S_n \times S_n$, $\theta^{-1}(A_n)= A_n \times A_n$, or $\theta^{-1}(A_n)= xA_n \times xA_n$ and $\theta^{-1}(xA_n)= A_n \times xA_n$ or $\theta^{-1}(xA_n)= xA_n \times A_n$. The map $\theta: S_n \times S_n \to S_n$, $\theta(x,y)=xy$ is continuous. We now show that γ: $S_n \times S_n \to S_n \times S_n$ is closed. Note that $U \subset S_n \times S_n$ is closed if and only if $U \in \sigma^*$. One can easily show that $\gamma(\phi)=\phi$, $\gamma(S_n \times S_n)= S_n \times S_n$, $\gamma(S_n \times A_n)= A_n \times S_n$, $\gamma(S_n \times xA_n)= xA_n \times S_n$, $\gamma(A_n \times S_n)= S_n \times S_n$, $\gamma(A_n \times A_n)= A_n \times A_n$, $\gamma(A_n \times xA_n)= xA_n \times A_n$, $\gamma(xA_n \times S_n)= S_n \times S_n$, $\gamma(xA_n \times A_n)= A_n \times xA_n$, $\gamma(xA_n \times xA_n)= xA_n \times A_n$. Thus γ is closed map. Let $w \in S_n \times S_n$. Since $\gamma^{-1}(w) \subset S_n \times S_n$ is finite, it follows that $\gamma^{-1}(w)$ is compact. Hence (S_n, τ^*) operates properly on itself.
(2) Let θ: $S_n \times S_n/A_n \to S_n/A_n$, $\theta(x,y)=xy$. Since $\theta^{-1}(\phi)=\phi$, $\theta^{-1}(S_n/A_n)= S_n \times S_n/A_n$, $\theta^{-1}(\{A_n\})= A_n \times \{A_n\}$, $\theta^{-1}(\{xA_n\})= xA_n \times xA_n$ and $\theta^{-1}(xA_n)= A_n \times \{xA_n\}$, it follow that θ is continuous. Hence (S_n, τ^*) operates continuously on $(S_n/A_n, \tau_*)$.

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