A Class of Explicit Fourth Order Method with Phase Lag of Order Six for Second Order Initial Value Problems

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Abstract: We derive the explicit fourth order two-step methods for the numerical integration of second order initial value problems \( \ddot{y} = f(t, y), y(t_0) = y_0, \dot{y}(t_0) = \dot{y}_0 \). Then we discuss the P-stability and phase lag of the methods. Next we discuss particular methods and present the numerical result of these methods.

Key words: Explicit fourth order method, numerical integration, P-stability, phase lag, initial value problem

INTRODUCTION

A class of two-step hybrid methods for the numerical integration of second order initial value problem

\[
\ddot{y} = f(t, y), y(t_0) = y_0, \dot{y}(t_0) = \dot{y}_0.
\]  

is defined in the following. Let \( h > 0 \) denote the stepsize \( t_n = t_{n-1} + h, n = 1, 2, \ldots \) and set \( y_n = y(t_n), \dot{y}_n = \dot{y}(t_n) \). We consider the method of the form

\[
y_{n+1} = 2y_n - y_{n-1} + h^2m\]

(2)

\[
y_{n+1} = A_1y_{n+1} + B_2y_n + C_3y_{n-1} + h^2m_1
\]

(3)

\[
\dot{y}_{n+1} = 2\dot{y}_n - \dot{y}_{n-1} + h^2\dot{y}_n
\]

(4)

\[
y_{n+1} = f(t_{n+1}, y_{n+1}), \quad \dot{y}_{n+1} = f(t_{n+1}, \dot{y}_{n+1})
\]

(5)

where:

\[
m_1 = \beta_0\dot{y}_{n+1} + \beta_1\dot{y}_{n-1} + \beta_3(\dot{y}_{n+1} + \dot{y}_{n-1}) + \gamma y_n
\]

(6)

and

\[
m_2 = s_k\dot{y}_{n+1} + \eta_k\dot{y}_{n-1} + \omega_k\dot{y}_{n-1}
\]

The methods are fourth order accurate if satisfy the following conditions.

i) \( 1 - \gamma - 2\beta_0 - 2\beta_3 = 0 \),

ii) \( \beta_1 n_1 = 0 \),

iii) \( \frac{1}{12} - \beta_0 - \beta_3 n_1 = 0 \),

iv) \( \frac{1}{6} - 2\beta_0 - \beta_3 n_2 = 0 \),

v) \( \frac{1}{12} - \beta_0 - \beta_3 n_1 = 0 \),

vi) \( \frac{1}{12} - \beta_0 - \frac{1}{2}\beta_3 n_1 = 0 \),

vii) \( \beta_1 n_1 = 0 \),

viii) \( \beta_1 n_1 = 0 \),

ix) \( \beta_3 n_1 = 0 \),

x) \( \beta_1 n_1 = 0 \),

xi) \( \beta_1 n_1 + \beta_3 n_1 n_1 = 0 \),

xii) \( \beta_1[ n_1^2 + n_1^2 ] = 0 \),

xiii) \( A_1 + A_2 + A_3 = 1 \)

where:

\[
n_1 = n_1 + n_2, \quad n_2 = n_1 n_1, \quad n_1 = n_2 + n_2, \quad n_2 = n_1 + n_1 n_1, \quad n_3 = n_1, \quad n_4 = A_1 + n_1, \quad n_5 = A_1 + n_1, \quad n_6 = A_1 + n_1 + n_1 + n_1, \quad n_7 = A_1 + n_1 + n_1 + n_1, \quad n_8 = A_1 + n_1 + n_1 + n_1 + n_1 + n_1,
\]

\[
n_{12} = \frac{1}{6}(A_1 + n_1) + n_1, \quad n_{13} = \frac{1}{6}(A_1 + n_1) + n_1 + n_1
\]

and the local truncation error is

\[
LTE = \lambda h^{4} + O(h^{5})
\]

\[
\lambda = \left( \frac{1}{360} - \frac{1}{12} \beta_0 - \frac{1}{12} \beta_3 n_1 \right) \frac{\partial f}{\partial y}
\]

(6)
\[
\text{LTE} = \frac{1}{720} \left[ 3y^{(0)} - \sum_j \frac{\partial f}{\partial y_j} y_j^{(0)} \right] h^6 + \mathcal{O}(h^7). \tag{7}
\]

**Case 2**: If \( \beta_j \neq 0 \) then either \( \alpha_i = 0 \) or \( \alpha_j = 0 \).
(a) If \( \alpha_i = 0 \) then we have from the conditions \( \alpha = 0, s, + s, = u, + s, \beta_j = 1/12, \gamma = 5/6 - 2 \beta_j, C = A_1, B_2 = 1 - 2 A_2, q_i, + q_i = - (A_1 + A_i) - 2 (u_i + u) \), \( \gamma = 0 \) and \( \alpha = 1/300 \) \( \tag{8} \)

and the LTE becomes
\[
\text{LTE} = \left[ -\frac{1}{240} y^{(0)} + m_s \right] h^4 + \mathcal{O}(h^5) \tag{9}
\]

where, \( m_s = \left[ \frac{5}{720} \beta_j (s_i + s) \right] \sum_j \frac{\partial f}{\partial y_j} y^{(0)} \)

and \( m_s = \beta_j A_i + s_i + q_i + u_i \sum_j \sum_i \frac{\partial f}{\partial y_j \partial y_i} y^{(0)} \).

Observe that Chawla and Rao's method is of this class. They choose the parameters as \( \alpha_1 = 0, A_1 = A_i = C, - C, - C, - C = 0, B_i = 1, q_i = q_i - 2 \alpha_i, u = u - u, - s, - s, - s = - \alpha, \beta_j = 1/12, \beta_i = 5/12, \gamma = 0 \) and \( \alpha = -1/300 \) \( \tag{10} \)

and LTE
\[
= -\frac{1}{720} \left[ 3y^{(0)} - 5y^{(0)} (f_i, s_i) - 600 \alpha y^{(0)} (f_i, s_i) \right] h^6 + \mathcal{O}(h^7) \tag{11}
\]

(b) If \( \alpha_i \neq 0 \) then fourth order conditions becomes
\[
\beta_0 = \frac{1}{12} \beta_i \alpha_i, \gamma = \frac{5}{6} + 2 \beta_i (\alpha_i - 1), s_i = u_i + s_i, B_i = 1 + \alpha_i - 2 A_i, C = A_i, - \alpha_i, C = A_i, - \alpha_i,\]

\[
q_i = \frac{1}{2} (\alpha_i - \alpha_i) A_i, - s_i, - u_i, q_i = \frac{1}{2} (\alpha_i - \alpha_i) A_i, - s_i, - u_i
\]

and the local truncation error becomes
\[
\text{LTE} = \lambda_i h^4 + \mathcal{O}(h^5) \tag{12}
\]

where
\[
\lambda_i = \frac{1}{12} \left( \frac{1}{30} - \beta_i \beta_i \alpha_i \right) y^{(0)} + 2 \beta_i \alpha_i \left[ \frac{1}{6} \alpha_i (\alpha_i - 1) - (s_i - u_i) \right] \sum_j \frac{\partial f}{\partial y_j} y^{(0)} + \sum_j \sum_i \frac{\partial f}{\partial y_j \partial y_i} y^{(0)} y^{(0)} + \left[ \frac{1}{12} (\beta_i + \beta_i \alpha_i) \beta_i (s_i + s_i) \right] \sum_j \frac{\partial f}{\partial y_j} y^{(0)}.
\]

**Case 1**: If \( \beta_i = 0 \) then from condition (iii) \( \beta_i = 1/12 \) and condition (i) \( \gamma = 5/6 \). Then the local truncation error from (6) becomes

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To analyses the stability properties of the method, we apply the method (2), (3), (4) to the scalar test equation:

\[ y + c'y = 0, \quad c \text{ real.} \quad (13) \]

We obtain the stability polynomial

\[ p(r) = r^4 - \left[2 Ki, H^2 + K_i, H^4 - K_i, H^6\right] + \left[1 + \beta_{i}, K_i, H^2 + \beta_{i}, K_i, H^4\right] \quad (14) \]

where:

\[ K_i = \left[2 \beta_{i} + \gamma + \beta_{i} (2 A_i + 2 A_i + B_i + B_i)\right], \]

\[ K_i = \left[\beta_{i} + \beta_{i} (A_i + A_i + 2 A_i + 2 A_i + q_i + q_i)\right], \]

\[ K_i = \beta_{i} (s_i + s_i), K_i = (C_i + C_i - A_i - A_i), \]

\[ K_i = (s_i + s_i, u_i, u_i) \text{ and } H = c h \]

Equation (14) is a difference equation of the form[1]

\[ y_{n+1} - \tau_i(H) y_n + \tau_0(H) y_{n-1} = 0 \quad (15) \]

where:

\[ \tau_i = \tau_{i-1} = 1 \text{ and } \tau_i(H) = 2 + H^2 - \frac{1}{12} H^4 + \beta_{i} (s_i + s_i) H^6. \]

When we apply the fourth order condition then the stability polynomial is of the form \[ p(r) = r^4 - 2 B(H) r + 1, \]

where, \[ B(H) = \frac{1}{2} H^2 + \frac{1}{24} H^4 + \frac{1}{2} \beta_{i} (s_i + s_i) H^6. \]

Making the Routh-Herwitz transformation

\[ r = \frac{1+z}{1-z}, \quad (15) \]

\[ (\tau_2 - \tau_1) z^2 + 2(\tau_1 - \tau_0) z + (\tau_2 + \tau_1 + \tau_0) = 0. \]

Thus, the necessary and sufficient conditions for P-stability are \( (\tau_2 - \tau_1 + \tau_0) \geq 0, (\tau_1 - \tau_0) = 0, (\tau_2 + \tau_1 + \tau_0) \geq 0. \)

Since \( \tau_1 = \tau_2, \) then the condition becomes

\[ (2 \tau_2 - \tau_1)^2 + (2 \tau_2 + \tau_1) = 0 \]

and the necessary and sufficient conditions for P-stability are \( 2 \tau_2 - \tau_1 \geq 0 \) and \( 2 \tau_2 + \tau_1 \geq 0 \)

\[ 2 \tau_2 + \tau_1 \geq 0 \quad (16) \]

where:

\[ 2 \tau_2 - \tau_1 = 4 - H^2 + \frac{1}{12} H^4 + \beta_{i} (s_i + s_i) H^6 \geq 0 \quad (17) \]

and

\[ 2 \tau_2 + \tau_1 = H^2 - \frac{1}{12} H^4 + \beta_{i} (s_i + s_i) H^6 \geq 0 \quad (18) \]

Condition (18) is satisfied if \( \frac{1}{144} - 4 \beta_{i} (s_i + s_i) \leq 0 \)

\[ \beta_{i} (s_i + s_i) \geq \frac{1}{576} \]

Thus the methods are P-stable if satisfy the condition (17) and (18) with \( \beta_{i} (s_i + s_i) \geq \frac{1}{576} \).

Phase LAG: When the method (2), (3) and (4) is applied to the scalar equation (13), we have the recurrence relation[6]:

\[ \frac{y_{n+1}}{y_n} = \left[2 - H^2 + \frac{1}{12} H^4 + \beta_{i} (s_i + s_i) H^6\right] y_n + y_{n+1} = 0 \]

Substituting \( y_n = e^{\theta i n} \), we have

\[ e^{\theta i n} \left[2 - H^2 + \frac{1}{12} H^4 + \beta_{i} (s_i + s_i) H^6\right] e^{\theta i n} + 1 = 0 \]

on expansion of \( e^{\theta i n} \) and \( e^{\theta i n} \), we have the form

\[ 0 = (1 + \theta^i) H^2 + (\theta^i + \theta) H^4 \]

\[ + \left(\frac{7}{12} \theta^i + \frac{1}{12} \theta^i - \frac{1}{12}\right) H^6 \]

\[ + \left(\frac{1}{4} \theta^i + \frac{1}{6} \theta^i - \frac{1}{12}\right) H^8 \]

\[ + \left(\frac{1}{40} \theta^i + \frac{1}{120} \theta^i - \frac{1}{72}\right) H^{10} \]

\[ + \left(\frac{127}{20160} \theta^i + \frac{1}{720} \theta^i - \frac{1}{288}\right) H^{12} + \beta_{i} (s_i + s_i) \theta H^{14} \quad (19) \]

Let \( \theta = \eta_0 + \eta_1 H + \eta_2 H^3 + \eta_3 H^5 + \eta_4 H^7 \). Substitutes in (19) and then comparing the coefficients of \( H, j = 2, 3, 4, 5, 6 \) to zero, we get

\[ \eta_0 = i, \quad \eta_1 = 0, \quad \eta_0 = 0, \quad \eta_1 = 0, \quad \eta_1 = -\frac{1}{2} \beta_{i} (s_i + s_i), \quad \eta_4 = -\frac{1}{2} \beta_{i} (s_i + s_i) \]

Thus we have
\[
\vartheta = \frac{i}{2} \left[ \frac{1}{360} \beta_i (s_\alpha + s_\beta) \right] \mathbf{H}^\alpha + O(\mathbf{H}^\beta).
\]

If \( \beta_i (s_\alpha + s_\beta) = \frac{1}{360} \) which is greater than \(1/576\) (P-stability condition), we may write \( \vartheta = \frac{i}{2} \eta^\alpha H^\alpha + \eta^\beta H^\beta \) and substituting in (19) and then comparing the coefficients of \( H^\alpha \) and \( H^\beta \) to zero, we get \( \eta^\alpha = 0 \) and \( \eta^\beta = -\frac{i}{40320} \). Thus \( \vartheta = \frac{i}{40320} H^\beta + O(\mathbf{H}^\beta) \).

Thus the quantity \( b - 1 \) in the definition of phase lag is
\[
b - 1 = \frac{i}{40320} H^\beta + O(\mathbf{H}^\beta).
\]

**PARTICULAR METHODS**

**Case 1:** \( \beta_i \neq 0 \) and \( \alpha_i = 0 \). Observe that Chawla and Rao\(^{11} \) method is of this class with parameters.

\[
\alpha_i = 0, \quad A_\alpha = A_\alpha^* = C_\alpha = C_\alpha^* = 0, \quad B_\alpha = B_\alpha^* = 1,
\]
\[
q_\alpha = q_\alpha^* = -\frac{1}{150}, \quad u_\alpha = u_\alpha^* = s_\alpha = s_\alpha^* = \frac{1}{360},
\]
\[
\beta_i = \frac{1}{12}, \quad \beta_i^* = \frac{5}{12}, \quad \gamma = 0.
\]

In this case \( \beta_i (s_\alpha + s_\beta) = \frac{1}{360} > \frac{1}{576} \) satisfy the P-stability condition and also phase lag given by (20).

We choose the parameters in two different ways

(a) The points \( \left( t_n, x_{n+1}, y_{n+1} \right) \) and \( \left( t_n, x_n, y_n \right) \) are coincident and

(b) \( \tilde{y}_{n+1} = \tilde{y}_n \).

(a) If the points \( \left( t_n, x_{n+1}, y_{n+1} \right) \) and \( \left( t_n, x_n, y_n \right) \) are coincident then

\[
A_\alpha - A_\alpha^* = B_\alpha - B_\alpha^*, \quad C_\alpha - C_\alpha^* = s_\alpha - s_\alpha^*, \quad q_\alpha - q_\alpha^*, \quad u_\alpha - u_\alpha^*.
\]

then from (8)
\[
C_\alpha = A_\alpha = A_\alpha^*, \quad B_\alpha = 1 - 2A_\alpha, \quad u_\alpha = s_\alpha = u_\alpha^* = s_\alpha^*.
\]
\[
q_\alpha = q_\alpha^* = -A_\alpha - 2s_\alpha, \quad \beta_i = \frac{1}{12}, \quad \gamma = \frac{5}{6} - 2\beta_i.
\]

for P-stability we have \( \beta_i (s_\alpha + s_\beta) > \frac{1}{576} \) and if we have \( \beta_i (s_\alpha + s_\beta) = \frac{1}{360} \) then the method has phase lag given by (20). Thus we take \( \beta_i (s_\alpha + s_\beta) = \frac{1}{360} > \frac{1}{576} \) for P-stable method. We choose \( \beta_i = 1 \), then \( s_\alpha = 1/720 \). Also let \( A_\alpha = 1/2 \).

(b) For \( y_{n+1} = \tilde{y}_n \) we have

\[
\alpha_i = 0, \quad A_\alpha = 0, \quad B_\alpha = 1, \quad C_\alpha = 0, \quad s_\alpha = 0, \quad q_\alpha = 0, \quad u_\alpha = 0.
\]

then from (8)
\[
s_\alpha = u_\alpha, \quad \beta_\alpha = \frac{1}{12}, \quad \gamma = \frac{5}{6} - 2\beta_\alpha, \quad B_\alpha = 1 - 2A_\alpha, \quad C_\alpha = A_\alpha.
\]

and for P-stable with phase lag given by (20), we must have \( \beta_i (s_\alpha + s_\beta) = \frac{1}{360} \). Thus we choose \( \beta_i = 1 \), then \( s_\alpha = 1/360 \). Also let \( A_\alpha = 1/2 \).

Also we test the method with \( A_\alpha = 0 \) and \( \beta_i = 1/3 \), then \( s_\alpha = 1/120 \). The local truncation error for all these methods is

\[
\text{LTE} = \frac{1}{240} \left[ -y^{(\alpha)} + \sum \frac{\partial y^{(\alpha)}}{\partial y} y^{(\alpha)} \right] h^\alpha + O(h^\beta).
\]

**Case 2:** If \( \beta_i \neq 0 \) and \( \alpha_i \neq 0 \), we have chosen the method for \( \alpha_i = 1 \). In this case \( y_{n+1} = \tilde{y}_n \). Then

\[
A_\alpha = 0, \quad B_\alpha = 0, \quad C_\alpha = 1, \quad s_\alpha = 0, \quad q_\alpha = 0, \quad u_\alpha = 0.
\]

then from (11), we have
\[
u_\alpha = s_\alpha, \quad \beta_\alpha = 1 - 2A_\alpha, \quad \gamma = \frac{5}{6}, \quad B_\alpha = 2 - 2A_\alpha, \quad C_\alpha = A_\alpha - 1, \quad q_\alpha = 1 - A_\alpha - 2s_\alpha.
\]

For P-stable method with phase lag of order eight given by (20), we must have \( \beta_i (s_\alpha + s_\beta) = \frac{1}{360} \) or in this case, \( \beta_\alpha = \frac{1}{360} \). We choose \( \beta_i = 1 \) and \( A_\alpha = 1/2 \), then, \( s_\alpha = \frac{1}{360} \). For these methods, LTE is given by (26).

Methods for different examples were tested. The methods are given below.

The methods derived above and used for companion purpose in next section.
M 1: Given by equations (22) and (23) with parameters $\beta_1 = 1$, $S_1 = 1/720$ and $\Lambda_1 = 1/2$
M 2: Given by equations (24) and (25) with parameters $\beta_1 = 1$, $S_1 = 1/360$ and $\Lambda_1 = 1/2$
M 3: Given by equations (24) and (25) with parameters $\beta_1 = 1/3$, $S_1 = 1/120$ and $\Lambda_1 = 0$
M 4: Given by equations (27) and (28) with parameters $\beta_1 = 1$, $S_1 = 1/360$ and $\Lambda_1 = 1/2$
M 5: Given by equations (21).

**NUMERICAL ILLUSTRATION**

We have tried a number of explicit scalar (nonstiff) test problems of the form (1). They give similar results and so we restrict our attention to one oscillatory example.

**Example 1:** $\ddot{y} + \sinh y = 0$, $y(0) = 1$, $\dot{y}(0) = 0$.

This is a pure oscillation problem whose solution has maximum amplitude unity and period approximately six. We have calculated error as $|\text{Error at } t = 6|$

In Table 1-3 we present the following statistics:

1) Number of evaluation of the differential equation right hand side $f$, FCN;
2) Number of steps overall, NST;
3) Number of successful steps to complete the integration, NSST;
4) Number of steps where the stepsize is changes, NCST;
5) Number of failed steps, NFST;
6) Number of steps on which the iteration diverged, NDIV;
7) Number of steps where the stepsize is halved, NHIST;

**Table 1:** Comparison of the method for example 1 with Tol = $10^{-2}$

<table>
<thead>
<tr>
<th>Methods</th>
<th>Error</th>
<th>FCN</th>
<th>NST</th>
<th>NSST</th>
<th>NFST</th>
<th>NCST</th>
<th>NDIV</th>
<th>NHIST</th>
</tr>
</thead>
<tbody>
<tr>
<td>M 1</td>
<td>1.641$\times 10^{-1}$</td>
<td>55</td>
<td>18</td>
<td>13</td>
<td>5</td>
<td>6</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>M 2</td>
<td>1.650$\times 10^{-1}$</td>
<td>55</td>
<td>18</td>
<td>13</td>
<td>5</td>
<td>6</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>M 3</td>
<td>3.434$\times 10^{-1}$</td>
<td>55</td>
<td>18</td>
<td>13</td>
<td>5</td>
<td>6</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>M 4</td>
<td>1.832$\times 10^{-1}$</td>
<td>55</td>
<td>18</td>
<td>13</td>
<td>5</td>
<td>6</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>M 5</td>
<td>1.654$\times 10^{-1}$</td>
<td>55</td>
<td>18</td>
<td>13</td>
<td>5</td>
<td>6</td>
<td>1</td>
<td>1</td>
</tr>
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</table>

**Table 2:** Comparison of the method for example 1 with Tol = $10^{-4}$

<table>
<thead>
<tr>
<th>Methods</th>
<th>Error</th>
<th>FCN</th>
<th>NST</th>
<th>NSST</th>
<th>NFST</th>
<th>NCST</th>
<th>NDIV</th>
<th>NHIST</th>
</tr>
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<tbody>
<tr>
<td>M 1</td>
<td>2.980$\times 10^{-4}$</td>
<td>139</td>
<td>46</td>
<td>35</td>
<td>11</td>
<td>12</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>M 2</td>
<td>2.980$\times 10^{-4}$</td>
<td>139</td>
<td>46</td>
<td>35</td>
<td>11</td>
<td>12</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>M 3</td>
<td>2.959$\times 10^{-4}$</td>
<td>139</td>
<td>46</td>
<td>35</td>
<td>11</td>
<td>12</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>M 4</td>
<td>2.979$\times 10^{-4}$</td>
<td>139</td>
<td>46</td>
<td>35</td>
<td>11</td>
<td>12</td>
<td>3</td>
<td>3</td>
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<tr>
<td>M 5</td>
<td>2.980$\times 10^{-4}$</td>
<td>139</td>
<td>46</td>
<td>35</td>
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**Table 3:** Comparison of the method for example 1 with Tol = $10^{-6}$

<table>
<thead>
<tr>
<th>Methods</th>
<th>Error</th>
<th>FCN</th>
<th>NST</th>
<th>NSST</th>
<th>NFST</th>
<th>NCST</th>
<th>NDIV</th>
<th>NHIST</th>
</tr>
</thead>
<tbody>
<tr>
<td>M 1</td>
<td>2.050$\times 10^{-6}$</td>
<td>367</td>
<td>122</td>
<td>108</td>
<td>14</td>
<td>13</td>
<td>5</td>
<td>5</td>
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<tr>
<td>M 2</td>
<td>2.050$\times 10^{-6}$</td>
<td>367</td>
<td>122</td>
<td>108</td>
<td>14</td>
<td>13</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>M 3</td>
<td>2.044$\times 10^{-6}$</td>
<td>367</td>
<td>122</td>
<td>108</td>
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<td>13</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>M 4</td>
<td>2.050$\times 10^{-6}$</td>
<td>367</td>
<td>122</td>
<td>108</td>
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<td>13</td>
<td>5</td>
<td>5</td>
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<tr>
<td>M 5</td>
<td>2.050$\times 10^{-6}$</td>
<td>367</td>
<td>122</td>
<td>108</td>
<td>14</td>
<td>13</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

Next we consider example for system, where we use a test problem a moderately stiff system of two equations

**Example 2:** $\ddot{y} + \sinh (y_1 + y_2) = 0$, $y_1(0) = 1$, $\dot{y}_1(0) = 0$, $\ddot{y}_2 + y_1^2 y_2 = 0$, $y_2(0) = 10^{-4}$, $\dot{y}_2(0) = 0$.

For this example we have deliberately introduced coupling from the stiff (linear) equation to the nonstiff (nonlinear) equation. For this example again we have calculated error as $||\text{Error at } t = 6||$. 

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Table 4: Comparison of the method for example 2 with Tol = 10^{-2}

<table>
<thead>
<tr>
<th>Methods</th>
<th>Error</th>
<th>FCN</th>
<th>NOST</th>
<th>NSST</th>
<th>NPST</th>
<th>NCST</th>
<th>NDIV</th>
<th>NFST</th>
</tr>
</thead>
<tbody>
<tr>
<td>M 1</td>
<td>9.832x10^{-3}</td>
<td>1502</td>
<td>501</td>
<td>480</td>
<td>21</td>
<td>32</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>M 2</td>
<td>9.894x10^{-3}</td>
<td>1505</td>
<td>502</td>
<td>480</td>
<td>22</td>
<td>34</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>M 3</td>
<td>4.059x10^{-2}</td>
<td>1475</td>
<td>493</td>
<td>466</td>
<td>27</td>
<td>38</td>
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<td>4</td>
</tr>
<tr>
<td>M 4</td>
<td>9.904x10^{-2}</td>
<td>1517</td>
<td>506</td>
<td>484</td>
<td>22</td>
<td>33</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>M 5</td>
<td>9.730x10^{-2}</td>
<td>1505</td>
<td>502</td>
<td>483</td>
<td>19</td>
<td>28</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 5: Comparison of the method for example 2 with Tol = 10^{-3}

<table>
<thead>
<tr>
<th>Methods</th>
<th>Error</th>
<th>FCN</th>
<th>NOST</th>
<th>NSST</th>
<th>NPST</th>
<th>NCST</th>
<th>NDIV</th>
<th>NFST</th>
</tr>
</thead>
<tbody>
<tr>
<td>M 1</td>
<td>4.816x10^{-3}</td>
<td>1024</td>
<td>344</td>
<td>317</td>
<td>24</td>
<td>34</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>M 2</td>
<td>3.538x10^{-3}</td>
<td>1756</td>
<td>585</td>
<td>558</td>
<td>27</td>
<td>43</td>
<td>3</td>
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<tr>
<td>M 3</td>
<td>1.727x10^{-3}</td>
<td>1357</td>
<td>452</td>
<td>429</td>
<td>23</td>
<td>32</td>
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<td>3</td>
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<tr>
<td>M 4</td>
<td>5.807x10^{-4}</td>
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<td>327</td>
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<td>15</td>
<td>19</td>
<td>3</td>
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<tr>
<td>M 5</td>
<td>5.121x10^{-4}</td>
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<td>301</td>
<td>290</td>
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</tbody>
</table>

Table 6: Comparison of the method for example 2 with Tol = 10^{-4}

<table>
<thead>
<tr>
<th>Methods</th>
<th>Error</th>
<th>FCN</th>
<th>NOST</th>
<th>NSST</th>
<th>NPST</th>
<th>NCST</th>
<th>NDIV</th>
<th>NFST</th>
</tr>
</thead>
<tbody>
<tr>
<td>M 1</td>
<td>3.615x10^{-4}</td>
<td>994</td>
<td>331</td>
<td>311</td>
<td>20</td>
<td>23</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>M 2</td>
<td>3.615x10^{-4}</td>
<td>994</td>
<td>331</td>
<td>311</td>
<td>20</td>
<td>23</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>M 3</td>
<td>1.344x10^{-4}</td>
<td>1300</td>
<td>433</td>
<td>421</td>
<td>12</td>
<td>10</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>M 4</td>
<td>3.614x10^{-4}</td>
<td>994</td>
<td>331</td>
<td>311</td>
<td>20</td>
<td>23</td>
<td>5</td>
<td>5</td>
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<tr>
<td>M 5</td>
<td>3.615x10^{-4}</td>
<td>994</td>
<td>331</td>
<td>311</td>
<td>20</td>
<td>23</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

Example 3: Consider the almost periodic linear problem studied by Stiefel and Bettis[3].

\[ \dot{Z} + Z = 0.001e^t, \ Z(0) = 1, \ \dot{Z}(0) = 0.99951, \]

with analytic solution.

\[ Z(t) = \cos t + 0.0005t \sin t - 0.0005t \cos t. \]

This solution represents motion on a perturbed circular orbit with the distance from the origin \( |Z(t)| \). We have computed solution to this problem using our fourth order method. We use the system

\[ \dot{u} + u = 0.001 \cos t, \ u(0) = 1, \ \dot{u}(0) = 0, \ \dot{v} + v = 0.001 \sin t, \ v(0) = 1, \ \dot{v}(0) = 0.99951, \]

Where, \( Z = u + iv \). After computing \( u \) and \( v \), we have calculated \( |Z| \) at \( t = 40\pi \) with stepsize \( h = \pi/4 \). Result are presented in Table 7-9 and comparison of approximation in Table 10.

Table 7: Comparison of the method for example 3 with Tol = 10^{-2}

<table>
<thead>
<tr>
<th>Methods</th>
<th>Error</th>
<th>FCN</th>
<th>NOST</th>
<th>NSST</th>
<th>NPST</th>
<th>NCST</th>
<th>NDIV</th>
<th>NFST</th>
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</thead>
<tbody>
<tr>
<td>M 1</td>
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<td>322</td>
<td>321</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>M 2</td>
<td>1.311x10^{-2}</td>
<td>967</td>
<td>322</td>
<td>321</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>M 3</td>
<td>5.155x10^{-4}</td>
<td>967</td>
<td>322</td>
<td>321</td>
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<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>M 4</td>
<td>1.180x10^{-2}</td>
<td>967</td>
<td>322</td>
<td>321</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>M 5</td>
<td>6.879x10^{-3}</td>
<td>967</td>
<td>322</td>
<td>321</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 8: Comparison of the method for example 3 with Tol = 10^{-3}

<table>
<thead>
<tr>
<th>Methods</th>
<th>Error</th>
<th>FCN</th>
<th>NOST</th>
<th>NSST</th>
<th>NPST</th>
<th>NCST</th>
<th>NDIV</th>
<th>NFST</th>
</tr>
</thead>
<tbody>
<tr>
<td>M 1</td>
<td>1.234x10^{-3}</td>
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<td>1284</td>
<td>1281</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>M 2</td>
<td>3.087x10^{-3}</td>
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<td>1284</td>
<td>1281</td>
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<td>3</td>
</tr>
<tr>
<td>M 3</td>
<td>2.424x10^{-4}</td>
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<td>1284</td>
<td>1281</td>
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<td>3</td>
</tr>
<tr>
<td>M 4</td>
<td>3.688x10^{-3}</td>
<td>3853</td>
<td>1284</td>
<td>1281</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>M 5</td>
<td>1.345x10^{-4}</td>
<td>3853</td>
<td>1284</td>
<td>1281</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 9: Comparison of the method for example 3 with Tol = 10^{-4}

<table>
<thead>
<tr>
<th>Methods</th>
<th>Error</th>
<th>FCN</th>
<th>NOST</th>
<th>NSST</th>
<th>NPST</th>
<th>NCST</th>
<th>NDIV</th>
<th>NFST</th>
</tr>
</thead>
<tbody>
<tr>
<td>M 1</td>
<td>6.161x10^{-3}</td>
<td>7693</td>
<td>2564</td>
<td>2560</td>
<td>4</td>
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<td>4</td>
<td>4</td>
</tr>
<tr>
<td>M 2</td>
<td>1.540x10^{-4}</td>
<td>7693</td>
<td>2564</td>
<td>2560</td>
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<td>4</td>
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<tr>
<td>M 3</td>
<td>8.542x10^{-1}</td>
<td>7693</td>
<td>2564</td>
<td>2560</td>
<td>4</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>M 4</td>
<td>1.543x10^{-4}</td>
<td>7693</td>
<td>2564</td>
<td>2560</td>
<td>4</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>M 5</td>
<td>7.650x10^{-4}</td>
<td>7693</td>
<td>2564</td>
<td>2560</td>
<td>4</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>


Table 10: Comparison of the approximation produced at $Z=40\pi$, with $h=\pi/4$

   | Methods | TOL=10^{-2} | TOL=10^{-4} | TOL=10^{-6} |
---|---------|------------|------------|------------|
M 1 | 1.0524749 | 1.0143116 | 1.0081329 |
M 2 | 1.0150825 | 1.0050591 | 1.0035116 |
M 3 | 1.0024875 | 1.0019744 | 1.0019711 |
M 4 | 0.9901767 | 0.9988838 | 1.0004287 |
M 5 | 1.0088506 | 1.0035170 | 1.0027414 |

REFERENCES


