Mathematical Inventory Model under Trade Credits Linked to Payment Time

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Abstract: This study is to extend Chou et al.'s model to assume that the unit selling price and unit purchasing price are not necessarily equal. Then, reformulate retailer's mathematical inventory replenishment model and obtain the optimal cycle time and optimal payment time for item so that the annual total relevant cost is minimized. One theorem is developed to efficiently determine the optimal replenishment and payment policy for the retailer.

Key words: Inventory, trade credit, payment time

INTRODUCTION

In 2005, Chou et al. modeled the retailer's inventory system under payment delay and cash discount and developed an efficient procedure to determine the retailer's optimal ordering policy. Chou et al. assumed that the retailer can obtain fully permissible delay in payments and cash discount if the payment is paid before the period of full delay payments permitted by the supplier. Otherwise, the retailer will just obtain partially permissible delay in payments within the period of partial delay payments permitted by the supplier. The supplier uses the credits policy linked to payment time to attract the retailer to pay the payment as soon as possible to shorten its collection period. However, they implicitly make the following assumptions:

- The unit selling price and the purchasing price per unit are assumed to be equal. However, as we know, the unit selling price for the retailer is usually significantly higher than the purchasing price per unit in order to obtain profit.
- At the end of the credit period, the account is settled. The retailer starts paying for higher interest charges on the items in stock and returns money of the remaining balance immediately when the items are sold. What the above statement describes is just one arrangement of capitals of enterprises. Based on considerations of profits, costs and developments of enterprises, enterprises may invest their capitals to the best advantage.

According to the above arguments, this article will adopt the following assumptions to modify Chou et al. model:

- The selling price per unit and the unit purchasing price are not necessarily equal to match the most practical situations.
- The retailer needs cash for business transactions. At the end of the credit period, the retailer pays off all units sold and keeps his/her profits for business transactions or other investment use. This viewpoint also can be found by Teng.

The main purpose of this study is to incorporate the assumptions (i) and (ii) to modify Chou et al.'s model. That is, we incorporate both Chou et al. and Teng to develop the retailer's inventory model. Then, we develop one easy-to-use theorem to efficiently determine the optimal inventory policy for the retailer.

MODEL REFORMULATION

For convenience, most notation and assumptions similar to Chou et al. will be used in this study.

Notation:

\[ D = \text{Demand rate per year} \]
\[ A = \text{Cost of placing one order} \]
\[ c = \text{Unit purchasing price} \]
\[ s = \text{Unit selling price} \]
\( h \) = Unit stock holding cost per year excluding interest charges
\( r \) = Cash discount rate, \( 0 \leq r < 1 \)
\( \alpha \) = The fraction of the total amount owed payable at the time of placing an order, \( 0 < \alpha \leq 1 \)
\( I_c \) = Interest earned per $ per year
\( I_c \) = Interest charges per $ investment in inventory per year
\( M_i \) = The period of full delay payments permitted in years
\( M_i \) = The period of partial delay payments permitted in years, \( M_i < M_i' \)
\( T \) = The cycle time in years
\( TRC_{1i}(T) \) = The annual total relevant cost when payment is paid at time \( M_i \) and \( T \geq M_i \)
\( TRC_{2i}(T) \) = The annual total relevant cost when payment is paid at time \( M_i \) and \( T < M_i \)
\( TRC_{1}(T) \) = The annual total relevant cost when payment is paid at time \( M_i \) and \( T > 0 \)
\begin{align*}
TRC_{1i}(T) & \text{ if } T \geq M_i \\
TRC_{2i}(T) & \text{ if } T < M_i
\end{align*}
\( TRC_{2i}(T) \) = The annual total relevant cost when payment is paid at time \( M_i \) and \( T \geq M_i / \alpha \)
\( TRC_{2i}(T) \) = The annual total relevant cost when payment is paid at time \( M_i \) and \( M_i \leq T < M_i / \alpha \)
\( TRC_{3i}(T) \) = The annual total relevant cost when payment is paid at time \( M_i \) and \( T < M_i \)
\( TRC_{4i}(T) \) = The annual total relevant cost when payment is paid at time \( M_i \) and \( T > 0 \)
\begin{align*}
TRC_{4i}(T) & \text{ if } T \geq M_i \\
TRC_{4i}(T) & \text{ if } M_i \leq T < M_i / \alpha \\
TRC_{4i}(T) & \text{ if } T < M_i
\end{align*}
\( TRC(T) \) = The annual total relevant cost when \( T > 0 \)
\begin{align*}
TRC(T) & \text{ if } \text{the payment is paid at time } M_i \\
TRC(T) & \text{ if } \text{the payment is paid at time } M_i'
\end{align*}
\( T^* \) = The optimal cycle time of \( TRC(T) \)
\( Q^* \) = The optimal order quantity = \( DT^* \).

**Assumptions:**
- Demand rate is known and constant.
- Shortages are not allowed.
- Time horizon is infinite.
- Replenishments are instantaneous.
- \( I_c \geq I_c \).
- Supplier offers a cash discount and fully delayed payment to the retailer if payment is paid within \( M_i \), otherwise just partially delayed payment if payment is paid within \( M_i' \).
- If payment is paid within \( M_i \), when the account is settled the retailer starts paying for the interest charges on the items in stock. If payment is paid behind \( M_i \), as the order is received, the retailer must make a partial payment \( \alpha c DT \) to the supplier. Then the retailer must pay off the remaining balance \( (1-\alpha) c DT \) at the end of the partially permissible delay period \( M_i' \).
- During the time the account is not settled, generated sales revenue is deposited in an interest-bearing account. When the account is settled the retailer pays off all units sold and keeps his/her profits, and starts paying for the higher interest charges on the items in stock.

**The model:**
- The annual total relevant cost consists of the following elements.

  - Annual ordering cost = \( \frac{A}{T} \).
  - Annual stock holding cost (excluding interest charges) = \( \frac{DTh}{2} \).
  - Annual purchasing cost:
    Since the supplier offers a cash discount if payment is paid within \( M_i \), there are two payment policies for the retailer.

**Case 1:** Payment is paid at time \( M_i \), the annual purchasing cost = \( c(1-r)D \).

**Case 2:** Payment is paid at time \( M_i' \), the annual purchasing cost = \( cD \).

**Case 1:** Payment is paid at time \( M_i \), according to assumptions (7) and (8); there are two sub-cases in terms of annual opportunity cost of the capital.

Case 1.1: \( T \geq M_i \).

Annual opportunity cost of the capital =

\[ c(1-r)l_1 D(T - M_i)^2 / 2T - s l_2 \left( \frac{D M_i^2}{2} \right) / T \]
Case 1.2: $T < M_i$

In this case, annual opportunity cost of the capital

$$-s M_i \left( \frac{DT^2}{2} + DT(M_i - T) \right) / T = -s M_i \left( \frac{DT^2}{2} + DT \frac{T}{2} \right) / T$$

Case 2: Payment is paid at time $M_i$, according to assumptions (7) and (8), there are three sub-cases in terms of annual opportunity cost of the capital.

Case 2.1: $\frac{M_i}{\alpha} \leq T$, (Fig. 1 and 2).

Annual opportunity cost of the capital

$$= c M_i \left( \frac{DT^2}{2} - (1 - \alpha)DTM_i \right) / T - s M_i \left( \frac{(1 - \alpha)DTM_i^2}{2} \right) / T$$

Case 2.2: $\frac{M_i}{\alpha} \leq T < \frac{M_i}{\alpha}$, (Fig. 3 and 4).

Annual opportunity cost of the capital
\[ \text{Case 2: Payment is paid at time } M_i \]
\[
\begin{align*}
\text{Case 2: Payment is paid at time } M_i \quad &\begin{cases} 
\text{TRC}_{2i}(T) & \text{if } T \geq \frac{M_i}{\alpha}, \\
\text{TRC}_{2i}(T) & \text{if } M_i \leq T < \frac{M_i}{\alpha}, \\
\text{TRC}_{2i}(T) & \text{if } 0 < T < M_i
\end{cases} \quad (4a) \\
&
\end{align*}
\]

where
\[
\begin{align*}
\text{TRC}_{2i}(T) &= \frac{A}{T} + \frac{DTh}{2} + cD + cI_i \frac{DT^2}{2} - (1 - \alpha_i)DTM_i \\
&\quad / T - sI_i \frac{(1 - \alpha_i)DM_i^2}{2} / T \\
&\quad \text{and } T - sI_i \frac{DM_i(M_i - \alpha_i T)}{2} / T \\
&\quad \text{and } T - sI_i \frac{DM_i(M_i - \alpha_i T)}{2} / T
\end{align*}
\]

Since \( \text{TRC}_{3i}(M_i) = \text{TRC}_{2i}(M_i) = \text{TRC}(T) \) and \( \text{TRC}_{2i}(M_i) \) is continuous and well-defined. All \( \text{TRC}_{2i}(T), \text{TRC}_{2i}(T), \text{TRC}_{2i}(T) \) and \( \text{TRC}_{2i}(T) \) are defined on \( T > 0 \).

**Optimality conditions:** From Eq. 2, 3, 5-7 yield

\[
\begin{align*}
\text{TRC}_{1i}(T) &= -\frac{(2A + DM_i^2(\alpha(1-r)I_k - sI_k))}{2T} \\
&\quad + \frac{D[2 + \alpha(1-r)I_k]}{2T} \\
&\quad + \frac{\alpha(1-r)I_k (D(T - M_i))^2}{2T} - sI_i DM_i^2 \quad (2)
\end{align*}
\]

\[
\begin{align*}
\text{TRC}_{1i}(T) &= -\frac{(2A + DM_i^2(\alpha(1-r)I_k - sI_k))}{2T} \\
&\quad + \frac{D[2 + \alpha(1-r)I_k]}{2T} \\
&\quad + \frac{\alpha(1-r)I_k (D(T - M_i))^2}{2T} - sI_i DM_i^2 \quad (2)
\end{align*}
\]

At \( T = M_i \), we find \( \text{TRC}_{1i}(M_i) = \text{TRC}_{1i}(M_i) \). Hence \( \text{TRC}_{1i}(T) \) is continuous and well-defined. All \( \text{TRC}_{1i}(T) \), \( \text{TRC}_{1i}(T) \) and \( \text{TRC}_{1i}(T) \) are defined on \( T > 0 \).
\[ \text{TRC}_{21}''(T) = \frac{2A - s(1-\alpha)DM_2^3I_x}{T^3}, \quad (13) \]
\[ \text{TRC}_{22}''(T) = \frac{[2A + DM_2^2(cl_k - sl_x)] + D[h + cl_k(1+\alpha^2)]}{2T^3}, \quad (14) \]
\[ \text{TRC}_{21}''(T) = \frac{2A + DM_2^2(cl_k - sl_x)}{T^3}, \quad (15) \]
\[ \text{TRC}_{22}''(T) = -\frac{A}{T^3} + \frac{D[h + \alpha\sigma^2l_k + s(1+\alpha^2)l_x]}{2}, \quad (16) \]
and
\[ \text{TRC}_{22}''(T) = \frac{2A}{T^3} > 0. \quad (17) \]

Equation 11 and 17 imply that all TRC\(_{21}(T)\) and TRC\(_{22}(T)\) are convex on \( T > 0 \). Equation 9 implies TRC\(_{11}(T)\) is convex on \( T > 0 \) if \( 2A + DM_2^2[\alpha(1-r)l_k - sl_x] > 0 \).

Equation 13 implies TRC\(_{22}(T)\) is convex on \( T > 0 \) if \( 2A - s(1-\alpha)DM_2^3I_x > 0 \). Equation 15 implies TRC\(_{21}(T)\) is convex on \( T > 0 \) if \( 2A + DM_2^2[cl_k - sl_x] > 0 \).

Furthermore, we have \( \text{TRC}_{21}''(M_1) = \text{TRC}_{12}''(M_1) \), \( \text{TRC}_{21}''(\frac{M_1}{\alpha}) = \text{TRC}_{22}''(\frac{M_1}{\alpha}) \) and \( \text{TRC}_{22}''(M_2) = \text{TRC}_{21}''(M_2) \).

Therefore, Eq. 1a,b imply that TRC\(_{21}(T)\) is convex on \( T > 0 \) if \( 2A + DM_2^2[\alpha(1-r)l_k - sl_x] > 0 \) and Eq. 4a-c imply that TRC\(_{22}(T)\) is piecewise convex but not convex in general on \( T > 0 \) if \( 2A - s(1-\alpha)DM_2^3I_x > 0 \).

**DECISION RULE OF THE OPTIMAL CYCLE TIME T* AND OPTIMAL PAYMENT TIME**

Let \( \text{TRC}_{21}''(T_{11}^*) = \text{TRC}_{12}''(T_{11}^*) \), \( \text{TRC}_{21}''(T_{12}^*) = \text{TRC}_{22}''(T_{12}^*) = 0 \). We can obtain

\[ T_{11}^* = \sqrt[3]{\frac{2A + DM_2^2[\alpha(1-r)l_k - sl_x]}{D[h + cl_k(1+\alpha^2)]}} \]
if \( 2A + DM_2^2[\alpha(1-r)l_k - sl_x] > 0 \) \quad (18)

\[ T_{11}^* = \sqrt[3]{\frac{2A}{D[h + cl_k(\alpha^2)]}} \]
if \( 2A + DM_2^2[\alpha(1-r)l_k - sl_x] < 0 \) \quad (19)

\[ T_{21}^* = \sqrt[3]{\frac{2A - s(1-\alpha)DM_2^3I_x}{D[h + cl_k(1+\alpha^2)]}} \]
if \( 2A - s(1-\alpha)DM_2^3I_x > 0 \) \quad (20)

\[ T_{21}^* = \sqrt[3]{\frac{2A + DM_2^2[cl_k - sl_x]}{D[h + cl_k(1+\alpha^2)]}} \]
if \( 2A + DM_2^2[cl_k - sl_x] > 0 \) \quad (21)

and

\[ T_{21}^* = \sqrt[3]{\frac{2A}{D[h + \alpha\sigma^2l_k + s(1+\alpha^2)l_x]}} \]
\quad (22)

Equation 18 implies that the optimal value of \( T \) for the case of \( T > M_1 \), that is \( T_{11}^* > M_1 \). We substitute Equation 18 into \( T_{11}^* > M_1 \) then we can obtain the optimal value of \( T \) if and only if \( -2A + DM_2^2(\alpha^2l_x) \leq 0 \).

Likewise, Eq. 19 implies that the optimal value of \( T \) for the case of \( T < M_2 \), that is \( T_{12}^* < M_2 \). We substitute Eq. 19 into \( T_{12}^* < M_2 \) then we can obtain the optimal value of \( T \) if and only if \( -2A + DM_2^2(\alpha^2l_x) > 0 \).

Similar discussion, we can obtain following results:

\[ M_2/\alpha \leq T_{11}^* \] if and only if
\[ -2A + DM_2^2[\alpha(1-r)l_k - sl_x] \leq 0 \]

\[ M_2 \leq T_{12}^* < M_2/\alpha \] if and only if
\[ -2A + DM_2^2(\alpha^2l_x) > 0 \] and
\[ -2A + DM_2^2(\alpha^2l_x + h + cl_k) \leq 0 \]

Furthermore, we let

\[ \Delta_{i1} = -2A + DM_2^2(h + sl_x), \]
\[ \Delta_{i1} = -2A + DM_2^2[\alpha(sl_x + \frac{h + cl_k}{\alpha^2})] \]
\quad (23)

\quad (24)
\[ \Delta_{22} = -2A + DM_{22} \left( s_i + \frac{h + c_i k_i}{\alpha^2} \right), \]  

(25)

\[ \Delta_{21} = -2A + DM_{21} (h + \alpha^2 I_k + s_i I_i), \]  

(26)

and

\[ \Delta_{24} = -2A + DM_{24} [h + \alpha^2 I_k + s(1 + \alpha^2) I_i]. \]  

(27)

Since \( M_i < M_j \), we can get \( \Delta_{22} > \Delta_{21} > \Delta_{20} > \Delta_{23} = \Delta_{24} \) from Eq. 23-27. Summarized above arguments, the optimal cycle time \( T^* \) and optimal payment time \( (T_i, T_j) \) can be obtained as follows.

**Theorem 1:**

(A) If \( \Delta_{22} < 0, \Delta_{21} > 0, \Delta_{20} > 0 \) and \( \Delta_{23} > 0 \), then \( TRC(T^*) = \min \{ TRC_i(T_{i1}^*), TRC_2(T_{12}^*) \} \). Hence \( T^* = T_{i1}^* \) or \( T_{12}^* \), optimal payment time is \( T_i \) or \( T_j \) associated with the least cost.

(B) If \( \Delta_{22} < 0, \Delta_{21} > 0, \Delta_{20} > 0 \) and \( \Delta_{23} > 0 \), then \( TRC(T^*) = \min \{ TRC_i(T_{i1}^*), TRC_2(T_{12}^*) \} \). Hence \( T^* = T_{i1}^* \) or \( T_{12}^* \), optimal payment time is \( T_i \) or \( T_j \) associated with the least cost.

(C) If \( \Delta_{22} < 0, \Delta_{21} > 0, \Delta_{20} > 0 \) and \( \Delta_{23} > 0 \), then \( TRC(T^*) = \min \{ TRC_i(T_{i1}^*), TRC_2(T_{12}^*) \} \). Hence \( T^* = T_{i1}^* \) or \( T_{12}^* \), optimal payment time is \( T_i \) or \( T_j \) associated with the least cost.

(D) If \( \Delta_{22} < 0, \Delta_{21} > 0, \Delta_{20} > 0 \) and \( \Delta_{23} > 0 \), then \( TRC(T^*) = \min \{ TRC_i(T_{i1}^*), TRC_2(T_{12}^*) \} \). Hence \( T^* = T_{i1}^* \) or \( T_{12}^* \), optimal payment time is \( T_i \) or \( T_j \) associated with the least cost.

(E) If \( \Delta_{22} < 0, \Delta_{21} > 0, \Delta_{20} > 0 \) and \( \Delta_{23} > 0 \), then \( TRC(T^*) = \min \{ TRC_i(T_{i1}^*), TRC_2(T_{12}^*) \} \). Hence \( T^* = T_{i1}^* \) or \( T_{12}^* \), optimal payment time is \( T_i \) or \( T_j \) associated with the least cost.

(F) If \( \Delta_{22} < 0, \Delta_{21} > 0, \Delta_{20} > 0 \) and \( \Delta_{23} > 0 \), then \( TRC(T^*) = \min \{ TRC_i(T_{i1}^*), TRC_2(T_{12}^*) \} \). Hence \( T^* = T_{i1}^* \) or \( T_{12}^* \), optimal payment time is \( T_i \) or \( T_j \) associated with the least cost.

(G) If \( \Delta_{22} < 0, \Delta_{21} > 0, \Delta_{20} > 0 \) and \( \Delta_{23} > 0 \), then \( TRC(T^*) = \min \{ TRC_i(T_{i1}^*), TRC_2(T_{12}^*) \} \). Hence \( T^* = T_{i1}^* \) or \( T_{12}^* \), optimal payment time is \( T_i \) or \( T_j \) associated with the least cost.

(H) If \( \Delta_{22} < 0, \Delta_{21} > 0, \Delta_{20} > 0 \) and \( \Delta_{23} > 0 \), then \( TRC(T^*) = \min \{ TRC_i(T_{i1}^*), TRC_2(T_{12}^*) \} \). Hence \( T^* = T_{i1}^* \) or \( T_{12}^* \), optimal payment time is \( T_i \) or \( T_j \) associated with the least cost.

(I) If \( \Delta_{22} < 0, \Delta_{21} > 0, \Delta_{20} > 0 \) and \( \Delta_{23} > 0 \), then \( TRC(T^*) = \min \{ TRC_i(T_{i1}^*), TRC_2(T_{12}^*) \} \). Hence \( T^* = T_{i1}^* \) or \( T_{12}^* \), optimal payment time is \( T_i \) or \( T_j \) associated with the least cost.

(J) If \( \Delta_{22} < 0, \Delta_{21} > 0, \Delta_{20} > 0 \) and \( \Delta_{23} > 0 \), then \( TRC(T^*) = \min \{ TRC_i(T_{i1}^*), TRC_2(T_{12}^*) \} \). Hence \( T^* = T_{i1}^* \) or \( T_{12}^* \), optimal payment time is \( T_i \) or \( T_j \) associated with the least cost.

**CONCLUSIONS**

This study extends Chou et al.'s model to assume that the unit selling price and unit purchasing price are not necessarily equal. This article reinvestigates the retailer's replenishment and payment policy under supplier offered cash discount and permissible delay in payments linked to retailer payment time. Then, we reformulate the retailer's mathematical inventory replenishment model and provide a very efficient solution procedure to determine the optimal cycle time and optimal payment time for the retailer.

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