On the Approximative Solution of Boundary Value Problems by Collocation

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ABSTRACT

This paper concerns with the use of B-Splines to approximate a solution of a differential equations by collocation. The effect of knot placement on the accuracy of approximation is considered and numerical examples are given to illustrate the effectiveness of knot sequence.

Key words: Collocation methods, B-splines, knot sequence

INTRODUCTION

The numerical solution of boundary value problem is a topic in which active research is currently underway. There are number of methods used to solve boundary value problems. The most important of these probably the collocation method. For discussion referring to collocation method (Reddien, 1979) (Deuflhard, 1979) and (Ascher et al., 1985). For an important collocation computer code, (Ascher et al., 1981) and (Ascher et al., 1981). In this study we use B-splines in the numerical solution of an initial boundary value problems by collocation. This method provides a strategy by which we can attack many problems in applied mathematics. Rayleigh Ritz method or Galerkin’s method could be made quite effective if one were to give up on using polynomials or other analytic functions as trial function and used piecewise polynomial instead.

Numerical solution technique

Now let us consider how this method works (Boor et al., 1973). We look for approximating a function g on [a,b], which is given to us implicitly, as a solution of the differential equation

$$(D^n g)(x) = F(x; g(x), K, (D^{m-1} g), \text{ for } x \in [a, b])$$

with boundary conditions

$$\sum_{j=1}^{m} w_j (D^{j-1} g)(x_j) = \beta_j g - c_p, \ i=1,2, K, m.$$  \hspace{1cm} (2.2)

Where $F[x; Z, K, Z_m]$ is a real valued function on $R^{m-1}$ and we will assume it to be sufficiently smooth and $w_j$ are constants and the points $x_j$ satisfy $a \leq x_j \leq b$. $c_p, \beta_j, K$ and $c_m$ are continuous linear functionals in $C^{(m-1)}$ and $c_p, c_m, K, c_m$ are known constants. These boundary conditions are linear, the differential equations is nonlinear. Since we will linearize (2.1) in the computations, we could have made these conditions nonlinear as well (Wittenbrink, 1973).

Since (2.1) is nonlinear, (2.1-2.2) may have many solutions. Therefore we require that there be a neighborhood around the specific solution $g$ and we will start our iterative process within this neighborhood in order to converge to this particular solution.

We intent to approximate $g$ by piecewise polynomial (pp) functions using collocation. That is, we determine a pp function $f$ so that it exactly satisfies the differential equations at certain points, the collocation points. We look for $f \in P_{k,m} \cap C^{(m-1)}$ for which
\[(D^{m}f)(t_{j})=F(t_{j};f(t_{j}),K,(D^{m-1}f)(t_{j})), \text{ for } i=1, K, l\]
\[(2.3)\]
\[\beta_{i}, f=c_{i}, \text{ for } i=1, 2, K, m. \]
\[(2.4)\]

Here \(P_{k, l}\) denotes the linear space of \(p^{l}\) functions of order \(k\) with breakpoint sequence \(\xi_{i}\).

We choose the collocation points per subinterval and distributed the same in each subinterval with \(-1 \leq P < P_{2} < K < P_{3} \leq 1\). We calculate these points as follows:
\[t_{i, j} = \frac{\xi_{i, j} \pm P_{i,j} (\xi_{i+1, j} - \xi_{i, j})}{2}, j = 1, 2, \ldots, K; i = 1, 2, K, l.\]

We choose \(P\) as the zeros of the \(k\)-th Legendre polynomial. The reason for such a selection can be given with the following theorem (Boor et al., 1978).

**Theorem.** Assume that the function \(F\) in (2.1) is sufficiently smooth in a neighborhood of the curve \([a, b] \times \{x \mid x = q(x), \text{ g}(x), K, D^{m-1}g(x)\}\).

Assume further that the collocation points \(P = P_{i,j}^{k, l}\) in \([-1, 1]\) has been chosen such that \[\int_{-1}^{1} q(x) \frac{d}{dx} (x-p_{i,j}) dx = 0\]
for every \(q \in P_{i,j}\). Then the solution \(f\) near \(g\) of the approximate problem (2.3-2.4) satisfies
\[\|D^{i} (g-f)\| \leq \text{const} \|\xi\|^{i}, i = 0, K, m.\]
\[(2.5)\]

At the breakpoints, the approximation is of even higher order and satisfies
\[\|D^{i} (g-f)\| \leq \text{const} \|\xi\|^{i}, i = 0, K, m.\]
\[(2.6)\]

Here \(\text{const}\) depends on \(F, g\) and \(k\), but does not depend on \(\xi\).

Since the problem (2.3-2.4) is linear in general, we need to use some iterative scheme for its solution. We can solve by Newton’s method starting with a sufficiently close initial guess \(f_{0}\), that is (2.3-2.4) has a solution
\[f_{+} = \lim_{i \to \infty} f_{i}\]
with \(f_{+}\) the solution \(P_{k,m} \cap C^{(m)}\) of the linear problem.

\[(D^{m}y)(t_{j})+\sum_{j=1}^{m} y_{j}(t_{j})(D^{j}y)(t_{j})=h(t_{j}), i=1, K, l,\]
\[(2.7)\]

where
\[y_{j}(x) = \left(\frac{\partial F}{\partial z_{j}}(x; f_{j}(x), (D^{m-1}f_{j})(x))\right), j=0, K, m-1; \]
\[(2.8)\]

and
\[h(x) = F(x; f_{j}(x), (D^{m-1}f_{j})(x)) + \sum_{j=1}^{m} y_{j}(x)(D^{j}f_{j})(x)\]
\[(2.9)\]

The function \(y\) in (2.7-2.9) is a linear combination of appropriate B-splines. Let \(t_{j}\) be the nondecreasing sequence which contains each of \(\xi_{i}\) and \(\xi_{i+1}\) \(k+m\) times and each interior breakpoint \(\xi_{i}, K, \xi_{l}\) \(k\) times. Then \(n = k+m\) and
\[P_{k,m} \cap C^{(m)} \rightarrow S_{k,m, 1}\]

Therefore the unknown function \(y\) can be written in the form

\[f(t_{j}) = \sum_{j=1}^{m} \sum_{i=0}^{m} b_{i,j} \delta_{i,j} \]
\[(2.10)\]

where \(b_{i,j}\) is the linear combination of appropriate B-splines.
\[
\gamma = \sum_{i=1}^{n} \alpha_i B_{i,k+m,l}
\]

We can determine \( \gamma \) by determining its \( B \) coefficient vector \( \alpha \). This gives the linear system

\[
\sum_{i=1}^{n} (B_j)_{ij} \alpha_i = h_{ij}, i=1,k,l,
\]

\[
\sum_{i=1}^{n} (\beta_i B_j) \alpha_i = c_{ip}, i=1,k,m
\]

(2.10)

Where linear differential operator \( L \) is defined by

\[
L = D^n y + \sum_{j=1}^{m} V_j(x) D^j y
\]

The following theorem gives sufficient condition for the existence of discrete solutions of boundary value problems.

Theorem. Let \( g(x), F(x, z_0, z_1, K, z_m) \) and \( \left( \frac{\partial F}{\partial z_k} \right) (x, z_0, z_1, K, z_m) \) be functions defined and continuous for

\[
\varepsilon_k g^{(m)}(x) \leq \delta, a \leq x \leq b (0 \leq k \leq m-1, \delta \geq 0)
\]

Let 0 = \( y \) be only trivial solution of the homogeneous equation 0 = \( \{m\} \) \( y \) satisfying the boundary condition (2.2). If the linear homogeneous equation.

\[
y^{(m)}(x) \sum_{k=0}^{m-1} \frac{\partial F(x, z_0, z_1, K, z_m)}{\partial z_k} y^{(k)} = 0
\]

has only trivial solution under the boundary condition (2.2), then there exist a number \( \alpha > 0 \) so that unique solution of the problem (2.1-2.2) can be found inside the sphere \( |w - g^{(m)}| \leq \alpha \)

The Effect of knot placement on the accuracy of the Spline approximation

Construction of the piecewise polynomials depends on partition of the interval which is an important matter since every partition leads to a different approximation. It was suggested by (Boor et al., 1978) that we place the breakpoints \( \xi_0, K, \xi_m \) so as to minimize

\[
\max_i |s(\xi_i)(D^k g)|_{\xi_i}
\]

(3.1)

For this purpose the following analysis should be considered.

\[
s(\alpha, \beta) = \|D^k g\| \sup_{a \leq x \leq \beta} \|D^k g(x)\|
\]

is a continuous function of \( \alpha \) and \( \beta \) and monotone, increasing in \( \alpha \) and decreasing in \( \alpha \) when \( D^k g \) is continuous. In order to minimize (3.1) we choose \( \xi_0, K, \xi_m \) so that

\[
|A(\xi_i)| D^k g|_{\xi_i} = \text{constant for} \ i = 1, K, l.
\]

(3.2)

It is not easy task to find appropriate placement of \( \xi_i \)’s since we don’t know \( D^k g \). Let us rewrite (3.2) as
The last equality reduces the appropriate determining of \( \xi_2, K, \xi_3 \) such that

\[
\int_\xi^\xi (D^k g)(x)^{1/\alpha} \, dx = \frac{1}{\alpha} \int_\xi^\xi (D^k g)^\alpha \, dx, \quad i = 1, k.
\]

This latter problem can be easily solved by replacing the function \( D^k g \) by some piecewise constant function \( h = D^k g \). Then

\[
h(x) = \int_a^x (h(s))^{1/\alpha} \, ds
\]

is continuous and monotone increasing piecewise linear function. Hence its inverse \( l^{-1} \) is defined. It is required to evaluate the function \( l^{-1} \) at the \( l-1 \) points \( l(l(b)/l, l = 1, K, l-1). \)

Then we first determine a piecewise constant approximation \( h \) to the function \( D^k g \). It makes no difference whether we construct the piecewise constant function \( h \) or the continuous piecewise linear function:

\[
H(x) = \int_a^x (h(s)) \, ds
\]

It is possible to determine the function \( H(x) \) as an element belonging to \( P_{2,1} \cap C \) by considering

\[
\var_{\{a(x)D^{k-1}f_{i+1} = V \var_{\{a(x)D^{k-1}\} D^k g} \}} ds
\]

We choose \( h \in P_{2,1} \) such that

\[
h(x) = \begin{cases} 
2 \frac{\Delta f_{i+1/2}}{\xi_{i+1} - \xi_{i+1}} & \text{on } [\xi_{i+1} - \xi_{i+1}], \\
\frac{\Delta f_{i+1/2}}{\xi_{i} - \xi_{i}} & \text{on } [\xi_{i} - \xi_{i}], \quad i = 1, K, l-1, \\
\frac{\Delta f_{i+1/2}}{\xi_{i+1} - \xi_{i+1}} & \text{on } [\xi_{i+1} - \xi_{i+1}]
\end{cases}
\]

where we have used the abbreviation \( f_{i+1/2} = D_{c_i} f_{i} \) on \([\xi_{i+1} - \xi_{i+1}], \) all it

If we sum up the process;

i) Choose the breakpoints \( \xi_i \) 's and an initial solution \( f_0 \).

ii) Obtain \( f_i \) by using Newton’s method.

iii) For better approximation, obtain \( h \in P_{2,1} \) with the use of \( f_i \).

iv) Determine the number \( l \) and the breakpoints \( \xi_{i,1} = l^{-1}(l(b)/l), l = 1, k, l-1.

v) Replace \( f_0 \) by \( f_i \) and repeat the process.

The method discussed above have been applied to the following problems and the results obtained are given below.

**Numerical results**

In this section the method discussed above were tested on two problems. Example.

\[
0.005y'' + y'' = 1, \quad 0 \leq x \leq 1
\]

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with the following boundary condition:
\[ y'(0) = y(1) = 0 \]

If we linearize the problem about the point \( y = y_0 \) by Newton's method we obtain
\[
\begin{align*}
0.005 y'' + 2y_0 y = 1 + y_0 \\
y'(0) = y(1) = 0
\end{align*}
\]

Let \( y_0 = x^2 - 1 \) be initial solution.
Let \( I = P_{0,2} \cap C^1 \). We subdivide the interval \([1, 0]\) into five subintervals and select the following points initially:
\[ 0.00 \ 0.25 \ 0.5 \ 0.75 \ 1. \]

In each iteration we have used the most recent approximation to the solution as the current guess \( f \), together with a different knot sequence, which is obtained via \( f \).

The knot sequences obtained are shown as below.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
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<tbody>
<tr>
<td>0.00</td>
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<td>0.63868</td>
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<td>1.00</td>
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</tr>
</tbody>
</table>

The best results are obtained using the last knot sequence. The solution changes rapidly in the interval \([0.75, 1]\). Therefore numerical results obtained are given for this interval.

<table>
<thead>
<tr>
<th>Point</th>
<th>Exact value</th>
<th>Approx. value</th>
</tr>
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<tbody>
<tr>
<td>0.760</td>
<td>-0.9690401</td>
<td>-0.9900425</td>
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<td>0.775</td>
<td>-0.9985633</td>
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<td>0.790</td>
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<td>-0.9670580</td>
<td>-0.9670580</td>
</tr>
<tr>
<td>0.835</td>
<td>-0.9556199</td>
<td>-0.9556212</td>
</tr>
<tr>
<td>0.850</td>
<td>-0.9402484</td>
<td>-0.9402495</td>
</tr>
<tr>
<td>0.865</td>
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<tr>
<td>0.880</td>
<td>-0.892056</td>
<td>-0.8920472</td>
</tr>
<tr>
<td>0.910</td>
<td>-0.8061453</td>
<td>-0.8060719</td>
</tr>
<tr>
<td>0.925</td>
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<td>-0.7412800</td>
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<tr>
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<tr>
<td>0.955</td>
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</tr>
<tr>
<td>1.000</td>
<td>-0.0000000</td>
<td>-0.0000000</td>
</tr>
</tbody>
</table>

Example: \( y'' = e^y \), \( 0 \leq x \leq 1 \)

with the following conditions
\[ y(0) = y(1) = 0 \]

If we linearize the problem about the point \( y = y_0 \) by Newton's method we obtain
\[
\begin{align*}
y'' - e^y y = (1 - y_0) e^y \\
y'(0) = y(1) = 0
\end{align*}
\]
Let $y_0 = x^2 - x$ be initial solution.

Let $f \in P_{1,2} \cap C^1$. The interval $[1, 0]$ is divided into five subintervals and we choose the following points initially:

$$0.00 \quad 0.20 \quad 0.40 \quad 0.60 \quad 0.80 \quad 1.00$$

The knot sequences obtained are as follows

$$
\begin{array}{cccccc}
0.00 & 0.19640 & 0.39710 & 0.60290 & 0.80360 & 1.00 \\
0.00 & 0.19727 & 0.39787 & 0.60213 & 0.80273 & 1.00 \\
0.00 & 0.19712 & 0.39753 & 0.60200 & 0.80271 & 1.00 \\
0.00 & 0.19707 & 0.39741 & 0.60205 & 0.80279 & 1.00 \\
0.00 & 0.19709 & 0.39747 & 0.60207 & 0.80275 & 1.00 \\
\end{array}
$$

The following results are found for the last knot sequence given above.

<table>
<thead>
<tr>
<th>Point</th>
<th>Exact value</th>
<th>Approx. value</th>
</tr>
</thead>
<tbody>
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<td>0.80</td>
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</tr>
<tr>
<td>0.96</td>
<td>-0.0177504</td>
<td>-0.017751</td>
</tr>
</tbody>
</table>

Better results can be obtained by increasing the order of polynomials and accuracy in the iterative process. But the most effective method beyond these is the repositioning of the breakpoints. As a different approximation to the solution we change the place of the knots in each iteration and we observed that accuracy is increased and the number of iteration is reduced.

REFERENCES


