On Multi-valued Semantics for Logic Programs

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Abstract: For a general program P, multi-valued interpretations and models are defined, considering a set of truth logic values and an undefined value. The program P may contain constant propositions, which are defined for each truth logic value. Two orderings between the set of all multi-valued interpretations are considered: one is Fitting ordering and the other is standard ordering. The semantics of type well-founded and of type stable for a program P are introduced. This study showed that the well-founded model is the least stable model with respect to Fitting ordering.

Key words: Fixed points, stable models, well-founded models

INTRODUCTION

The well-founded semantics has been introduced by Van Gelder et al.[1]. It is a 3-valued semantics. They use as truth values "true", "false" and "⊥" (an unknown truth value). They have shown that if a logic program P has a 2-valued well-founded model, then this model is the unique stable model of P.

The stable model semantics has been introduced by Gelfond and Lifschitz[2] and by Eberhard and Frodevaux[3]. Przymusinski[4] has introduced 3-valued stable models as a generalization of 2-valued stable models. He also found that the well-founded model of any program P coincides with the smallest 3-valued stable model of P.

Lucy[5] has defined a new semantics for Datalog programs, which includes the well-founded models and all stable models.

Fitting[6] has studied the structure of the family of all stable models for a logic program using two orderings; one is called the knowledge ordering based on degree of definedness, the other is called truth ordering based on degree of truth. In the first ordering every logic program has a smallest stable model, which coincides with the well-founded model.

Przymusinski[7] has introduced the well-founded model semantics for disjunctive logic programs and deductive databases. For normal programs, the partial disjunctive stable semantics coincides with the well-founded semantics.

Loyd and Umberto[8] proposed a well-founded semantics for deductive databases with uncertainty frameworks.

MalFon[9] gives a new characterization of Fitting model and of the well-founded model.

Lalloue[10] has defined a semantics for normal logic programs based on the property of composition. This semantics extends well-founded semantics and Fitting semantics.

This study defines a semantics of type well-founded and a stable semantics for the case multi-valued interpretations and points out a relationship between them.

Interpretations and models: Let P be a general logic program in sense Gelder[11]. Let H be the Herbrand base associated to P. We consider a total ordered set of truth values Lw = (0, v1,...,vn,1), where, value 0 corresponds to false, value 1 is for true and the values v1,...,vn-1 are intermediate between false and true. For every truth value v from Lw, we used a constant proposition denoted by cv and defined by cv(A) = v for every ground atom A from H. The undefined value will be denoted by u and corresponding constant by cu where, cu(A) = u for every A ∈ H. Let us denote 0 by v0 and 1 by v1. The constant propositions cv as well cu may appear in the bodies of rules from P.

Definition 1: By a multi-valued Herbrand partial interpretation I we mean a partial function from H into Lw. For an interpretation I, let us denote by V(I) the vector of sets from H:

\[ V(I) = (S_0, S_1, ..., S_n) \]

where, \( S_i = \{A/A \in H \text{ and } I(A) = v_i\}, 0 \leq i \leq n \).

We denote by S the set of all remaining atoms from H - \( \bigcup_{0 \leq j \leq n} S_j \). If V = (S0, S1, ..., Sn) where, Si are disjoint sets of H, 0 ≤ i ≤ n, then there is an interpretation I, such that V(I) = V. In the case S is empty, then I is called total interpretation.

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Assume that $L_n$ admits a negation, denoted $\overline{\cdot}$, which satisfies the following properties: $\overline{\overline{\cdot}} = \cdot$, $\overline{0} = 1$, $\overline{1} = 0$ and $\overline{v_i < v_j}$ implies $k_i < k_j$, for every $i,j, 0 \leq i,j \leq n$. Moreover, we consider $\overline{u} = u$.

For an atom $A$, such that $A \in S_n$, we write $I(A) = u$ and $I(\overline{A}) = u$, otherwise.

For a ground instantiated rule $r$ of $P$, having the form: $r = A \rightarrow L_1, \ldots, L_m$, let us denote by $\overline{I}$ (body $(r)$) = $\min \{ I(L_i), \overline{I(L_i)} \neq u, 1 \leq i \leq m \}$. We consider $\min \varnothing = 1$, where $\varnothing$ is empty set.

Let $M_n$ be the set of all ground instantiated rule of $P$, having $A$ as its head. Let $v_{i,k}$ be the truth value from $L_n$ defined by: $v_{i,k} = \max \{ \overline{I} (\text{body}(r))/r \in M_n \}$.

The interpretation $I$ is extended to ground literals denoted $\overline{I}$ by: $\overline{I(A)} = I(A)$ and $\overline{I(\overline{A})} = I(A)$ for every ground atom $A \in H$. In the following we define the notion of model for $P$.

**Definition 2:** An interpretation $I$ satisfies the ground instantiated rule of $P$ having the form $A \rightarrow L_1, \ldots, L_m$, if one of the following relations holds:

a. there is $j, 1 \leq j \leq m$, such that $\overline{I(L_j)} = 0$ or
b. $I(A) \neq u$ and $\min \{ I(L_i), \overline{I(L_i)} \neq u, 1 \leq i \leq m \} \leq I(A)$ or
c. $I(A) = u \rightarrow (I(\text{body}(r)) \setminus v_{i,k} \rightarrow (\exists i, 1 \leq i \leq m, \text{such that } \overline{I(L_i)} = u))$.

An interpretation $I$ is a model for $P$ if $I$ satisfies every ground instantiated rule of $P$.

In the following we need to specify two ordering between interpretations. The first one denoted $\leq_s$ is of type Fitting and the second one denoted $\leq_{S_n}$ is of type standard.

**Definition 3:** Let $I$ and $J$ be two interpretations, such that $V_I = (S_0, \ldots, S_n)$ and $V_J = (T_0, \ldots, T_n)$. We say that $I \leq_s J$ if $S_j \subseteq T_j$ for every $j, 0 \leq j \leq n$.

We say that $I \leq_{S_n}$ if $T_0 \subseteq S_n$, $S_n \subseteq T_0$ and $S_n \subseteq T_0 \cup \ldots \cup T_{n+1}$ for every $j, 1 \leq j \leq n-1$.

**Remark 1:** In the case $n = 1$, the ordering $\leq_s$ is the Fitting ordering and $\leq_{S_n}$ is the standard ordering. These orderings were used by Przymusinski to study the well-founded semantics and three-valued stable models.

**Stable semantics:** Study defines here multi-valued stable models. Firstly, define an operator between the set of all interpretations of the program $P$. This operator will be denoted by $S_n$.

**Definition 4:** Let $P$ be a logic program and $I$ be an interpretation of $P$. We define the interpretation $S_n(I)$ in the following manner: if $v_{n+1} = (T_0, T_1, \ldots, T_n)$ then:

i. For a ground atom $A$, $A \in T_0$ if for every ground instantiated rule of $P$, having the form $A \rightarrow L_1, \ldots, L_m$, there exists $i, 1 \leq i \leq m$, such that $\overline{I(L_i)} = 0$.

ii. For every $h, 1 \leq h \leq n$, a ground atom $A$ is considered in $T_h$ if a) and b) hold:

a. for every ground instantiated rule of $P$ having the form: $A \rightarrow V_1, \ldots, V_m$, we have: $\min \{ I(V_i), \overline{I(V_i)} \neq u, 1 \leq i \leq m \} = v_h$.

b. there is a ground instantiated rule of $P$ of the form: $A \rightarrow V_1, \ldots, V_m$, such that: $\min \{ I(V_i), \overline{I(V_i)} \neq u, 1 \leq i \leq m \} = v_h$.

iii. For a ground atom $A$, $A$ is considered in $T_n$ if there is a ground instantiated rule of $P$ having the form $A \rightarrow C_1, \ldots, C_n$, such that $\overline{I(C_i)} = 1$ for every $j, 1 \leq j \leq q$.

**Proposition 2:** Let $P$ be a positive program. The operator $S_n$ as it is defined in the definition 4 is monotonic with respect to the standard ordering $\leq_{S_n}$.

The proof results from the definition of the operator $S_n$ and the standard ordering $\leq_{S_n}$.

The existence of the least model with respect to $\leq_{S_n}$ for a positive program $P$ is emphasized by the following theorem.

**Theorem 1:** For a positive program $P$, there is the least fixed point of the operator $S_n$ with respect to the ordering $\leq_{S_n}$, denoted $L_n$. Moreover, $L_n$ is the least model of $P$ with respect to the ordering $\leq_{S_n}$.

**Proof:** Consider $\perp = (H, \varnothing, \ldots, \varnothing)$ the least interpretation with respect the ordering $\leq_{S_n}$. The model $L_n$ is obtained applying the operator $S_n$ $\omega$ times: $\perp, S_n(\perp), \ldots, S_n^{\omega}(\perp)$, where $\omega$ is the first ordinal.

The rest of the proof is classical, therefore it is skipped.

Now, we need to introduce an operator $I^\ast$ defined on the set of all interpretations, which extends the operator $I$ defined by Przymusinski.

**Definition 5:** Let $P$ be a general logic program and $I$ an interpretation. We denote by $P_I$ the positive program, which is obtained from $P$ by replacing in every ground instantiated clause of $P$, all negative literals of the form $\neg A$ by $C$, if $I(A) \neq u$ and by $u$ otherwise, where $v = I(A)$. The program $P_I$ is positive, hence applying the Theorem 1, it results that $P_I$ admits a unique least model $J$ with respect ordering $\leq_{S_n}$. The operator $I^\ast$ is defined by: $I^\ast(I)$ = $J$.

**Proposition 3:** Let $M$ be a fixed point of the operator $I^\ast$ from the definition 5. Then $M$ is a minimal model of $P$ with respect to ordering $\leq_{S_n}$. 

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Proof: Let M be a fixed point of $\Gamma^*$, hence M is the least model of $P(M)$ with respect to $\leq_{odr}$. Firstly, we show that M is a model for P. Let r be an arbitrary ground instantiated clause from $P$ of the form:

$$r = A - B_1, \ldots, B_p - D_1, \ldots, -D_q$$

(1)

The corresponding clause $r'$ from $P(M)$ has the form:

$$r' = A - B_1, \ldots, B_p, c_1, \ldots, c_q$$

(2)

where, $v_i = M(D_i)$ if $M(D_i) \neq u$ and $u$ otherwise.

It results that $M(D_j) = M(D_j) = c_j$, therefore M satisfies $r'$ iff M satisfies r.

Secondly, it must show that M is a minimal model for $P$ with respect to $\leq_{odr}$. Let $M_i$ be a model for $P$, such that $M_i \leq M$. It is sufficient to show that $M_i$ is also a model for $P(M)$ since M is the least model of $P(M)$ with respect to $\leq_{odr}$ ordering, we obtain that $M_i \leq M_i$, hence $M_i \leq M$.

For the ground instantiated clause $r$ having the form (1), let us denote by $r'$ the corresponding clause to $r$ from $P(M)$:

$$r' = A - B_1, \ldots, B_p, c_1, \ldots, c_q$$

(3)

where, $w_j = M_i(D_j)$ for every $j, 1 \leq j \leq q$.

As before, since $M_i$ is a model for $P$, it obtains that $M_i$ is a model for $P(M)$, hence M satisfies $r'$.

Since $M_i \leq M$ and using the definition of $v_i$ and $w_i$, the following statements are satisfied:

i. if $w_i = 0$ then $v_i = 0$, for every $j, 1 \leq j \leq q$.

ii. if $v_i = 1$ then $w_i = 1$, $1 \leq j \leq q$.

iii. when $w_i = 1$ then we have: $v_i \leq w_i$, whenever $v_i \neq u$, $1 \leq j \leq q$.

iv. if $0 < w_i < 1$, then we have: $v_i \neq u$ and $v_i \leq w_i$, $1 \leq j \leq q$.

These statements imply the inequality:

$$\min \{M_i(B_i), M_i(B_j) \neq u, 1 \leq i \leq p, c_1, \ldots, c_q, v_i \neq u, 1 \leq j \leq q\} \leq \min \{M_i(B_i), M_i(B_j) \neq u, 1 \leq i \leq p, c_1, \ldots, c_q, w_i \neq u, 1 \leq j \leq q\}. \quad (4)$$

The relation (4) and the fact that $M_i$ is a model for $r'$ involve that $M_i$ is a model for $P(M)$ for fixed point of $\Gamma^*$.

Definition 6: A multi-valued interpretation M for a program P is called a multi-valued stable model for P if M is a fixed point of $\Gamma^*$.

Well-founded models: For definition of well-founded models we need to introduce an operator, denoted W, defined on the set of all multi-valued interpretations.

For an interpretation $I$, if $J = W(I)$ and $V_J = (S_0, S_1, \ldots, S_n)$, we define the sets $S_j, 0 \leq j \leq n$.

Definition 7: Let I be an interpretation. We define the sets $S_j, 0 \leq j \leq n$ in the following manner:

a. for every $j, 1 \leq j \leq n$, a ground atom $A$ is included in $S_j$ iff

a1. for every ground instantiated rule $r$ of $P$ of the form:

$$r = A - L_1, \ldots, L_m,$$

we have: $\min \{I(L_i), I(L_j) \neq u, 1 \leq j \leq m\} \leq v_i$ and

a2. there exists a ground instantiated rule $r_i$ of $P$ with the form:

$$r_i = A - Q_1, \ldots, Q_n,$$

such that: $I(Q_i) \neq u$, for every $i, 1 \leq i \leq n$ and $\min \{I(Q_i), I(L_i) \} = v_i$.

b. A set of atoms V from H is called an unfounded set of P with respect to I if every atom A from V satisfies the following property:

for each ground instantiated rule $r$ of $P$, having the form:

$$r = A - L_1, \ldots, L_m,$$

one of the following statements holds:

b1. there is $i, 1 \leq i \leq m$, such that $I(L_i) = 0$ or

b2. there is $i, 1 \leq i \leq n$, such that $L_i$ is an atom and $L_i \in V$.

We consider $S_j$ as the union of all unfounded sets of $P$ with respect to $I$.

Remark 2: If $V_i$ and $V_j$ are unfounded sets of $P$ with respect to $I$, then their union $V_i \cup V_j$ is also an unfounded set with respect to $I$.

Proposition 4: The operator W is monotonic with respect to Fitting ordering $\leq_F$. Proof. Let $I$ and $J$ be two interpretations, such that $I \leq_F J$. Let $V_i = (S_0, S_1, \ldots, S_n)$ and $V_j = (T_0, T_1, \ldots, T_n)$.

We have $S_j \subseteq T_j$ for every $j, 0 \leq j \leq n$. That means: if $I(L_i) \neq u$ then $I(L_i) \neq u$ and $I(L_j) = I(L_i)$, for every literal $L_i$.

If $V_{\leq F} = (S_0', S_1', \ldots, S_n')$ and $V_{\leq F} = (T_0', T_1', \ldots, T_n')$, then it obtains that $S_j' \subseteq T_j'$ for every $j, 1 \leq j \leq n$. (1)

The relations $S_0 \subseteq T_0$ and $S_n \subseteq T_n$ imply the following statement: every unfounded set of P with respect to $I$ is an unfounded set of P with respect to $J$.

We obtain $S_j \subseteq T_j$. This relation and those from (1) involve $W(I) \leq_F W(J)$.

Now, we define a sequence of interpretations using the operator W defined above.

Definition 8: Let $\alpha$ range over countable ordinals. We define recursively the interpretations $I_\alpha$ and $\Gamma$ as follows:
1. For ordinal $0$, $I_0 = (\varnothing, \ldots, \varnothing)$, where $\varnothing$ is the empty set.

2. For the limit ordinal $\alpha : I_\alpha = \bigcup_{\beta < \alpha} I_\beta$.

3. For successor ordinal $\alpha = \gamma + 1$, $I_\alpha = W(I_\gamma)$.

4. $I^\kappa = \bigcup_{\alpha} I_\alpha$.

**Remark 3**

i. The interpretation $\Gamma$ is the least fixed point of $W$ with respect to the Fitting ordering $\varepsilon_r$.

ii. There exists a countable ordinal $\alpha$, such that $\Gamma = I_\alpha$.

Let us denote the interpretation $\Gamma$ by $I_\psi$.

**Theorem 2:** The sequence of interpretations $I_\alpha$ as defined in the Definition 8 is a monotonic sequence of interpretations with respect to $\varepsilon_r$-ordering and moreover it is a sequence of models for $P$.

**Proof:** The monotonicity of the sequence of interpretations results from the Proposition 4.

By the Definition 2, $I_\psi$ is a model for $P$. Since the operator $W$ is monotonic with respect to ordering $\varepsilon_r$, it results by induction on ordinals $\alpha$ the following statement:

for every ground literal $L$ and $\gamma < \alpha$, if $I_\alpha (L) \neq \sigma$ then $I_\gamma (L) \neq \sigma$ and $I_\gamma (L) = I_\alpha (L)$

Assume that $I_\psi$ is a model for $P$. Let us show that $I_{\alpha + 1}$ is also a model for $P$, where $\alpha$ is an arbitrary ordinal. Let $\kappa = A - I_{\alpha + 1}$, $I_{\alpha - 1}$ be a ground instantiated rule of $P$. If $I_{\alpha + 1} (A) = \sigma$, then $I_{\alpha + 1}$ satisfies $\kappa$.

In the case $I_{\alpha + 1} (A) \neq \sigma$, let $V_{I_{\alpha + 1}} = (S_0, S_1, \ldots, S_n)$. There exists $i \leq j \leq n$, such that $A \in S_i$. We have $I_{\alpha + 1} (A) = \sigma$. Using the Definition 7 and the relation (1), we obtain that $I_{\alpha + 1}$ satisfies $\kappa$.

Now, let $A$ be a limit ordinal. Assume that $I_\psi$ for every $\beta < \alpha$ are models for $P$. Let us show that $I_\alpha$ is model.

Let $V_{I_\alpha} = (S_0^{\alpha}, \ldots, S_n^{\alpha})$. We have $V_{I_\alpha} = \left\{ U \in S_0^{\alpha}, \ldots, U \in S_n^{\alpha} \right\}$

Let $r$ be defined as above. If $I_\alpha (A) = \sigma$, then $I_\alpha$ satisfies $r$. In the case $I_\alpha (A) \neq \sigma$, there is $h$, $0 \leq h \leq n$, such that $A \in \bigcup_{\beta \leq \alpha} S_\beta^{\alpha}$. The sequence of sets $S_\beta^{\alpha}$, $\beta < \alpha$ is ascending monotonic with respect to the inclusion. Let $\beta_i$ be the first ordinal such that $A \in S_\beta^{\alpha}$. We have $I_\alpha (A) \neq \sigma$ and $I_\alpha$ is a model for $r$. Since $\beta_i < \alpha$ and using the relation (1), it results that $I_\alpha$ satisfies $r$.

**Stable Semantics versus well-founded semantics:** In this section we point out a relation between the stable semantics and the well-founded semantics, namely the well-founded model of $P$ is the least stable model of $P$ with respect to $\varepsilon_r$-ordering.

**Theorem 3:** Let $P$ be a normal logic program. Then $P$ admits $\varepsilon_\psi$-least stable model. Moreover, this model coincides with the well-founded model of $P$.

**Proof:** Let $I_\psi$ be the well-founded model for $P$ and $\lambda$ be the minimum ordinal such that $I_\lambda = I_{\psi + 1}$ (from the Definition 8).

Firstly, we show that $I_\psi$ is a stable model for $P$. Let $P'$ be $P[I_\psi]$ and $M_\psi$ be an arbitrary model for $P'$, such that $M_\psi \subseteq I_\psi$. It must show that $M_\psi = I_\psi$. Let $V_{M_\psi}$ be the vector $(T_\psi, \ldots, T_n)$ and $V_{I_\psi} = (S_0^{\lambda}, \ldots, S_n^{\lambda})$.

The relation $M_\psi \subseteq I_\psi$ is equivalent with:

i. $S_0^{\lambda} \cap T_\psi$

ii. $T_\psi \subseteq S_0^{\lambda}$

iii. $T_0 \subseteq S_0^{\lambda} \cup \ldots \cup S_{h - 1}^{\lambda}$ for every $h$, $1 \leq h \leq n - 1$.

Assume that $M_\psi \neq I_\psi$. Then, we have one of the following assertions:

a. $S_0^{\lambda} \cap T_\psi$ or

b. $S_0^{\lambda} \cap T_\psi$ or

c. there is $h$, $1 \leq h \leq n - 1$ such that $S_0^{\lambda} \cap T_{h + 1}$.

The sign "$\subset$" denotes the strict inclusion and "$\cap$" means "not included".

In the case a) let us consider $\alpha$ the least ordinal such that $S_\alpha^{\lambda} \cap T_\psi$, where, $V_{I_\psi} = (S_0^{\alpha}, \ldots, S_n^{\alpha})$ and $I_\alpha$ is specified in the Definition 8, for every ordinal $\alpha$. It results that $S_\alpha^{\lambda} \cap T_\psi$ and there exists a ground atom $A$, such that $A \subseteq S_\alpha^{\lambda}$ and $A \subseteq T_\psi$. By the definition of $S_\alpha^{\lambda}$, there is a ground instantiated rule $r$ of $P$, having the form:

$A = B_1 \ldots B_n \ldots D_p$,

where, $B_1 \leq \gamma \leq m$ and $D_1 \leq p$ are ground atoms with the properties:

$\bar{I}_\alpha (E_j) = 1$ for every $j$, $1 \leq j \leq m$ and $\bar{I}_\alpha (D) = 0$ for every $1 \leq l \leq p$.
Let \( r'_j \) be the rule from \( P' \) corresponding to \( r_j \). Then \( r'_j \equiv A \rightarrow B_1,...,B_m \ c_1,...,c_q \), where \( v_j = \bar{I}_j(D_j) \) for every \( j, 1 \leq j \leq p \). Since \( S'_a \subseteq T_0 \) we obtain that \( M_i(B_j) = 1, \ j = 1, m \).
Since \( T_0 \subseteq I_0 \), it results \( I_0(D_j) = 0, \ j = 1, p \), hence \( v_j = 1 \), for every \( j, 1 \leq j \leq p \). We have: \( M_i \) is a model for \( r'_j \). This implies \( M_i(A) = 1 \), hence \( A \in T_0 \) which is impossible. Therefore, we have \( T_0 \subseteq S_a^1 \).

In the case a) let \( \alpha \) be the least ordinal, such that \( S_a^{\alpha+1} \subseteq T_h \cup \ldots \cup T_{n-1} \).

It results: \( S_a^\alpha \subseteq T_h \cup \ldots \cup T_{n-1} \).

Using the relation (1), we obtain: there is \( A, \) such that \( A \in S_a^{\alpha+1} \) and \( A \in T_h \cup \ldots \cup T_{n+1} \).

\( A \in S_a^{\alpha+1} \) implies: for every \( r \in M_0, r \equiv A \rightarrow B_1,...,B_m, I_{r} \), we have \( I_{r}(body(r)) \leq v_h \) and there is \( r_i \in M_0, r_i \equiv A \rightarrow Z_1,...,Z_q \), such that \( I_{r_i}(Z_j) = u \), for every \( j, 1 \leq j \leq p \) and \( \min \{ I_{r_i}(Z_j) \} = v_k \).

Let \( r_i \) from (5) be expressed as follows: \( r_i \equiv A \rightarrow B_1,...,B_m \rightarrow D_1,...,D_q \).

We have: \( I_{r_i}(B_j) = u, \ i = 1, m \) and \( I_{r_i}(D_j) = u, \ i = 1, q \), which implies: \( I_{r_i}(B_j) \leq v_h \) and \( I_{r_i}(D_j) \leq v_k \), \( i = 1, q \).

Let \( r'_j \) be the clause from \( P/M_0 \) corresponding to \( r_j \):

\( r'_j \equiv A \rightarrow B_1,...,B_m \ c_1,...,c_q \), where \( v_j = \bar{I}_j(D_j), \ j = 1, q \).

Since \( I_{r_j} \subseteq I_0 \) we have \( I_{r_j}(D_j) \neq I_{r_j}(D_j), \) for every \( j, 1 \leq j \leq q \), hence \( v_j \leq v_k \) for \( j = 1, q \).

We have \( I_{r_j}(B_j) = 0, \ i = 1, m \) if \( I_{r_j}(B_j) = 1, \) then \( I_{r_j}(B_j) = 1 \) and using \( T_0 = S_a^1, \) it obtains that \( M_i(B_j) = 1, \) if \( I_{r_j}(B_j) = 1 \), then using (2) it results: \( B_j \in T_h \cup \ldots \cup T_{n+1} \), hence \( M_i(B_j) \leq v_h \).
Since \( M_i \) satisfies \( r'_j \), we have \( M_i(A) \leq v_h \). We show that \( M_i(A) = 1 \). Assume the contrary: \( M_i(A) = 1 \). Using \( T_0 = S_a^1 \), we obtain \( A \in S_a^1 \).

From \( A \in S_a^{\alpha+1} \), it results \( A \in S_a^1 \), with \( h < n \).

But \( S_a^1 \cap S_a^\alpha = \emptyset \) for \( h < n \). The relations (8) and (9) constitute a contradiction.

From \( M_i(A) = v_h \) and \( M_i(A) < 1 \) we obtain \( A \in T_h \cup \ldots \cup T_{n+1} \) which contradicts the relation (3).

In conclusion for the case a), we have: \( S_a^{\alpha+1} \subseteq T_h \cup \ldots \cup T_{n+1} \), for every \( h = 1, n = 1 \).

Using (iii), it results \( S_a^\alpha \subseteq T_h, h = 1, n = 1 \).

In the case b), namely \( S_a^\alpha \subseteq T_h \), we show that \( T_\alpha \subseteq S_a^\alpha \), which will be a contradiction.

Let \( A \) be from \( T_\alpha \), hence \( M_i(A) = 0 \). Let \( r \) be a ground instantiated rule from \( P \), having the form: \( r \equiv A \rightarrow B_1,...,B_m \rightarrow D_1,...,D_q \).

The clause corresponding to \( r \) from \( P/M_0 \) is \( r' \):

\( r' \equiv A \rightarrow B_1,...,B_m \ c_1,...,c_q \), where \( v_j = \bar{I}_j(D_j), \ j = 1, p \).

Since \( M_i \) is a model for \( r' \), it follows that there exists \( i, 1 \leq i \leq m \) such that \( M_i(B_i) = 0 \) or there is \( j, 1 \leq j \leq p \), such that \( c_j = 0 \).

For every \( c_j = 0 \) we have \( I_j(D_j) = 0 \).

If \( c_j > 0 \) for all \( j, 1 \leq j \leq p \), then there is \( i, 1 \leq i \leq m \), such that \( M_i(B_i) = 0 \), hence \( B_i \subseteq T_\alpha \).

The assertions (10) and (11) say that \( T_\alpha \) is an unfounded set with respect to \( I_0 \).

If \( V_{\alpha}(T_\alpha) = (T_\alpha, T_\alpha, T_\alpha) \), then \( T_\alpha \subseteq T_\alpha \).

But \( W(I_\alpha) = I_\alpha \), hence we have \( T_\alpha = S_a^\alpha \), which implies \( T_\alpha \subseteq S_a^\alpha \) therefore a contradiction.

Thus, we have \( S_a \subseteq T_\alpha \), hence \( M_i \) - \( I_\alpha \) and \( I_\alpha \subseteq I_\alpha \) is a stable model for \( P \).

Secondly, we show that \( I_\alpha \) is \( \leq \alpha \)-least stable model for \( P \).

Let \( M_i \) be a stable model for \( P \). Let \( V_\alpha \) be defined as follows:

\( V_\alpha = (T_\alpha, T_\alpha, T_\alpha) \).

The model \( M_i \) is the least model of \( P/M_0 \), with respect to the ordering \( \leq \). Let \( I_\alpha \) be the interpretations as in the Definition 8. Let \( V_\alpha = (S_a^\alpha, \ldots, S_a^\alpha) \).

We show by induction on \( \alpha \) the following relations: \( S_a \subseteq T_\alpha \), for every \( k, 0 \leq k \leq n \).

Since \( I_\alpha = (\alpha, \ldots, \alpha) \), we have that (12) are true for every \( k = 0 \).

Assume that (12) is true for every ordinal \( \alpha < \beta \).

If \( \beta \) is limit ordinal, then (12) is true for \( \beta \).

Now let \( \beta \) be a successor ordinal, \( \beta = \alpha + 1 \).

It must show that \( S_a^\beta = T_\beta \), \( k = 0, n \).

Let us distinguish two cases:

1) \( k \geq 1 \),
2) \( k = 0 \).

In case 1) let \( A \) be from \( S_a^\beta \). We have: for every \( r \in M_\alpha \), having the form: \( r \equiv A \rightarrow Z_1,...,Z_q, \) \( \min \{ I_{r}(Z_j), \bar{I}_{r}(Z_j) \} \leq v_k \) and there is \( r_i \equiv A \rightarrow S_1,...,S_k \), such that \( I_{r_i}(S_j) = u \), for every \( j, 1 \leq j \leq p \) and \( \min \{ I_{r_i}(S_j), 1 \leq j \leq p \} = v_k \). From (12) it results:

(12)
\[ I_\alpha(L) = M_\alpha(L) \text{ for every ground literal } L. \] (14)

Since \( M_\alpha \) is also a model for \( P \), we have \( M_\alpha(A) \neq u \) and moreover \( M_\alpha(A) \neq v_\alpha \). Let us denote \( M_\alpha(A) = v_i \).

If we assume that \( v_i \neq v_\alpha \), then we define an interpretation \( M_i \) as follows:

\[ M_i'(B) = M_i(B) \text{ if } B \neq A \text{ and } v_i \text{ otherwise.} \]

It results that \( M_i' \) is a model for \( P/M_i \), \( M_i' \leq M_i \) and \( M_i' \neq M_i \), which contradicts the fact \( M_\alpha \) is the least model for \( P/M_\alpha \) with respect to the ordering \( \leq_{\alpha} \). Hence, we have \( v_i = v_\alpha \), i.e. \( \alpha \in T_\alpha \).

In the case 2) let \( A \) be form \( S_{i}^{\alpha^1} \).

In this case, for every \( r \in M_\alpha \) of the form: \( r = a - L_{a_{1}} \ldots L_{a_{n}} \), there is \( i \) such that \( I_\alpha(L_{a_i}) = 0 \), or there is \( i \), such that \( I_\alpha \) is atom and \( L_{a_i} \in S_{i}^{\alpha^1} \).

Using the hypothesis of induction (12), we obtain:

\[ I_\alpha(L_{a_i}) = 0 \text{ implies } M_i(L_{a_i}) = 0. \]

Let \( r \in M_\alpha \) be of the form \( r = a - B_{a_j} \ldots B_{a_k} \lor a_{a_{1}} \ldots a_{a_{k}} \).

The clause corresponding to \( r \) from \( P/M_i \) is \( r' = a - B_{a_j} \ldots B_{a_k} \lor a_{a_{1}} \ldots a_{a_{k}} \text{ where } v_i = \lnot M_i(D_{a_j}), \text{ } j = 1, \ldots, p \).

If \( M_i(A) = u \) and \( M_i(A) = v_\alpha \) with \( v_\alpha \neq 0 \), then we consider a model \( M_i' \) defined by following:

\[ T_0' = T_0 \cup S_{i}^{\alpha^1}, T_0'' = T_0 - S_{i}^{\alpha^1}, j = 1, \ldots, n \]

and \( V_{M_i'} = (T_0', \ldots, T_0'') \).

Since \( S_{i}^{\alpha^1} \) is an unfounded set with respect to \( M_i \), we have \( M_i' \) is a model for \( P/M_i \).

Moreover, since \( M_i' \leq M_i \) and \( M_i' \neq M_i \), it results a contradiction.

If \( M_i(A) = u \), we consider the same interpretation \( M_i' \) as it was described above, which implies a contradiction. It results the statement (13). Taking in (13) \( \alpha = \lambda \), it obtains that \( I_\alpha \leq_{\lambda} M_i \), therefore \( I_\alpha \) is the \( \leq_{\lambda} \)-least stable model for \( P \).

**CONCLUSION**

This study introduced new semantics for general logic programs considering a set of \( n \)-\( 1 \)-truth logic values and an undefined value. One of semantics is of type well-founded and the other is of type stable. We have studied a relationship between the two semantics. For \( n=1 \) and \( u=1/2 \), the results of Przymusinski[9] are obtained.

**REFERENCES**