Action of Subgroups of $G = \langle x, y; x^2 = y^4 = 1 \rangle$ on $Q^*(\sqrt{n})$

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Abstract: The study presents the action of subgroups of $G = \langle x, y; x^2 = y^4 = 1 \rangle$ on $Q^*(\sqrt{n})$ and discuss some number-theoretic properties of totally positive, totally negative and ambiguous numbers belonging to the orbit of subgroups $H_1 = \langle t, y; t^4 = 1 \rangle$, where $t = xy^2x$ and $H_2 = \langle t, y; y^4 = 1 \rangle$, where $t = xy^2x$, acting on $Q^*(\sqrt{n})$ and also compare our results obtained from $H_1$ with subgroup $H_3 = \langle t, y; t^4 = 1 \rangle$, where $t = xyx$.

Key words: Coset diagram, totally positive numbers, totally negative numbers, ambiguous numbers

INTRODUCTION

The group $G$ is $\langle x, y; x^2 = y^4 = 1 \rangle$, where $(ax) = -1/2a$, $(ay) = -1/(2a+1)$. The groups $H_1 = \langle t, y; t^4 = 1 \rangle$, where $(at) = (a)xyx = 1/2a$ and $(ay) = -1/(2a+1)$, $H_2 = \langle t, y; y^4 = 1 \rangle$, where $(at) = (a)xyx = 1/2a+1$ and $H_3 = \langle t, y; t^4 = 1 \rangle$, where $(at) = (a)xyx = -1/(2a+1)$ are thus subgroups of $G$.

Action of the subgroup $H_1 - \langle t, y; t^4 = 1 \rangle$, where $t = xyx$, of $G$ on $Q^*(\sqrt{n})$ has been discussed (Aslam, 1997) which is in fact the main inspiration of our study.

In this study we are interested in the action of subgroups $H_2 = \langle t, y; y^4 = 1 \rangle$ and $H_3 = \langle t, y; t^4 = 1 \rangle$ of $G - \langle x, y; x^2 = y^4 = 1 \rangle$ on $Q^*(\sqrt{n})$

We recollect that a quadratic irrational number $\alpha = \frac{a + \sqrt{n}}{c}$ has its algebraic conjugate $\bar{\alpha} = \frac{a - \sqrt{n}}{c}$ in $Q^*(\sqrt{n})$. If $\alpha$ and $\bar{\alpha}$ have different sign then $\alpha$ is called an ambiguous number. If they are both negative, then we call $\alpha$ is totally negative number and if $\alpha$ and $\bar{\alpha}$ both are positive then $\alpha$ is called totally positive number.

ACTION OF $H_1$ ON $Q^*(\sqrt{n})$

Let $\alpha \in Q^*(\sqrt{n})$. The transformation $t$ defined as $(at) = (a)xyx = (a) = (a) + (2a-1)$ has order 2 and $(ay) = -1/(2a+1)$ has order 4, then for each $\alpha$, $(a) = (a)(a)(a)(a)$ and $(a) = (a)$ are vertices of a small square. If $\alpha$ is totally positive number, then all of $(a)(a)$, $(a)(a)$, $(a)(a)$ are totally negative numbers (Mushtaq and Aslam, 1993). However, as we observe in our study that if $\alpha$ is totally negative number, then $(a)(a)$ is totally positive number and when $\alpha$ is totally positive number, then $(a)(a)$ may or may be not totally negative in general. Also we observe that if $\alpha$ is an ambiguous number, then $(a)(a)$ is not ambiguous number. Before proving our result, we reproduce the following results (Kausar et al., 1997) which are being used in our discussions.

Lemma 1: (Kausar et al., 1997) An $\alpha = \frac{a + \sqrt{n}}{c} \in Q^*(\sqrt{n})$ is totally positive if and only if either $a$, $b$, $c<0$ or $a$, $b$, $c<0$

Lemma 2: (Kausar et al., 1997) An $\alpha = \frac{a + \sqrt{n}}{c} \in Q^*(\sqrt{n})$ is totally negative number if and only if either $c$, $b<0$ and $a<0$ or $c$, $b<0$ and $a>0$

Lemma 3: (Kausar et al., 1997) An $\alpha = \frac{a + \sqrt{n}}{c} \in Q^*(\sqrt{n})$ is an ambiguous number if and only if either $c>0$ and $b<0$ or $c<0$ and $b>0$

Theorem 4: (Mushtaq and Aslam, 1993) If $\alpha = \frac{a + \sqrt{n}}{c}$ is a totally positive real quadratic irrational number, then $(\alpha)(a)$ for $n \leq 3$, are totally negative numbers.

Theorem 5: If $\alpha = \frac{a + \sqrt{n}}{c} \in Q^*(\sqrt{n})$, is totally negative real number, then $(\alpha)(a)$ is totally positive number.

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Proof: Let \( \alpha = \frac{a + \sqrt{n}}{c} \in Q^*(\sqrt{n}) \) is totally negative number. We are to prove that \( (\alpha) \) t is totally positive number. For this, we consider

\[
(\alpha) t = \frac{\alpha - 1}{2 \alpha - 1} = \frac{(2b - 3a + c) + \sqrt{n}}{4b - 4a + c} = a_t + \sqrt{n}
\]

(say)

where \( a_t = 2b - 3a + c \), \( c_t = 4b - 4a + c \) and \( b_t = (a_t^2 - n)/c_t = (2b - 3a + c)^2 - n/4b - 4a + c = 4b^2 + 8ac - 3a^2 + 12ab - 6ac + 4bc - a^2 - n/4b - 4a + c = 4b^2 + 5bc - 12ab - 6ac + 8a^2 + c^2/4b - 4a + c \)

By lemma 2, if \( \alpha = \frac{a + \sqrt{n}}{c} \) is totally negative number, then either \( c > 0 \) and \( a < 0 \) or \( c > 0 \) and \( b < 0 \) and \( a > 0 \). This gives that

- \( a_t, b_t, c_t > 0 \) (In case of \( c > 0 \) and \( a < 0 \))
- \( a_t, b_t, c_t > 0 \) (In case of \( c > 0 \) and \( b < 0 \) and \( a > 0 \))

But, then, by lemma 1 \( (\alpha) t \) is totally positive number. Hence the proof

Theorem 6: Let \( \alpha = \frac{a + \sqrt{n}}{c} \in Q^*(\sqrt{n}) \) be an ambiguous number, then \( (\alpha) t \) is not an ambiguous number.

Proof: Note that if \( \alpha = \frac{a + \sqrt{n}}{c} \) is an ambiguous number then, by definition either \( c > 0 \), \( b > 0 \) or \( a > 0 \), \( b > 0 \), that is \( b < 0 \). Thus \( b - (a^2 - n)/c \) implies that \((a - n)/c \). Which implies that \( a > 0 \) or \( a < 0 \). Using the calculations of Theorem 5, we have

\[
a_t = 2b - 3a + c, c_t = 4b + c - a, b_t = a_t^2 - n/c_t = 4b^2 + 5bc - 12ab - 6ac + 8a^2 + c^2/c_t,
\]

Following are the possible cases.

Case 1(a): \( a, b \) both are negative and \( c \) is positive \((c > 0, b < 0, a < 0)\)

Then \(-3a + c < 0 \) and \( 2b < 0 \), giving that \( a_t = 2b - 3a + c \), either positive or negative. If \( a_t > 0 \), then \(-3a + c + 2b < 0 \) this implies that \(-4a + c + 4b < 0 \) or \( 4b > -4a > 0 \). Since \( c_t = 4b + c - a \) therefore \( c_t > 0 \). But we are not sure about \( b_t \), whether it is positive or negative, we conclude that \( (\alpha) t \) is not an ambiguous number. For \( a_t > 0 \) we have that \(-3a < -2b \). Since \(-4a + c + 4b < 0 \). Therefore either \(-4a + c + 4b < 0 \) or \(-4a + c + 4b > 0 \) that is \( c_t < 0 \) or \( c_t < 0 \). But we are not sure about \( b_t \), whether it is positive or negative. So, we conclude that \( (\alpha) t \) is not an ambiguous number.

Case 1(b): \( a, c \) both are positive and \( b \) is negative \((c > 0, b < 0, a > 0)\)

Here note that 2 b - 3 a < 0 and \( c > 0 \) this implies that

\[
2b - 3a + c < 0 \text{ or } 2b - 3a + c > 0, \text{ that is a } 1 > 0 \text{ or a } 1 < 0
\]

For \( b_t > 0 \), we have \( c_t > 0 \). But we are not sure about \( b_t \), whether it is positive or negative. So we conclude that \( (\alpha) t \) is not an ambiguous number.

Similarly for \( a_t < 0 \), we have \( c < 2 b - 3 a \) which implies that \( c < 4 b - 4 a \) that is \( c_t < 0 \). But this does not implies that \( b_t > 0 \) or \( b_t < 0 \). Therefore we conclude again that \( (\alpha) t \) is not an ambiguous number.

Case 2(a): \( a, c \) are negative and \( b \) is positive \((c < 0, b > 0, a < 0)\)

In this case we note that 2 b - 3 a < 0 and \( c < 0 \) which implies that 2 b - 3 a + c < 0 or 2 b - 3 a + c > 0. That is either \( a_t \) is positive or \( a_t \) is negative. For \( a_t > 0 \) we have 4 b - 4 a + c. That is \( c_t > 0 \) and for \( a_t < 0 \) we have 4 b - 4 a + c that is \( c_t < 0 \). But this does not implies that \( b_t > 0 \) or \( b_t < 0 \). Therefore we conclude that \( (\alpha) t \) is not an ambiguous number.

Case 2(b): \( a, b \) are positive and \( c \) is negative \((c < 0, b > 0, a > 0)\)

We note that -3 a + c < 0 and 2 b > 0 which implies that 2 b - 3 a + c < 0 or 2 b - 3 a + c > 0. That is \( a_t > 0 \) or \( a_t < 0 \) for the case when \( a_t > 0 \) we have 4 b - 4 a + c which implies that 4 b - 4 a + c < 0. That is \( c_t < 0 \). Therefore \( a_t > 0 \) we have 4 b - 3 a + c < 0 or 4 b - 3 a + c > 0, that is \( a_t < 0 \) or \( b_t > 0 \). In both the cases we are sure, whether \( b_t > 0 \) or \( b_t < 0 \). Therefore we conclude that \( (\alpha) t \) is not an ambiguous number.

We observe that \( (\alpha) t \) is not an ambiguous number in all the cases. Hence the theorem is proved.

Theorem 7: Let \( \alpha = \frac{a + \sqrt{n}}{c} \) be totally positive number.

- If \( (\alpha) t \) is totally positive number, then either \( |2b + c| > 3a, |4b + c| > 4a \) or \( |2b + c| < 3a, |4b + c| < 4a \) and
- If \( (\alpha) t \) is totally negative number, then either \( |2b + c| > 3a, |4b + c| < 4a \) or \( |2b + c| < 3a, |4b + c| > 4a \)

Proof: Let \( \alpha = \frac{a + \sqrt{n}}{c} \) is totally positive number, then by lemma 1 either \( a, b, c < 0 \) or \( a, b, c > 0 \). Since we know that

\[
(\alpha) t = (\alpha - 1)/2\alpha - 1 = \frac{a + \sqrt{n}}{c_t}, \text{ where } a_t = 2b + 3a, c_t = 4b + c - a, b_t = (a_t^2 - n)/c_t = (4b^2 + 5bc - 12a - 6ac + 8a^2 + c^2)/4b - 4a + c
\]
Suppose \((\alpha) t\) is totally positive number then, by lemma 1, either \(a_i > 0, b_i > 0, c_i > 0\) or \(a_i < 0, b_i < 0, c_i < 0\).

Now suppose that \((\alpha) t\) is totally negative then, by lemma 2, either \(a_i > 0, b_i < 0, c_i < 0\) or \(a_i < 0, b_i > 0, c_i > 0\).

Now for the case of \(a_i > 0, b_i > 0, c_i < 0\) we observe that

\[
\begin{align*}
a_i &= 2b+c < 3 \Rightarrow 2b+c < 3 \quad (a) \\
2a_i &
\end{align*}
\]

and for \(a_i < 0, b_i < 0, c_i < 0\), we observe that

\[
\begin{align*}
a_i &= 2b+c < 3 \Rightarrow 2b+c < 3 \quad (a') \\
a_i &
\end{align*}
\]

From (a) and \((a')\) we see that \(a_i > 0\) only when

\[
|2b+c| > 3
\]

From (b) and \((b')\) we see that \(a_i < 0\) only when

\[
|2b+c| < 3
\]

From (c) and \((c')\) we see that \(b_i > 0\) only when

\[
|4b+c| > 4
\]

From (d) and \((d')\) we see that \(b_i < 0\) only when

\[
|4b+c| < 4
\]

When \((\alpha) t\) is totally positive we have either

\[
\begin{align*}
\begin{cases}
|2b+c| > 3 \text{ and } |4b+c| > 4 & \text{(In case of } a_i > 0, b_i > 0, c_i > 0) \\
|2b+c| = 3 \text{ and } |4b+c| > 4 & \text{(In case of } a_i > 0, b_i > 0, c_i < 0) \\
|2b+c| > 3 \text{ and } |4b+c| < 4 & \text{(In case of } a_i < 0, b_i < 0, c_i > 0) \\
|2b+c| = 3 \text{ and } |4b+c| < 4 & \text{(In case of } a_i < 0, b_i < 0, c_i < 0)
\end{cases}
\end{align*}
\]

\[
(A)
\]

When \((\alpha) t\) is totally negative we have either

\[
\begin{align*}
\begin{cases}
|2b+c| > 3 \text{ and } |4b+c| < 4 & \text{(In case of } a_i > 0, b_i > 0, c_i > 0) \\
|2b+c| = 3 \text{ and } |4b+c| < 4 & \text{(In case of } a_i > 0, b_i > 0, c_i < 0) \\
|2b+c| > 3 \text{ and } |4b+c| > 4 & \text{(In case of } a_i < 0, b_i < 0, c_i > 0) \\
|2b+c| = 3 \text{ and } |4b+c| > 4 & \text{(In case of } a_i < 0, b_i < 0, c_i < 0)
\end{cases}
\end{align*}
\]

\[
(B)
\]

From (A) and (B) theorem is proved.

\[
\text{Coset diagram of } H_3 = \langle t, y : t^4 = y^4 = 1 \rangle \text{ where } t = x y^2 x:
\]

It has been shown (Mushtaq and Aslam, 1993) while studying the action of \(G = \langle x, y : x^4 = 1 = y^4 \rangle\) on \(Q^*(\sqrt{n})\) if \(\alpha\) is an ambiguous vertex of a square in coset diagram for \(\alpha^2\) then \((\alpha)t\) is an ambiguous number and one of the vertices \((\alpha)y, (\alpha)y^2, (\alpha)y^3, (\alpha)y^4\) is ambiguous and other two are totally negative. In other words each ambiguous number is joined by \(x\)-edge or by \(y\)-edge, to other two ambiguous numbers. Since there are finite ambiguous numbers in \(Q^*(\sqrt{n})\) (Mushtaq, 1988), therefore ambiguous vertices form a closed path in the Coset diagram.

In our case we have proved that if \(\alpha\) is an ambiguous number, then \((\alpha) t\) is not ambiguous number (Theorem 6). Hence each square which have two vertices ambiguous, will not form the closed path in the diagram. But then if \(\alpha\) is totally negative number then \((\alpha) t\) is totally positive number (Theorem 5), therefore the general fragment of coset diagram will be as shown in Fig. 1.

**Action of** \(H_3 = \langle t, y : t^4 = y^4 = 1 \rangle\) on \(Q^*(\sqrt{n})\): We have discussed some number-theoretic properties of totally positive, totally negative and ambiguous numbers belonging to the orbit of subgroup \(H_3 = \langle t, y : t^4 = 1 \rangle\), where \(t = x y^2 x\), acting on \(Q^*(\sqrt{n})\) and then compare our results obtained from \(H_3\) with subgroup \(H_1 = \langle t, y : t^4 = 1 \rangle = y^3\), where \(t = x y x\), which was discussed Aslam (1997) already. The results we have proved are;

**Theorem 8:** If \(\alpha = \frac{a + \sqrt{n}}{c} \in Q^*(\sqrt{n})\) is totally negative then \((\alpha)t\) is totally positive for each \(i = 1, 2\) or 3.

**Proof:** If \(\alpha = \frac{a + \sqrt{n}}{c} \in Q^*(\sqrt{n})\), then \((\alpha)t = -1/2(a-1) = a_1 - \sqrt{n}/c_1\) where \(a_i = a+c, b_i = c/2, c_i = -4a+2b+2\)

![Fig. 1: Coset Diagram of H_3 in which each square have two vertices ambiguous, will not form the closed path since (α) t is not ambiguous number for a an ambiguous number](image-url)
are new values of a, b, c respectively. Similarly in case of $(\alpha)^t = 1-\alpha/2$ and $(\alpha)^t = 1-2\alpha$, the new values of a, b, c are $a_t = -3a+2b+c$, $b_t = -2a+b+c$, $c_t = -4a+4b+c$ and $a_t = -a+2b$, $b_t = -4a+4b+c/2$, $c_t = 2b$, respectively.

If $\alpha = \frac{a + \sqrt{n}}{c} \in Q^*(\sqrt{n})$ is totally negative number then by lemma 2, either $a<0$ and $b>0$, $c<0$ or $a>0$ and $b<0$, $c>0$. When $a<0$ and $b>0$, $c<0$, we see that (for each case, $(\alpha)^i$, $i = 1, 2, 3$), the new values of a, b, and c are positive. Hence by lemma 1 $(\alpha)^i$ are totally positive. Similarly, when $a>0$ and $b<0$, $c<0$, we observe that the new values of a, b, and c for each $(\alpha)^i$, $i = 1, 2, 3$ are negative. Therefore by lemma 1 each $(\alpha)^i$, $i = 1, 2, 3$ is totally positive. Hence the proof.

**Theorem 9:** If $\alpha = \frac{a + \sqrt{n}}{c} \in Q^*(\sqrt{n})$ is an ambiguous number then, for $i = 1, 2, 3$ one of $(\alpha)^i$ is an ambiguous number and other two are totally positive.

**Proof:** (Case 1) Let $\alpha$ be a negative number then possibilities for signs of $\alpha = \frac{a - \sqrt{n}}{c}$, $(\alpha)^i$, $(\alpha)^i$ and $(\alpha)^i$ are given in Table 1.

(Case 2) Now consider the case when $\alpha$ is a positive number. In this case the possibilities for signs of $\alpha = \frac{a - \sqrt{n}}{c}$, $(\alpha)^i$, $(\alpha)^i$ and $(\alpha)^i$ are given in Table 2.

Therefore from the cases ((Case 1) and (Case 2)) above, we deduce that for $i = 1, 2, 3$, one of $(\alpha)^i$ is an ambiguous number and other two are totally positive. Hence the proof.

The coset diagram (Fig. 2) can also illustrate proof of the above theorem:

Further it has been observed (Mushtaq and Aslam, 1993) that if $k > 0, k < 1, 1$ or $\infty$ is one of the four vertices of a square in a coset diagram, then

- $z<0$ implies that $(z)>0$
- $z>1$ implies that $1/2<(z)<1$
- $1/2<z<1$ implies that $0<(z)<1/2$
- $0<z<1/2$ implies that $(z)<0$

that is if vertices $k, k, k, k$ of a square are not or then one of four vertices is negative and other three are positive.

In our case, we study that if $k > 0, 1/2, 1$ or $\infty$ is vertex of a square in Coset diagram, then

- $z<0$ implies that $(z)<1/2$
- $0<z<1/2$ implies that $1/2<(z)<1$
- $1/2<z<1$ implies that $(z)>1$
- $z<1$ implies that $(z)<0$

that is if vertices $k, k, k, k$ of a square are not $0, 1/2, 1$ or $\infty$, then one of these four vertices is positive and the other three are negative.

It has been studied by Aslam (1997) that if if $k > 0, 1/2, 1$ or $\infty$ is vertex of a square in Coset diagram, then

that is if vertices $k, k, k, k$ of a square are not $0, 1/2, 1$ or $\infty$, then one of these four vertices is positive and the other three are negative.

![Coset Diagram of H1 in which 4 cycles of y are defined by four edges (unbroken) and 4 cycles of the transformation t = x y^2 x are defined by four edges (broken) of a square, permuted counter clockwise by both y and t](image)

**Fig. 2: Coset Diagram of H1 in which 4 cycles of y are defined by four edges (unbroken) and 4 cycles of the transformation t = x y^2 x are defined by four edges (broken) of a square, permuted counter clockwise by both y and t**

![Table 1: Possibilities for sign of negative](image)

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**Table 2: Possibilities for sign of positive**

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that is if vertices $k, k, k, k$ of a square are not $0, 1, 2, 1$ or $\infty$, then one of these four vertices is negative and the other three are positive.
Fig. 3(a): Fragment of Coset diagram for $H_1$
(b): Fragment of Coset diagram for $H_2$

transformation $t = xy'^x$ are defined by four edges (broken) of a square, permuted counter clockwise by both $y$ and $t$. Fixed points by $y$ and $t$ are denoted by heavy dots) summarize the whole discussion. The vertices of 4 cycles of transformations $t = xy'^x$ are permuted in the opposite direction to the vertices of $t = xyx$. The general fragments of Coset diagrams for $H_1 = <t, y; t^4 = y^4 = 1>$ where $t = xyx$ and $H_2 = <t, y; t^4 = y^4 = 1>$ where $t = x y^3 x$, are illustrated in the Fig. 3(a, b).

Note that the Fig. 3(a) and 3(b) are reflections of each other.

REFERENCES


