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Stability Testing of 2-D Digital Filters Including Those Having Non-essential Singularities of the Second Kind

E. Morris Abraham Gnanamuthu and [1]P.S. Reddy
Faculty of Engineering, Multimedia University, Cyberjaya, Kuala Lumpur, Malaysia
[1]Department of Electronics and Communication Engineering,
SRM Engineering College, Chennai, Tamil Nadu, India

Abstract: This study aims a method of testing the Bounded Input Bounded Output (BIBO) stability of recursive 2-Dimensional (2-D) digital filters using the method for computation of variance or 2-D complex integrals. In the case when the non-essential singularities of the second kind (NSSK) are present, a method is given to determine whether the filter is l_1 stable. The method seems to be very simple for lower order filters and even for higher order filters.

Key words: Circuits and systems, signal processing, stability testing, digital filters

INTRODUCTION

Computations of variance is required in the statistical analysis of quantization effects in recursive digital filters. In the case of 1-D filters, a method using Laurent series[1] expansion is presented to evaluate the Complex integral or variance. In this method, one has to first take it for granted that the 1-D transfer function H(z) is BIBO stable and then proceed with the decomposition of $H(z)H(z^{-1})$ of a particular type and finally arrive at a value for $\sum_{n=0}^{\infty} h_n^2$,

where $h_n = Z^{-1}\, H(z)$ is the impulse response of the 1-D filter.

In the case of 2-D digital filter transfer function $H(z_1, z_2)$, though the same technique of decomposition of $H(z_1, z_2)\, H(z_1^{-1}, z_2^{-1})$ is not in general applicable, a modified method is presented by Hwang[2] where 2-D complex integral

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h_{mn}^2 = \frac{1}{(2\pi j)^2}\oint_{|z_1|=1}\oint_{|z_2|=1}$$

$$H(z_1,z_2)H(z_1^{-1},z_2^{-1})\frac{dz_1}{z_1}\frac{dz_2}{z_2}$$

is evaluated, by evaluating one integral at a time. This method always works and gives positive value for

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h_{mn}^2$$

where $h_{mn} = Z^{-1}\, H(z_1, z_2)$, if $H(z_1, z_2)$ is BIBO stable. But we find that this method does not give a positive value for the integral when the given $H(z_1, z_2)$ is not BIBO stable. We bank on this and give a method for testing the BIBO stability of 2-D recursive filters.

We give some concepts on stability and present a theorem on stability for ready reference. In section III, we present briefly the method of Hwang[2] to calculate the variance in the case of 2-D recursive digital filters and we discuss the way by which one can judge whether the given transfer function is BIBO stable or not depending on the nature of the value the 2-D Complex integral yields when we apply the method of Hwang[2]. In section V we make use of recently reported results on a simple method of testing a 2-D filters for BIBO stability and apply it to some already available filters having the NSSK and come to the same conclusion regarding their l_1-stability and their inverses.

Stability aspects: We presented two concepts of stability and discuss them as they apply to 2-D recursive filters. We assume that the 2-D Z-transform is defined with negative powers of the variable z.

Theorem 1: The 2-D transfer function $H(z_1, z_2)$ is given by

$$H(z_1,z_2) = \frac{A(z_1,z_2)}{B(z_1,z_2)} \quad \text{is BIBO stable iff}$$

$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |h_{mn}| < \infty$, where h_{mn} is the inverse Z-transform of $H(z_1, z_2)$.

Corresponding Author: Ezra Morris Abraham Gnanamuthu, Lecturer, Faculty of Engineering, Multimedia University, Jalan Multimedia, 63100 Cyberjaya, Selangor, Malaysia Tel: +603-83183029, Fax: +603-8318029
E-mail: ezra.morris@mum.edu.my

Theorem 1 is well known and can be found at many places in the literature[3]. The BIBO stability is also known as l_1-stability. A 2-D recursive filter transfer function H (z_1, z_2) is l_2-stable if

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h_{mn}^2 < \infty$$

A 2-D recursive filter may be l_2-stable but l_1-unstable and not the other way. For example a 2-D transfer function is said to have non-essential singularities of the second kind[4] when both A (z_1, z_2) and B (z_1, z_2) are zero for some z_1 and z_2, $|z_1|=1$, $|z_2|=1$. In this case the filter may be l_2 stable but not l_1 stable. Whenever a filter having non-essential singularities of the second kind is l_1 stable, it is also l_2 stable.

Condition for stability: In the Hwang's method of calculating variance from the 2-D transfer function H (z_1, z_2), one has to decompose the function H (z_1, z_2) H (z_1^{-1}, z_2^{-1}) as

$$\begin{aligned} &H(z_1,z_2)H(z_1^{-1},z_2^{-1}) \\ &= \frac{p_1(z_1)z_2^{N-1}+\cdots+p_N(z_1)}{b_0(z_1)z_2^{N}+\cdots+b_N(z_1)} \\ &+ \frac{q_0(z_1^{-1})z_2^{-N}+\cdots+q_N(z_1^{-1})}{b_0(z_1^{-1})z_2^{-N}+\cdots+b_N(z_1^{-1})} \qquad (1) \\ &= \frac{P(z_1,z_2)}{B(z_1,z_2)} + \frac{Q(z_1^{-1},z_1^{-1})}{B(z_1^{-1},z_1^{-1})} \end{aligned}$$

by equating the like coefficients of the equation (1) on either side by expressing the numerator A (z_1, z_2) and denominator B (z_1, z_2) as polynomials in variable z_2, the coefficients being the polynomials in z_1. We get the matrix equation of the type Qq=a with q_i and p_i being the entries of the column matrix q. We have to then solve the matrix equation and get q_0 (z_1^{-1}). Then $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h_{mn}^2$ is obtained as equal to the 1-D integral

$$\frac{1}{2\pi j}\oint_{|z_1|=1} \frac{q_0(z_1^{-1})dz_1}{b_0(z_1^{-1})z_1}$$

When we talk about the variance calculation using the method suggested by Hwang[2], there is a possibility that we may end up with a negative value for variance. This happens when we evaluate the 2-D Complex integral without testing the 2-D transfer function for BIBO stability. So we can use the method used to evaluate the variance to test for BIBO stability. Let us now workout some examples.

Example 1:
Consider $$H(z_1,z_2) = \frac{z_1 z_2}{(z_1-0.7)z_2+(0.3-0.5z_1)}$$

We have found that H (z_1, z_2) clearly has no non-essential singularities of the second kind. Also we found by some other method that the filter is BIBO stable (l_1 stable). This is the same example which Hwang[2] has given and it has a positive value for variance.

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h_{mn}^2 = 2.981424002$$

Example 2:
Let $$H(z_1,z_2) = \frac{z_1 z_2}{0.5z_1 z_2+0.2z_2+0.5z_1+1}$$

This filter is found to be BIBO unstable. Let us now check what value we get for variance by using the method suggested by Hwang[2].

$$H(z_1,z_2) = \frac{z_1 z_2}{(0.5z_1+0.2)z_2+(0.5z_1+1)}$$

The matrix equation we get is[2]

$$\begin{bmatrix} (0.5z_1+0.2) & (0.5z_1+1) & (0.5z_1^{-1}+1) \\ 0 & (0.5z_1+0.2) & 0 \\ (0.5z_1+1) & 0 & (0.5z_1^{-1}+0.2) \end{bmatrix} \begin{bmatrix} q_0(z_1^{-1}) \\ q_1(z_1^{-1}) \\ p_1(z_1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solving the above matrix equation for q_0 (z_1^{-1}) we have,

$$\frac{q_0(z_1^{-1})}{b_0(z_1^{-1})z_1} = \frac{-2.5}{(z_1+1.8633)(z_1+0.5367)}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h_{mn}^2 = \frac{1}{2\pi j}\oint_{|z_1|=1}\left(\frac{-2.5}{(z_1+1.8633)(z_1+0.5367)}\right)$$

$$= \frac{-2.5}{1.3266}$$

Since $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h_{mn}^2$ is negative, We can say that the given filter is unstable since variance cannot be negative.

Example 3: Consider the transfer function[5]

$$H(z_1,z_2) = \frac{(1-z_1)(1-z_2)}{2z_1z_2-z_2-z_1}$$

The transfer function has non-essential singularities of the second kind since the numerator and denominator are equal to zero at $z_1=1$ and $z_2=1$. The matrix equation[2] corresponding to this $H(z_1, z_2)$ is

$$\begin{bmatrix} 2z_1-1 & -z_1 & -z_1^{-1} \\ 0 & 2z_1-1 & 0 \\ -z_1 & 0 & 2z_1^{-1}-1 \end{bmatrix} \begin{bmatrix} q_0(z_1^{-1}) \\ q_1(z_1^{-1}) \\ p_1(z_1) \end{bmatrix} = \begin{bmatrix} (-1+z_1)(-1+z_1^{-1})+(1-z_1)(1-z_1^{-1}) \\ (-1+z_1)(1-z_1^{-1}) \\ (1-z_1)(-1+z_1^{-1}) \end{bmatrix} \quad (2)$$

we get after solving the above matrix equation

$$\frac{q_0(z_1^{-1})1}{b_0(z_1^{-1})z_1} = \frac{3z_1^2-6z_1+3}{(-4Z_1+2)(2-z_1)z_1} \quad (3)$$

and the corresponding integral[2] value

$$= \frac{-0.75}{3} + 0.75 = 0.5$$

So in this case the filter is l_2 stable.

As mentioned earlier a 2-D filter of this type may be l_2 stable but need not be l_1 stable. If we use the same variance method, the determinant of the co-efficient matrix in equation (2) is given by $\Delta = \frac{-4z_1^3+10z_1^2-8z_1+2}{z_1}$

The Δ is equal to zero for $z_1=0$ since it has a factor $(z_1-1)^2$ in the numerator. So when we try to decompose the function $H(z_1,z_2)H(z_1^{-1},z_2^{-1})$ as in[2], we get values for $q_0(z_1^{-1})$, $q_1(z_1^{-1})$ and $p_1(z_1)$ which may be infinity for $z_1=1$. But we find that all $q_0(z_1^{-1})$, $q_1(z_1^{-1})$ and $p_1(z_1)$ have factors $(z_1-1)^2$ in the numerators cancelling with the factor $(z_1-1)^2$ in the numerator of Δ. So $H(z_1-z_2)$ may be BIBO stable.

Example 4: Consider the following example[5]

$$H(z_1,z_2) = \frac{(1-z_2)^2(1-z_1)}{z_2(2z_1z_2-z_1-z_2)}$$

This function $H(z_1-z_2)$ also has non-essential singularities of the second kind at $z_1=1$ and $z_2=1$. The matrix equation[2] is

$$\begin{bmatrix} 1-2z_1 & 0 & z_1 & 0 & z_1^{-1} \\ 0 & 1-2z_1 & 0 & z_1^{-1} & 0 \\ 0 & 0 & 1-2z_1 & 0 & 0 \\ 0 & z_1 & 0 & 1-2z_1^{-1} & 0 \\ z_1 & 0 & 0 & 0 & 1-2z_1^{-1} \end{bmatrix} \quad (4)$$

$$\begin{bmatrix} q_0(z_1^{-1}) \\ q_1(z_1^{-1}) \\ q_2(z_1^{-1}) \\ p_1(z_1) \\ p_2(z_1) \end{bmatrix} = \begin{bmatrix} 12-6z_1-6z_1^{-1} \\ 0 \\ 2z_1+2z_1^{-1}-4 \\ 0 \\ 0 \end{bmatrix}$$

The determinant of the coefficient matrix in (4) is

$$\Delta = -8Z_1^3 + 36z_1^2 - 62z_1 + 4z_1^{-1} - 24z_1^{-1} + 55$$
$$= \frac{-8z_1^5+36z_1^4-62z_1^3+55z_1^2-24z_1^{-1}+4}{z_1^2}$$

This Δ does not have zeros on the unit circle at $z_1=1$. So we have a finite solution for equation (4) when $z_1=1$ and $z_2=1$. In this case the filter will be l_1 stable as well as l_2 stable as proved in[5].

BIBO stability of filters having the NSSK: We deal exclusively with the 2-D filter transfer functions having the NSSK.

We continue to assume that the 2-D transform is defined with negative powers of the complex variable z. We know the following theorem very well regarding the BIBO stability of 1-D recursive filters, given the denominator polynomial B(z) of a transfer function.

Theorem 1: A 1-D digital filter is BIBO stable iff
$B(z) \neq 0, |z| \geq 1$

In a recent study[6] we have proved that to test the stability of a 2-D filter transfer function whose denominator is $B(z_1,z_2)$ we need to test for stability only the stability of 1-D polynomials $B(z_1,z^{2N+1}, z)$ and/or $B(z,z^{2N+1})$ where N is the degree of both the variables in $B(z_1,z_2)$. We make use of this fact and test some well known transfer functions which have non-essential singularities of the second kind for BIBO stability (l_1- stability).

Example 5: Let

$$H(z_1,z_2) = \frac{(1-z_1)(1-z_2)}{2z_1z_2-z_2-z_1)}$$

be required to be tested for BIBO stability. We note that this is the same $H(z_1,z_2)$ as in example 3. We have

$$H(z_1,z_2)=\frac{(z_1z_2-z_1-z_2+1)}{2z_1z_2-z_1-z_2}$$

The filter is of order N=1.

$$H(z_1,z_2)\Big|_{\substack{z_2=z\\ z_1=z^3}}=\frac{z^4-z^3-z+1}{2z^4-z^3-z}$$
$$=\frac{(z-1)(z^3-1)}{z(z-1)(2z^2+z+1)}=\frac{z^3-1}{2(2z^2+z+1)} \tag{5}$$

Now look at the denominator polynomial factor $(2z^2+z+1)$. It has two complex conjugate poles at $\frac{+1\pm j\sqrt{7}}{4}$ having magnitudes of $\frac{1}{\sqrt{2}}$. Since the poles lie outside the unit circle, the filter is BIBO unstable. With this we can say that the 2-D transfer function $H(z_1,z_2)$ and hence the 2-D filter is BIBO unstable. Unfortunately in this method we are not able to comment anything on the l_2 stability of the filter. But on seeing (5) we can comment that the inverse transfer function

$$\frac{1}{H(z_1,z_2)}=\frac{2z_1z_2-z_2-z_1}{(1-z_1)(1-z_2)}$$

is also l_1 unstable since in (5), three zeros are on $|z|=1$, which is in conformity with the conjecture[7].

Example 6: Consider the following transfer function[8].

$$H(z_1,z_2)=\frac{2z_1z_2-z_1-z_2}{5z_1z_2-2z_1-3z_2}$$

we have,

$$H(z_1,z_2)\Big|_{\substack{z_2=z\\ z_1=z^3}}=\frac{2z^4-z^3-z}{5z^4-2z^3-3z}$$
$$=\frac{z(z-1)(2z^2+z+1)}{z(z-1)(5z^2+3z+3)}=\frac{2z^2+z+1}{5z^2+3z+3} \tag{6}$$

It is easy to see that the 1-D function in (6) has all its zeros and poles inside the unit circle. Thus the function is BIBO stable. Further on seeing the numerator of (6) we can say that the inverse of $H(z_1,z_2)$ is also stable. So the corresponding 2-D transfer function $H(z_1,z_2)$ and its inverse are BIBO stable as proved by Swamy and Roytman[8].

In this study, the method to find variance for 2-D recursive digital filters is applied to test whether the given 2-D transfer function $H(z_1,z_2)$ is BIBO stable or not. We also applied the same technique to 2-D transfer functions having non-essential singularities of the second kind and could conclude successfully whether these transfer functions are l_1 stable (BIBO stable) or l_2 stable or both. In summary if the recursive 2-D transfer function has no non-essential singularities of the second kind (NSSK), it will be l_1 unstable if the value of $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h_{mn}^2$ is negative. Unfortunately we have to restrict ourselves to lower order transfer functions since the complexity involved in evaluating the determinant of the matrix to obtain $q_0\,(z_1^{-1})$ and the consequent 1-D integral evaluation is too much in Hwang's method. We are trying to propose altogether a different simple approach to evaluate the variance in the case of 2-D recursive filters.

We also used the recently proved results[6] on the stability testing of 2-D recursive filters and shown with two examples, what was conjectured in[9] and proved in[8,10] are also true. This method is also very simple and can comment on the l_1 stability of the 2-D transfer function and its inverse when it has the NSSK.

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