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A Generalization of Cartesian Product of Fuzzy Subgroups and Ideals

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Abstract: In this paper we generalize Malik and Mordeson's paper (1991). I analysis the cartesian product of fuzzy subgroups (ideals) of different groups (different ideals). That is; if μ and σ are fuzzy subgroups (ideals) of G_1 and G_2 (R_1 and R_2) respectively then $\mu\times\sigma$ is a fuzzy subgroup (ideal) of $G_1\times G_2$ ( ). Conversely the opposite direction of the above statements is studied. We generalize the above statements for different Groups (Rings).

Key Words: Fuzzy Subset, Fuzzy Subgroup, Level Subgroup, Fuzzy Ideal, Level Ideal, Fuzzy Relation, Cartesian product

Introduction

The concept of a fuzzy subset was introduced by Zadeh(1965). Fuzzy subgroup and its important properties were defined and established by Rosenfeld (1971). Then many authors have studied about it. After this time it was necessary to define fuzzy ideal of a ring. The notion of a fuzzy ideal of a ring was introduced by Liu (1982). Malik, Mordeson and Mukherjee have studied fuzzy ideals. The concept of a fuzzy relation on a set was introduced by Zadeh(1971). Bhattacharya and Mukherjee have studied fuzzy relation on groups. Malik and Mordeson (1991) studied fuzzy relation on rings. Moreover Malik and Mordeson have written very important book for Fuzzy algebra which is "Fuzzy Commutative Algebra"(Malik and Mordeson, 1998).

In this paper G is a group and R is a commutative ring with identity. A fuzzy relation on R is the fuzzy subset of $R\times R$. We generalize malik and mordeson's paper. That is; if μ_1,μ_2 are fuzzy subgroups (ideals) of G_1 and G_2 (R_1 and R_2) respectively then $\mu_1\times\mu_2$ is a fuzzy subgroup (ideal) of $G_1\times G_2$ ($R_1\times R_2$). Let μ_1,μ_2 be fuzzy subsets of G_1,G_2 respectively such that $\mu_1\times\mu_2$ is a fuzzy subgroup (ideal) of $G_1\times G_2$ ($R_1\times R_2$). Then μ_1 or μ_2 is fuzzy subgroup (ideal) of G_1 or G_2 (R_1 or R_2) respectively. Let μ_1 and μ_2 be fuzzy subsets of R such that $\mu_1\times\mu_2$ is a fuzzy subgroup (ideal) of $G_1\times G_2$ ($R_1\times R_2$) . If $\forall x\in G_1,\forall y\in G_2$ $\mu_1(e_1)=\mu_2(e_2)$, $\mu_1(x)\le\mu_1(e_1)$ and $\mu_2(y)\le\mu_2(e_2)$ ($\forall x\in R_1,\forall y\in R_2$ $\mu_1(0_1)=\mu_2(0_2)$, $\mu_1(x)\le\mu_1(0_1)$ and $\mu_2(y)\le\mu_2(0_2)$) then both μ_1 and μ_2 are fuzzy subgroups (ideals) of G_1 and G_2 (R_1 and R_2). Also we extend these above theorems for n different Groups (Rings). That is if $\mu_1,\mu_2,\mu_3,\ldots,\mu_n$ are fuzzy subgroups (ideals) of $G_1,G_2,\ldots,G_n$ ($R_1, R_2,\ldots,R_n$) respectively, then $\mu_1\times\mu_2\times\mu_3\times\ldots\times\mu_n$ is fuzzy subgroup (ideal) of $G_1\times G_2\times\ldots\times G_n$ ($R_1, R_2,\ldots,R_n$). Then we prove the opposite direction of the previous statement under some conditions.

Preliminaries: In this section, we review some basic definitions and results.

Definition 1.1: A fuzzy subset of S is a function $\mu:S\to[0,1]$.

Definition 1.2: A fuzzy subset μ of G is called a fuzzy subgroup of G if

(i) $\mu(xy)\ge\min(\mu(x),\mu(y))$

(ii) $\mu(x^{-1})\ge\mu(x)$ for all $x,y\in G$

If μ is a fuzzy subgroup of G then $\mu(x^{-1})=\mu(x)$ for all $x\in G$.

Definition 1.3: If μ is a fuzzy subset of S, then for any $t\in\operatorname{Im}\mu$, the set $\mu_t=\{x\in S\,|\,\mu(x)\ge t\}$ is called the level subset of S with respect to μ.

Theorem 1.4: Let μ be fuzzy subset of G. μ is a fuzzy subgroup of G if and only if μ_t is an subgroup of G for $\forall t\in\operatorname{Im}\mu$.

Here, if μ is a fuzzy subgroup of G, then μ_t is called a level subgroup of μ.

Definition 1.5: A fuzzy subset μ of R is called a fuzzy left (right) ideal of R if

(i) $\mu(x-y)\ge\min(\mu(x),\mu(y))$

(ii) $\mu(xy)\ge\mu(y)$ ($\mu(xy)\ge\mu(x)$) for all $x,y\in R$.

A fuzzy subset μ of R is called a fuzzy ideal of R if μ is a fuzzy left and fuzzy right ideal of R.

Definition 1.6: If μ is a fuzzy subset of R, then for any $t \in \operatorname{Im}\mu$, the set $\mu_t = \{x \in R \mid \mu(x) \geq t\}$ is called the level subset of R with respect to μ.

Theorem 1.7: Let μ be fuzzy subset of R. μ is a fuzzy ideal of R if and only if μ_t is an ideal of R for $\forall t \in \operatorname{Im}\mu$.

Here, if μ is a fuzzy ideal of R, then μ_t is called a level ideal of μ.

Definition 1.8: A fuzzy relation μ on R is the fuzzy subset of $R \times R$.

Definition 1.9: Let μ and σ be fuzzy subsets of R. The Cartesian product of μ and σ is $\mu \times \sigma(x, y) = \min(\mu(x), \sigma(y))$ for all $x, y \in R$.

Fuzzy Subgroups and Fuzzy Ideals: Now we will generalize some theorems in (Malik and Mordeson, 1991).

Theorem 2.1: If μ_1 and μ_2 are fuzzy subgroups of G_1 and G_2 respectively, then $\mu_1 \times \mu_2$ is a fuzzy subgroup of $G_1 \times G_2$.

Proof: Let $(a_1,b_1),(a_2,b_2) \in G_1 \times G_2$.

$$\begin{aligned}
\mu_1 \times \mu_2((a_1,b_1),(a_2,b_2)) &= \mu_1 \times \mu_2(a_1a_2, b_1b_2) \\
&= \min(\mu_1(a_1a_2), \mu_2(b_1b_2)) \\
&\geq \min(\mu_1(a_1), \mu_1(a_2), \mu_2(b_1), \mu_2(b_2)) \\
&\geq \min(\min(\mu_1(a_1), \mu_2(b_1)), \min(\mu_1(a_2), \mu_2(b_2))) \\
&= \min(\mu_1 \times \mu_2(a_1,b_1), \mu_1 \times \mu_2(a_2,b_2))
\end{aligned}$$

and

$$\begin{aligned}
\mu_1 \times \mu_2((a_1,b_1)^{-1}) &= \mu_1 \times \mu_2(a_1^{-1}, b_1^{-1}) \\
&= \min(\mu_1(a_1^{-1}), \mu_2(b_1^{-1})) \\
&\geq \min(\mu_1(a_1), \mu_2(b_1)) \\
&= \mu_1 \times \mu_2(a_1,b_1).
\end{aligned}$$

Therefore $\mu_1 \times \mu_2$ is a fuzzy subgroup of $G_1 \times G_2$.

Theorem 2.2: If μ_1, μ_2 are fuzzy ideals of R_1, R_2 respectively, then $\mu_1 \times \mu_2$ is fuzzy ideal of $R_1 \times R_2$.

Proof:
$\mu_1 \times \mu_2\,(0_1, 0_2) = \min(\mu_1(0_1), \mu_2(0_2))$. Let $t \in \operatorname{Im}(\mu_1 \times \mu_2)$ then $t \leq \mu_1(0_1)$ and $t \leq \mu\,(0_2)$.

Thus μ_{1_t} and μ_{2_t} are ideals of R_1 and R_2 respectively. Hence for all

$$t \in \operatorname{Im}(\mu_1 \times \mu_2), \quad (\mu_1 \times \mu_2)_t = \mu_{1_t} \times \mu_{2_t}$$

is left ideal of $R_1 \times R_2$. Because $\forall (x,y),(z,t) \in (\mu_1 \times \mu_2)_t$. and $\forall (a,b) \in (R_1, R_2)$ we must show that $(x-z, y-t) \in (\mu_1 \times \mu_2)_t$ and $(xa, yb) \in (\mu_1 \times \mu_2)_t$.

$\mu_1 \times \mu_2(x-z, y-t) = \min(\mu_1(x-z), \mu_2(y-t))$ and since μ_{1_t} and μ_{2_t} are ideals of R_1 and R_2 respectively $\min(\mu_1(x-z), \mu_2(y-t)) \geq t$ then $(x-z, y-t) \in (\mu_1 \times \mu_2)_t$. Since $\mu_1 \times \mu_2\,(xa, yb) = \min(\mu_1(xa), \mu_2(yb))$ and

μ_{1_t} and μ_{2_t} are ideals of $R_1 \times R_2$ $\min(\mu_1(xa), \mu_2(yb)) \geq t$ then $(xa, yb) \in (\mu_1 \times \mu_2)_t$.

Hence $(\mu_1 \times \mu_2)_t$ is ideal of $R_1 \times R_2$.

Corollary 2.3:

i. If $\mu_1, \mu_2, \mu_3, \ldots, \mu_n$ are fuzzy subgroups of $G_1, G_2, \ldots, G_n$ respectively, then $\mu_1 \times \mu_2 \times \mu_3 \times \ldots \times \mu_n$ is fuzzy subgroups of $G_1 \times G_2 \times \ldots \times G_n$.

ii. If $\mu_1, \mu_2, \mu_3, \ldots, \mu_n$ are fuzzy ideals of $R_1, R_2, \ldots, R_n$ respectively, then $\mu_1 \times \mu_2 \times \mu_3 \times \ldots \times \mu_n$ is fuzzy ideal of $R_1, R_2, \ldots, R_n$.

Proof: One can easily show by induction method.

Theorem 2.4: Let μ_1, μ_2 be fuzzy subsets of G_1, G_2 respectively such that $\mu_1 \times \mu_2$ is a fuzzy subgroup of $G_1 \times G_2$. Then μ_1 or μ_2 is fuzzy subgroup of G_1 or G_2 respectively.

Proof: We know that
$\mu_1 \times \mu_2(e_1, e_2) = \min(\mu_1(e_1), \mu_2(e_2)) \geq \mu_1 \times \mu_2(x, y)$, $\forall (x, y) \in G_1 \times G_2$. Then $\mu_1(x) \leq \mu_1(e_1)$ or $\mu_2(y) \leq \mu_2(e_2)$. If $\mu_1(x) \leq \mu_1(e_1)$, then $\mu_1(x) \leq \mu_2(e_2)$ or $\mu_2(y) \leq \mu_2(e_2)$. Let $\mu_1(x) \leq \mu_2(e_2)$. Then $\forall x \in G_1$ $\mu_1 \times \mu_2(x, e_2) = \mu_1(x)$. $\forall x, y \in G_1$

$$\begin{aligned}
\mu_1(xy) &= \mu_1 \times \mu_2(xy, e_2) \\
&= \mu_1 \times \mu_2((x, e_2)(y, e_2)) \\
&\geq \min(\mu_1 \times \mu_2(x, e_2), \mu_1 \times \mu_2(y, e_2)) \\
&= \min(\mu_1(x), \mu_1(y))
\end{aligned}$$

and

$$\mu_1(x^{-1}) = \mu_1 \times \mu_2(x^{-1}, e_2)$$
$$= \mu_1 \times \mu_2(x^{-1}, e_2^{-1})$$
$$= \mu_1 \times \mu_2(x, e_2)^{-1}$$
$$\geq \min \mu_1 \times \mu_2(x, e_2)$$
$$= \mu_1(x).$$

Therefore μ_1 is fuzzy subgroup of G_1.

Now suppose that $\mu_1(x) \leq \mu_2(e_2)$ is not true for all $x_1 \in G_1$. If $\mu_1(x) > \mu_2(e_2)$ $\exists x \in G_1$, then $\mu_2(y) \leq \mu_2(e_2)$ $\forall y \in G_2$. Therefore $\mu_1 \times \mu_2(e_1, y) = \mu_2(y)$ for all $y \in G_2$. Similarly $\forall x, y \in G_2$

$$\mu_2(xy) = \mu_1 \times \mu_2(e_1, xy)$$
$$= \mu_1 \times \mu_2((e_1, x)(e_1, y))$$
$$\geq \min(\mu_1 \times \mu_2(e_1, x), \mu_1 \times \mu_2(e_1, y))$$
$$= \min(\mu_2(x), \mu_2(y))$$

and

$$\mu_2(x^{-1}) = \mu_1 \times \mu_2(e_1, x^{-1})$$
$$= \mu_1 \times \mu_2(e_1^{-1}, x^{-1})$$
$$= \mu_1 \times \mu_2(e_1, x)^{-1}$$
$$\geq \min \mu_1 \times \mu_2(e_1, x)$$
$$= \mu_2(x).$$

Hence μ_2 is fuzzy subgroup of G_2. Consequently either μ_1 or μ_2 is fuzzy subgroup of G_1 or G_2 respectively.

Theorem 2.5: Let μ_1, μ_2 be fuzzy subsets of R_1, R_2 respectively such that $\mu_1 \times \mu_2$ is a fuzzy ideal of $R_1 \times R_2$. Then μ_1 or μ_2 is fuzzy ideal of R_1 or R_2 respectively.

Proof: We can prove in a similar manner.

Corollay 2.6: Let $\mu_1, \mu_2, \mu_3, \ldots, \mu_n$ be similar fuzzy subsets of $R_1, R_2, \ldots, R_n$ such that $\mu_1 \times \mu_2 \times \mu_3 \times \ldots \times \mu_n$ is fuzzy subgroup (ideal) of $G_1 \times G_2 \times \ldots \times G_n$ ($R_1 \times R_2 \times \ldots \times R_n$). Then μ_1 or μ_2 or μ_3 or ... or μ_n is a fuzzy subgroup (ideal) of $G_1, G_2, \ldots, G_n$ ($R_1, R_2, \ldots, R_n$) respectively.

Corollay 2.7: let μ_1 and μ_2 be fuzzy subsets of R such that $\mu_1 \times \mu_2$ is a fuzzy subgroup (ideal) of $G_1 \times G_2$ ($R_1 \times R_2$). If $\forall x \in G_1, \forall y \in G_2$
$\mu_1(e_1) = \mu_2(e_2)$,
$\mu_1(x) \leq \mu_1(e_1)$ and $\mu_2(y) \leq \mu_2(e_2)$
($\forall x \in R_1, \forall y \in R_2$ $\mu_1(0_1) = \mu_2(0_2)$,
$\mu_1(x) \leq \mu_1(0_1)$ and $\mu_2(y) \leq \mu_2(0_2)$) then both μ_1 and μ_2 are fuzzy subgroups (ideals) of G_1 and G_2 (R_1 and R_2).

Corollary 2.8: Let $\mu_1, \mu_2, \mu_3, \ldots, \mu_n$ be fuzzy subsets of $G_1, G_2, \ldots, G_n$ ($R_1, R_2, \ldots, R_n$) such that $\mu_1 \times \mu_2 \times \mu_3 \times \ldots \times \mu_n$ is fuzzy subgroup ideal of $G_1, G_2, \ldots, G_n$ ($R_1, R_2, \ldots, R_n$). If $\forall x_1 \in G_1, \forall x_2 \in G_2, \ldots, \forall x_n \in G_n$
$\mu_1(e_1) = \mu_2(e_2) = \mu_3(e_3) = \ldots = \mu_n(e_n)$,
$\mu_1(x_1) \leq \mu_1(e_1)$, $\mu_2(x_2) \leq \mu_2(e_2)$, $\mu_3(x_3) \leq \mu_3(e_3)$,..., $\mu_n(x_n) \leq \mu_n(0_n)$
($\forall x_1 \in R_1, \forall x_2 \in R_2, \ldots, \forall x_n \in R_n$
$\mu_1(0_1) = \mu_2(0_2) = \mu_3(0_3) = \ldots = \mu_n(0_n)$,
$\mu_1(x_1) \leq \mu_1(0_1)$, $\mu_2(x_2) \leq \mu_2(0_2)$, $\mu_3(x_3) \leq \mu_3(0_3)$,..., $\mu_n(x_n) \leq \mu_n(0_n)$) then
then $\mu_1, \mu_2, \mu_3, \ldots, \mu_n$ is a fuzzy subgroups (ideals) of $G_1, G_2, \ldots, G_n$ ($R_1, R_2, \ldots, R_n$) respectively.

References

Liu, W. J., Fuzzy invariant subgroups and fuzzy ideals, Fuzzy Sets and Systems, 8, 1982, 133-139.

Malik, D.S. and J. N. Mordeson, 1991. Fuzzy relations on rings and groups, Fuzzy Sets and Systems 43 117-123.

Malik, D.S. and J. N. Mordeson, 1998. Fuzzy Commutative Algebra, World scientificPublishing.

Rosenfeld, A., 1971. Fuzzy groups, J. Math. Anal. Appl. 35, 512-517.

Zadeh, L. A., 1965. Fuzzy sets, Inform. Control 8 383-353.

Zadeh, L. A., 1971. Similarity relations and fuzzy ordering, Inform. Sci. 3 177-220.