Matrix Representation of the Tetrahedron Groups

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Abstract: We present the method by which one can get the matrix representation of the tetrahedron groups introduced in Lannér, (1950)

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Introduction
Lannér, (1950) and Best, (1971) listed the nine non-congruent tetrahedra. With each of these tetrahedra one can associate a subgroup of the whole group of isometries, \( \text{P} \Gamma \text{L}(2, C) \), of \( H^3 \), generated by reflections in faces of each tetrahedron. This subgroup we call it the tetrahedron group.

In \( \text{P} \Gamma \text{L}(2, C) \), we find the matrices representing the generators of each tetrahedron group through the consideration of its action on \( H^3 \). This matrix representation is found through finding the orientation preserving subgroup first and then present the matrices representing the reflections.

The Three – Dimensional Geometry:
Let \( H^3 = \{(x, y, w) \in \mathbb{R}^3 : w > 0 \} \).

We may regard \( H^3 \) as a subspace of the space \( H = \{x + yi + wj + tk : x, y, w, t \in \mathbb{R}\} \) of quaternion with the usual multiplication,

\[ \rho = \beta = k^2 = -1, \quad ij = ji = k. \]

The sub ring \( \mathbb{R} + R \) is isomorphic to \( C \) and will be identified with \( C \). Then

\[ H^3 = \{\zeta = z + wj : z \in C, w \in \mathbb{R}, w > 0\}. \]

Consider the group \( G = \text{PSL}(2, C) \) acting on \( H^3 \) as follows.

Let \( g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \), \( (\alpha, \beta, \gamma, \delta \in C, \alpha \delta - \beta \gamma = 1) \)
be an element of \( G \) and let \( \zeta \in \mathbb{H}^3 \). Then the action of \( G \) on \( \mathbb{H}^3 \) is given by:

\[ g: \zeta \mapsto \frac{\alpha \zeta + \beta}{\gamma \zeta + \delta}. \tag{1} \]

It is clear that the the image of \( \zeta \) is also in \( \mathbb{H}^3 \) (Harvey, 1977)
We can prove the following lemma.

Lemma 1: \( G \) is homeomorphic to the product of stabilizer of a point of \( \mathbb{H}^3 \) and the subgroup \( G_0 \) of \( G \) consisting of all matrices of the form

\[ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \quad \alpha \in \mathbb{R}, \quad \alpha > 0, \quad \beta \in \mathbb{C}. \]

Proof: The action of elements of \( G_0 \) on \( j \) is given by

\[ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} j \mapsto \alpha \beta + \alpha^2 j \in \mathbb{H}^3 \]

Therefore \( \mathbb{H}^3 \) is a \( G \)-orbit, and hence \( G \) is transitive in its action on \( \mathbb{H}^3 \).

In fact, the map \( \phi : G_0 \rightarrow \mathbb{H}^3 \) given by

\[ \phi(g) = g(j) \]

defines a homeomorphism between \( G_0 \) and \( \mathbb{H}^3 \). We consider now the stabilizer of \( j \) in \( G \).

\[ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G, \quad \alpha \beta + \alpha^2 j = j \]

means \( \gamma = -\beta, \delta = \alpha \).

Therefore, the stabilizer of \( j \) is the group

\[ \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{C}, \alpha \alpha + \beta \beta = 1 \right\} \]

Which is the group \( \text{PSU}(2, C) \) of all unitary matrices.

The map \( h : \mathbb{H}^3 \rightarrow \text{PSU}(2, C) \) given by

\[ h(\alpha + j) = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \]

for all \( \alpha + j \in \mathbb{H}^3 \), defines a homeomorphism between \( \mathbb{H}^3 \) and \( \text{PSU}(2, C) \).

Since \( G \) is transitive in its action on \( \mathbb{H}^3 \), then the stabilizer of any element \( \zeta \) of \( \mathbb{H}^3 \) is conjugate to the stabilizer of \( j \).

Let \( g \in G \), then

\[ g j = \phi^{-1}(g(j)) j = \psi(g) j, \text{ say.} \]

Therefore,

\[ \psi(g)^{-1} g j = j \]

that is to say

\[ \psi(g)^{-1} g \in \text{PSU}(2, C). \]

Now \( \psi(g) \in G_0 \), then

\[ g = \psi(g) \psi(g)^{-1} g. \]

This last formula sets up a homeomorphism between \( G \) and the product of \( G_0 \) and \( \text{PSU}(2, C) \).

Therefore \( G \) is the whole group orientation preserving isometries of \( \mathbb{H}^3 \) because it contains the whole orthogonal subgroup acting on tangent space at \( j \), and also it is transitive on \( \mathbb{H}^3 \).
Consider the transformation

\[ t : \zeta = z + wz \mapsto z + wz. \]

Consider \( t \) as

\[ (x, y, z) \mapsto (x, -y, w), \]

so \( t \) is an isometry and it reverses orientation.

Let \( u \) be any other orientation reversing transformation, then \( ut \) is an orientation preserving element and therefore it is in \( \text{PSL}(2, \mathbb{C}) \).

Adjoin \( t \) to \( \text{PSL}(2, \mathbb{C}) \) we get the whole group of isometries \( \Gamma \). \( L(2, \mathbb{C}) \).

\[ \begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{hyperbolic_tetrahedron}
\caption{The Hyperbolic Tetrahedron Groups}
\end{figure} \]

The Hyperbolic Tetrahedron Groups: Lannér, (1950) has shown that there are exactly nine non-congruent hyperbolic tetrahedra with all dihedral angles equal to an integer sub multiple of \( \pi \).

Let \( \pi / \lambda_i \) and \( \pi / \mu_i \), \( i = 1, 2, 3 \), be the dihedral angles at opposite edges of the tetrahedron, where \( \pi / \lambda_i \), \( i = 1, 2, 3 \) are the angles at the edges of a face.

We list the nine non-congruent tetrahedra by writing \( [\lambda_1, \lambda_2, \lambda_3 : \mu_1, \mu_2, \mu_3] \) in each case.


With each tetrahedron we associate a group of isometries of \( \mathbb{H} \), call it the tetrahedral group, generated by the reflections in its faces. Because of the angle condition, this group is discrete in each case listed and has this presentation:

\[
< a, b, c, d : a^2 = b^2 = c^2 = d^2 = (bd)^4 = (cd)^4 = (ad)^4 = 1 >
\]

Where \( a, b, c \) and \( d \) are reflections in faces of tetrahedron.

The orientation preserving subgroup has presentation:

\[
< a, b, c : a^2 = b^2 = c^2 = (bc)^2 = (ab)^2 = (ad)^2 = (bd)^2 = (cd)^2 = 1 >
\]

where \( a_1 = ax \), \( a_2 = bx \), \( a_3 = cx \), \( a_4 = dx \).

Matrix representation of the tetrahedral groups:

Now \( T_1, ..., T_9 \) are to denote not the tetrahedra, but the corresponding tetrahedral groups.

We find matrices representing the orientation preserving subgroup of the tetrahedral group \( T_1 \),

that is, \( x_1, x_2, x_3 : x_1^2 = x_2^2 = x_3^2 = (x_2 x_3)^j = 1 \).

Let \( X_1, X_2, X_3 \) be the matrices representing \( x_1, x_2, x_3 \) respectively. The action of \( X_i \) is defined by (1). Each \( X_i \) has determinant \( 1 \) and \( x_i \)'s being rotations through \( \pm 2 \cos(\pi / n_i) \). So the matrices \( X_i \) are found by solving the equations obtained from the determinants and traces of \( X_1, X_2, X_3, X_1 X_2, X_3 X_1, X_2 X_3 \), and \( X_1 X_2 X_3 \).

From these matrices one can find the matrices associated with the reflections \( a, b, c \) and \( d \) of the tetrahedron group \( T_1 \).

If \( \begin{pmatrix} p & q \\ r & s \end{pmatrix} \) is a matrix associate of a reflection,

then it is associated with the orientation reversing isometry

\[
\zeta = \pm (p + d)(s + d)^{-1} \quad p, q, r, s \in \mathbb{C}
\]

Where \( \zeta j = t \zeta = j \zeta = j^{-1} = -j \zeta j \).

For (2) to be associated with a reflection, we have

\[
ps - qr = -1, \quad s = -p.
\]

If \( A \) and \( B \) are two matrix associates of reflections given in this way, then the orientation preserving isometry which is the product of these two reflections will be associated in the ordinary way (1) with the matrix \( \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \) (where the bar denotes complex conjugation).

So having the matrices \( X_1, X_2, X_3 \) known, then if \( A, B, C, D \) are the matrix associates of the reflections \( a, b, c, d \) respectively, then we have

\[
T_1 = < a, b, c, d : a^2 = b^2 = c^2 = d^2 = (bc)^2 = (ca)^2 = (ab)^2 = (ad)^2 = (bd)^2 = (cd)^2 = 1 >
\]

and can be represented by:
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\[
\begin{align*}
A_5 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & B_5 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \\
C_5 &= \begin{pmatrix} 1 \sqrt{5+1} & -1 \sqrt{5-1} \\ 2 & 2 \end{pmatrix} & D_5 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{align*}
\]

A similar method can be used to find the matrices associated with the generators of \( T_2 - T_9 \), and we list them below.

\[T_2 = \langle a, b, c, d : a^2 = b^2 = c^2 = d^2 = (bc)^2 = \rangle \]

\[
\begin{align*}
A_6 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & B_6 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\end{align*}
\]

\[T_6 = \langle a, b, c, d : a^2 = b^2 = c^2 = d^2 = (bc)^2 = \rangle \]

\[
\begin{align*}
A_7 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & B_7 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\end{align*}
\]

\[T_7 = \langle a, b, c, d : a^2 = b^2 = c^2 = d^2 = (bc)^2 = \rangle \]

\[
\begin{align*}
A_8 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & B_8 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\end{align*}
\]

\[T_8 = \langle a, b, c, d : a^2 = b^2 = c^2 = d^2 = (bc)^2 = \rangle \]

\[
\begin{align*}
A_9 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & B_9 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\end{align*}
\]

\[T_9 = \langle a, b, c, d : a^2 = b^2 = c^2 = d^2 = (bc)^2 = \rangle \]

References


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