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Travelling Wave Solutions for the Generalized Special Type of the Dodd-Bullough-Mikhailov Equation

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Abstract: In this study, the generalized special Dodd-Bullough-Mikhailov Equations $u_{tt}-u_{xx} = \alpha e^{mu} + \beta e^{-nu}$ is studied. The existence of periodic wave and unbounded wave solutions of this equation is proved by using the method of bifurcation theory of dynamical systems. Some exact explicit parametric representations of the above traveling solutions are also obtained.

Key words: Unbounded travelling wave solution, periodic travelling wave solution, the generalized special Dodd-Bullough-Mikhailov equation

INTRODUCTION

In this study, we consider the following Generalized Special Dodd-Bullough-Mikhailov (GSDBM) equation

$$u_{tt}-u_{xx}= \alpha e^{mu} + \beta e^{-nu} \tag{1}$$

where α, β are two non-zero real number, m, n≥1 are positive integer. Specially, when m = 1, n = 2 and α = β = −1, (1) is called the Dodd-Bullough-Mikhailov equation (DBM), when m = 1, n = 1 and α = β = −1/2, (1) is called the sinh-Gordon equation. This equation appears in problems varying from fluid flow to quantum field theory. Recently, by using the tanh method (Wazwaz, 2005) considered some solitary wave and periodic wave solutions for the special DBM equation. In this study, we investigate dynamical behavior of solutions of Eq. (1). To answer this question, we shall consider the bifurcations of travelling wave solutions of (1) in the five-parameter space (α, β, m, n, c).

Making the transformations u(x, t) = In v(x, t), (1) becomes

$$vv_{tt} - v_t^2 - vv_{xx} + v_x^2 = \alpha v^{m+2} + \beta v^{-n+2} \tag{2}$$

Let v(x −ct) = ϕ(x − ct) = ϕ(ξ). Substituting ϕ(x − ct) into (2), we obtain

$$(c^2-1)(\phi\phi'' - (\phi')^2) = \alpha\phi^{m+2} + \beta\phi^{-n+2} \tag{3}$$

where "/" is the derivative with respect to ξ. Clearly, (3) is equivalent to the following two-dimensional system:

$$\frac{d\phi}{d\xi} = y, \frac{dy}{d\xi} = \frac{(c^2-1)\phi^{n-2}y^2 + \alpha\phi^{m+n} + \beta}{(c^2-1)\phi^{n-1}} \tag{4}$$

System (4) has the first integrals

$$y^2 = \phi^2 h + \frac{2\alpha}{m(c^2-1)}\phi^{m+2} - \frac{2\beta}{n(c^2-1)}\phi^{2-n} \tag{5}$$

System (4) is planar dynamical systems defined in the 5-parameter space (α, β, m, n, c).

Because the phase orbits defined by the vector fields of (4) determine all travelling wave solutions, we will investigate bifurcations of phase portraits of this system as these parameters are varied.

Usually, a solitary wave solution of a non-linear wave equation corresponds to a homoclinic orbit of its travelling wave equation; a kink (or anti-kink) wave solution corresponds to a heteroclinic orbit (or connecting orbit). Similarly, a periodic orbit of a travelling wave equation corresponds to a periodic travelling wave solution of the non-linear wave equation. To find all possible bifurcations of solitary waves, periodic waves, kink and anti-kink wave of a non-linear wave equation, we need to investigate the existence of all homoclinic, heteroclinic orbits and periodic orbits for its travelling wave equation in the parameter space. In doing so, the bifurcation theory of dynamical systems (Chow and Hale, 1981) is very important and useful.

We notice that by using transformation u(x, t) = In ϕ(x, t), we make (1) and become the traveling Eq. (4). Therefore, we are only interesting the positive boundary solutions of ϕ(ξ). In addition, if a solution ϕ(ξ) of (4)

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can approach to $\phi(\xi) = 0$, then $\text{In}(\phi(\xi))$ approach to $-\infty$. In other words, this solution determines an unbounded travelling wave solution of (1).

It is easy to see that the right-hand side of the second equation in (4) is generally not continuous when $\phi(\xi) = 0$. In other words, on such straight lines in the phase plane (ϕ, y), the function ϕ''_{ξ} is not well-defined. It implies that the smooth system (1) sometimes have non-smooth travelling wave solutions. This phenomenon has been considered before by (Li and Chen, 2005; Li and Zhenrong, 2000, 2002) in which the authors had already pointed out that the existence of such a singular straight line for a travelling wave equation is the very reason why travelling waves can lose their smoothness.

BIFURCATIONS OF PHASE PORTRAITS OF SYSTEMS (4)

System (4) have the same phase orbits for the cases $n = 1$, or $n \geq 2$ as the following systems, respectively

$$\frac{d\phi}{d\tau} = (c^2 - 1)\phi y, \frac{dy}{d\tau} = (c^2 - 1)y^2 + \alpha\phi^{m+2} + \beta \tag{6}$$

and

$$\frac{d\phi}{d\tau} = (c^2 - 1)\phi^{n-1} y, \frac{dy}{d\tau} = (c^2 - 1)\phi^{n-2}y^2 + \alpha\phi^{m+n} + \beta \tag{7}$$

except for the straight line $\phi = 0$, where $d\xi = (c^2 - 1)\phi d\tau$ and $d\xi = (c^2 - 1)\phi^{n-1}d\tau$. By using (5) for $\phi \neq 0$, we define

$$H(\phi, y) = \frac{y^2}{\phi^2} - \frac{2\alpha}{m(c^2 - 1)}\phi^m + \frac{2\beta}{n(c^2 - 1)}\phi^{-n} = h \tag{8}$$

Without loss of generality, we can assume that $c^2 - 1 > 0$. We see from (6) and (7) that for the equilibrium points of these two systems, the following conclusions hold.

- When $n = 1$, $m = 2k$, $k \in Z^+$, system (6) has two equilibrium points at $A_+(\phi_+, 0)$ and $S(0, 0)$, where

$$\phi_+ = \left(-\frac{\beta}{\alpha}\right)^{\frac{1}{m+1}}.$$

- When $n = 1$, $m = 2k - 1$, $k \in Z^+$, if $\alpha\beta < 0$, (6) has three equilibrium points at $A_\pm(\phi_\pm, 0)$ and $S(0, 0)$, where

$$\phi_+ = \left(-\frac{\beta}{\alpha}\right)^{\frac{1}{m+1}};$$

  if $\alpha\beta > 0$, (6) has one equilibrium point at $S(0, 0)$.
- When $n = 2$, $m = 2k - 1$, $k \in Z^+$, if $(c^2 - 1)\beta < 0$, (6) has three equilibrium points at $A_+(\phi_+, 0)$ and $S_\pm(0, Y_\pm)$, where

$$\phi_+ = \left(-\frac{\beta}{\alpha}\right)^{\frac{1}{m+2}}, \; Y_\pm = \pm\sqrt{\frac{-\beta}{c^2 - 1}}; \text{ if } (c^2 - 1)\beta > 0,$$

  (7) has one equilibrium point at $A_+(\phi_+, 0)$.
- When $n = 2$, $m = 2k$, $k \in Z^+$, if $\alpha\beta < 0$, $(c^2 - 1)\beta < 0$, (7) has four equilibrium points at $A_\pm(\phi_\pm, 0)$ and $S_\pm(0, Y_\pm)$, where

$$\phi_+ = \left(-\frac{\beta}{\alpha}\right)^{\frac{1}{m+2}}, \; Y_\pm = \pm\sqrt{\frac{-\beta}{c^2 - 1}};$$

  if $\alpha\beta > 0$, $(c^2 - 1)\beta < 0$, (6) has two equilibrium points at $S_\pm(0, Y_\pm)$; if $\alpha\beta > 0$, $(c^2 - 1)\beta > 0$, (7) has no equilibrium point.
- When $n \geq 3$, $m + n = 2k + 1 \geq 5$, $k \in Z^+$, (7) has one equilibrium point at $A_+(\phi_+, 0)$, where

$$\phi_+ = \left(-\frac{\beta}{\alpha}\right)^{\frac{1}{m+n}}.$$

- When $n \geq 3$, $m + n = 2k \geq 4$, $k \in Z^+$, if $\alpha\beta < 0$, (7) has two equilibrium points at $A_\pm(\phi_\pm, 0)$, where

$$\phi_+ = \left(-\frac{\beta}{\alpha}\right)^{\frac{1}{m+n}};$$

  if $\alpha\beta > 0$, (7) has no equilibrium point.

Let $M_1(\phi_i, y_i)$ and $M_2(\phi_i, y_i)$ be the coefficient matrix of the linearized system of (6) and (7) at an equilibrium point (ϕ_i, y_i), respectively. Then we have Trace $(M_1(\phi_\pm, 0)) = 0$ and $J_1(M_1(\phi_\pm, 0)) = \det M_1(\phi_\pm, 0)) = (c^2 - 1)(m + 1)\beta\phi_\pm$, $J_1(M_1(0, 0)) = 0$. For $n = 2$, we have Trace $M_2(\phi_\pm, 0) = 0$, Trace $(M_2(0, \pm Y_+)) = 0$ and $J_2(M_2(\phi_\pm, 0)) = \det(M_2(\phi_\pm, 0)) = (c^2 - 1)(m + 2)\beta$, $J_2(M_2(0, \pm Y_+)) = 2(c^2 - 1)^2 Y_+^2$. For $n \geq 3$, we have, Trace $(M_2(\phi_\pm, 0))$ and $J_2(M_2(\phi_\pm, 0)) = \det(M_2(\phi_\pm, 0)) = \beta(c^2 - 1)^2(m + 2)(\pm\phi_+)^{n-2}$.

By the theory of planar dynamical systems, we know that for an equilibrium point (ϕ_i, y_i), of a planar integrable system, if $J < 0$ then the equilibrium point is a saddle point; if J > 0 and Trace $(M(\phi_i, y_i) = 0$ then it is a center point; if J > 0 and (Trace $(M(\phi_i, y_i)))^2 - 4J(M(\phi_i, y_i)) > 0$ then it is a node; if J = 0 and the index of the equilibrium point is zero then it is a cusp; if J = 0 and the index of the equilibrium point is not zero then it is a high order equilibrium point.

For $H(\phi, y)$ defined by (8), we have

$$h_1 = H_1(\pm\phi_+, 0) = -\frac{2\beta(m+n)}{mn(c^2-1)}(\phi_+)^{-n}.$$

For a fixed h the level curve $H(\phi, y) = h$ defined by (8) determines a set of invariant curves of (6) and (7), except for the straight line $\phi = 0$, which contains different branches of curves. As h is varied, it defines different families of orbits of (6) and (7) with different dynamical behaviors.

From the above analysis we obtain the different phase portraits of (4) shown in Fig. 1 (k, l ≥ 1).

EXACT EXPLICIT TRAVELLING WAVE SOLUTIONS OF (1) FOR m = 1, n = 1 OR m = 1, n = 2 OR m = 2, n = 1 OR m = 2, n = 2

For m = n = 1, (1) becomes

$$u_{tt} - u_{xx} = \alpha e^{u} + \beta e^{-u} \tag{9}$$

When $\alpha = \beta = -1/2$, (9) was called the sinh-Gordon equation by Wazwaz (2005). In this case, (4) have the following forms:

$$\frac{d\phi}{d\xi} = y, \frac{dy}{d\xi} = \frac{(c^2-1)y^2 + \alpha\phi^3 + \beta\phi}{(c^2-1)\phi} \tag{10}$$

with the first integrals

$$y^2 = \phi^2 h + \frac{2\phi(\alpha\phi^2 - \beta)}{(c^2-1)}, H(\phi, y) = \frac{y^2}{\phi^2} - \frac{2\alpha}{c^2-1}\phi + \frac{2\beta}{(c^2-1)\phi} = h \tag{11}$$

For m = 1, n = 2, (1) becomes

$$u_{tt} - u_{xx} = \alpha e^{u} + \beta e^{-2u} \tag{12}$$

When $\alpha = \beta = -1$, (12) was called the special (DBM) equation by Wazwaz (2005). In this case, (4) have the following forms:

$$\frac{d\phi}{d\xi} = y, \frac{dy}{d\xi} = \frac{(c^2-1)y^2 + \alpha\phi^3 + \beta}{(c^2-1)\phi} \tag{13}$$

with the first integrals

$$y^2 = \phi^2 h + \frac{2\alpha\phi^3 - \beta}{(c^2-1)}, H(\phi, y) = \frac{y^2}{\phi^2} - \frac{2\alpha}{c^2-1}\phi + \frac{\beta}{(c^2-1)\phi^2} = h \tag{14}$$

For m = 2, n = 1, (1) becomes

$$u_{tt} - u_{xx} = \alpha e^{2u} + \beta e^{-u} \tag{15}$$

In this case, (4) have the following forms:

$$\frac{d\phi}{d\xi} = y, \frac{dy}{d\xi} = \frac{y^2}{\phi} + \frac{\alpha\phi^3 + \beta}{c^2-1} \tag{16}$$

with the first integrals

$$y^2 = \phi^2 h + \frac{\phi(\alpha\phi^3 - 2\beta)}{c^2-1}, H(\phi, y) = \frac{y^2}{\phi^2} - \frac{\alpha\phi^3 - 2\beta}{(c^2-1)\phi} = h \tag{17}$$

For m 2, n = 2, (1) becomes

$$u_{tt} - u_{xx} = \alpha e^{2u} + \beta e^{-2u} \tag{18}$$

In this case, (4) have the following forms:

$$\frac{d\phi}{d\xi} = y, \frac{dy}{d\xi} = \frac{(c^2-1)y^2 + \alpha\phi^4 + \beta}{(c^2-1)\phi} \tag{19}$$

with the first integrals

$$H(\phi, y) = \frac{y^2}{\phi^2} - \frac{\alpha\phi^2}{c^2-1} + \frac{\beta}{(c^2-1)\phi^2} = h \tag{20}$$

By using (11) and (14) and (17) and (20) and the first equations of (10) and (13) and (16) and (19) to do integrations, we can obtain some exact explicit parametric representations for the breaking wave solutions and periodic wave solutions of (9) and (12) and (15) and (18). Because the singular straight line $\phi = 0$ intersects at two node points with other orbits of (11) and (14) and (17) and (20), so that, corresponding to these orbits, the travelling wave solutions of (9) and (12) and (15) and (18) is breaking waves.

Unbounded wave solutions of (9) and (12) and (15) and (18): For system (10), when $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$ or $\alpha\beta < 0$, $\beta(c^2 - 1) > 0$ (Fig. 1 (1), (2)), we have

$$\pm\phi_+ = \pm\left(-\frac{\beta}{\alpha}\right)^{\frac{1}{2}}, \quad h_1 = \frac{\pm 4\sqrt{-\alpha\beta}}{c^2-1}.$$

Corresponding to $H_1(\pm\phi, 0) = h_1$ defined by (11), system (10) has two orbits connecting the saddle $A_+(\phi_+, 0)$. Two orbits have the same algebraic equation for $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$

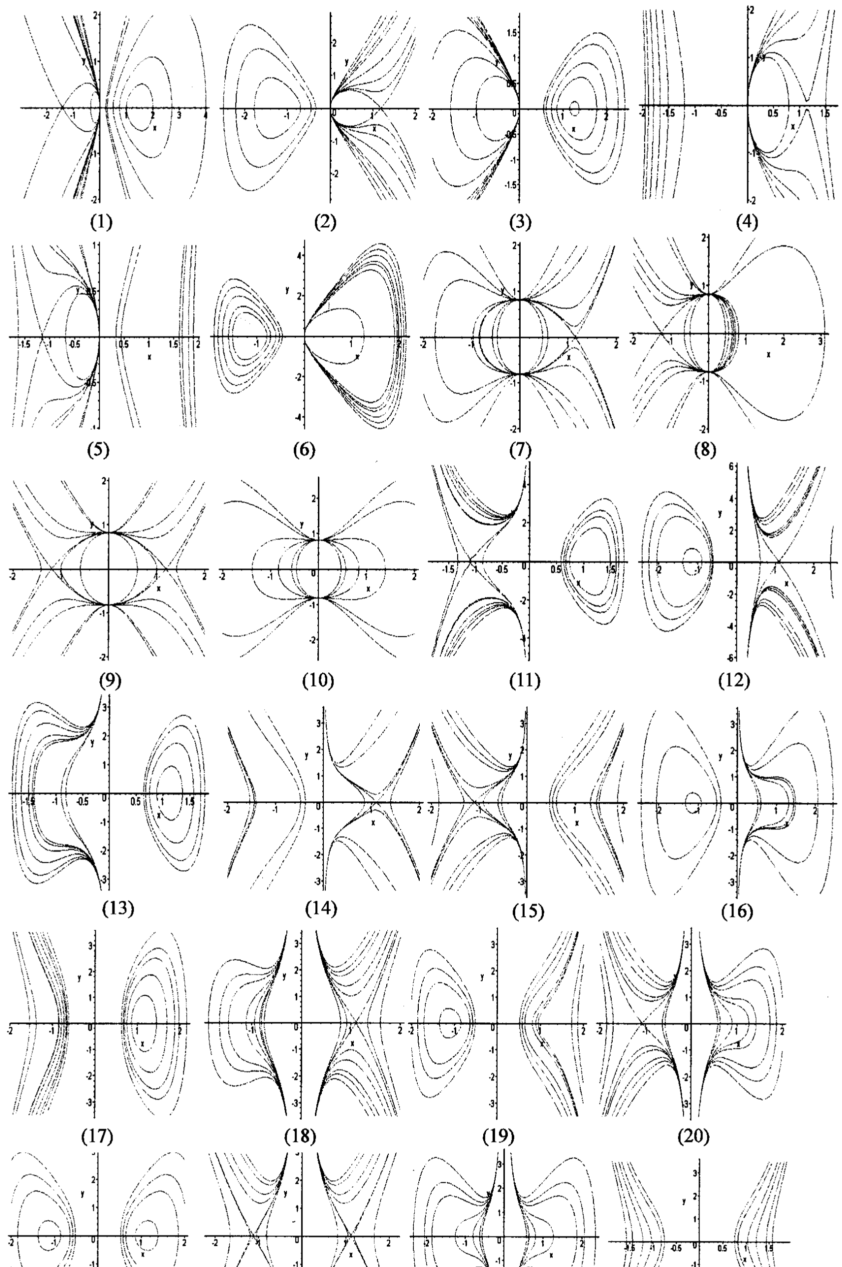

Fig. 1: The phase portraits of (1) $n = 1$, $m = 2k - 1$, $\alpha\beta < 0$, $\beta(c^2 - 1) > 0$, (2) $n = 1$, $m = 2k + 1$, $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$, (3) $n = 1$, $m = 2k$, $\alpha\beta < 0$, $\beta(c^2 - 1) > 0$, (4) $n = 1$, $m = 2k$, $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$, (5) $n = 2$, $m = 2k$, $\alpha\beta > 0$, $\beta(c^2 - 1) > 0$, (6) $n = 2$, $m = 2k$, $\alpha\beta > 0$, $\beta(c^2 - 1) < 0$, (7) $n = 2$, $m = 2k - 1$, $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$, (8) $n = 2$, $m = 2k - 1$, $\alpha\beta > 0$, $\beta(c^2 - 1) < 0$, (9) $n = 2$, $m = 2k$, $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$, (10) $n = 2$, $m = 2k$, $\alpha\beta > 0$, $\beta(c^2 - 1) < 0$, (11) $n = 2l + 1$, $m = 2k + 1$, $\alpha\beta < 0$, $\beta(c^2 - 1) > 0$, (12) $n = 2l + 1$, $m = 2k + 1$, $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$, (13) $n = 2l - 1$, $m = 2k$, $\alpha\beta < 0$, $\beta(c^2 - 1) > 0$, (14) $n = 2l - 1$, $m = 2k$, $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$, (15) $n = 2l - 1$, $m = 2k$, $\alpha\beta > 0$, $\beta(c^2 - 1) > 0$, (16) $n = 2l - 1$, $m = 2k$, $\alpha\beta > 0$, $\beta(c^2 - 1) > 0$, (17) $n = 2l$, $m = 2k - 1$, $\alpha\beta < 0$, $\beta(c^2 - 1) > 0$, (18) $n = 2(l + 1)$, $m = 2k - 1$, $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$, (19) $n = 2l$, $m = 2k - 1$, $\alpha\beta > 0$, $\beta(c^2 - 1) > 0$, (20) $n = 2(l + 1)$, $m = 2k$, $\alpha\beta > 0$, $\beta(c^2 - 1) < 0$, (21) $n = 2l$, $m = 2k$, $\alpha\beta < 0$, $\beta(c^2 - 1) > 0$, (22) $n = 2(l + 1)$, $m = 2k$, $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$, (23) $n = 2l$, $m = 2k$, $\alpha\beta > 0$, $\beta(c^2 - 1) < 0$, (24) $n = 2l \geq 2$, $m = 2k$, $\alpha\beta > 0$, $\beta(c^2 - 1) > 0$

$$y^2 = \frac{2\alpha\phi}{c^2-1}(\phi-\phi_+)^2 \tag{21}$$

and for $\alpha\beta < 0$, $\beta(c^2 - 1) > 0$

$$y^2 = \frac{-2\alpha(-\phi)}{c^2-1}(\phi-\phi_+)^2 \tag{22}$$

Thus, from (21), we obtain the parametric representations of the arch orbit, for $\xi\epsilon(0, \infty)$ as follows:

$$\phi(\xi) = \phi_+ \tanh^2(\omega_1\xi) \tag{23}$$

where

$$\omega_1 = \frac{1}{2}\sqrt{\frac{2\alpha}{c^2-1}}$$

clearly, we have that $\phi(0) = 0$, $\phi(\pm\infty) = \phi_+$.

It follows that Eq. (9) has one unbounded wave solution with the parametric representations

$$u(x,t) = \ln\phi(x-ct) = \ln(\phi_+ \tanh^2(\omega_1(x-ct))), x-ct \in (0,\infty) \tag{24}$$

From (22), we obtain the parametric representations of the arch orbit for $\xi\epsilon(0, T)$ as follows:

$$\phi(\xi) = \phi_+ \tan^2(\omega\xi) \tag{25}$$

where

$$\omega = \frac{1}{2}\sqrt{\frac{-2\alpha\phi_+^2}{c^2-1}}, T = \pi\sqrt{\frac{c^2-1}{-2\alpha\phi_+^2}}$$

clearly, we have that $\phi(0) = 0$, $\phi(T) = \infty$.

It follows that Eq. (9) has one unbounded wave solution with the parametric representations

$$u(x,t) = \ln(\phi_+ \tan^2(\omega(x-ct))), x-ct \in (0,T) \tag{26}$$

For system (13), when $\alpha <, \beta > 0$, $c^2 - 1 < 0$ or $\alpha < 0$, $\beta < 0$, $c^2 - 1 > 0$ (Fig. 1 (7), (8)), we have

$$\phi_+ = \left(-\frac{\beta}{\alpha}\right)^{\frac{1}{3}}, \; h_1 = \frac{-3\alpha}{c^2-1}\phi_+ \cdot$$

Corresponding to $H_1(\phi_+, 0)$ defined by (14), system (13) has two orbits connecting the node points $S_\pm$ and the saddle $A_+(\phi_+, 0)$ and also has an arch orbit connecting two node $S_\pm$ in the left (or right) side of the straight line $\phi = 0$. Three orbits have the same algebraic equation for $\alpha < 0$, $\beta > 0$, $c^2 - 1 < 0$

$$y^2 = \frac{2\alpha}{c^2-1}(\phi + \frac{1}{2}\phi_+)(\phi_+ - \phi)^2 \tag{27}$$

or $\alpha < 0$, $\beta < 0$, $c^2 - 1 > 0$

$$y^2 = \frac{-2\alpha}{c^2-1}(-\phi - \frac{1}{2}\phi_+)(\phi - \phi_+)^2 \tag{28}$$

Thus, from (27), we obtain the parametric representations of the arch orbit of (13), respectively, for $\xi\epsilon(-T_1, T_1)$, $\xi\epsilon(T_1, \infty)$ and $\xi\epsilon(-\infty, -T_1)$ as follows:

$$\phi(\xi) = -\frac{1}{2}\phi_+ + \frac{3}{2}\phi_+ \tanh^2(\omega_2\xi) \tag{29}$$

where

$$\omega_2 = \sqrt{\frac{3\alpha\phi_+}{4(c^2-1)}}, \; T_1 = \frac{1}{\omega_2}\tanh^{-1}(\frac{1}{\sqrt{3}}) \cdot$$

Clearly, we have that $\phi(T_1) = 0$, $\phi(\pm\infty) = \phi_+$.

It follows that Eq. (12) has two unbounded wave solutions with the parametric representations

$$u(x,t) = \ln(-\frac{1}{2}\phi_+ + \frac{3}{2}\phi_+ \tanh^2(\omega_2(x-ct))), \tag{30}$$

for $x - ct\epsilon(-T_1, T_1)$, $x - ct\epsilon(T_1, \infty)$ and $x - ct\epsilon(-\infty, -T_1)$.

From (18), we obtain the parametric representations of the arch orbit of (13), for $\xi\epsilon(-T_2, T_2)$, $\xi\epsilon(T_2, \infty)$ and $\xi\epsilon(-\infty, -T_2)$ as follows:

$$\phi(\xi) = -\frac{1}{2}\phi_+ + \frac{3}{2}\phi_+ \tanh^2(\omega_3\xi) \tag{31}$$

where

$$\omega_3 = \sqrt{\frac{3\alpha\phi_+}{4(c^2-1)}}, \; T_2 = \frac{1}{\omega_3}\tanh^{-1}(\frac{1}{\sqrt{3}}) \cdot$$

Clearly, we have that $\phi(T_2) = 0$, $\phi(\pm\infty) = \phi_+$.

It follows that Eq. (12). has two unbounded wave solutions with the parametric representations for $x - ct\epsilon(-T_2, T_2)$, $x - ct\epsilon(T_2, \infty)$ and $x - ct\epsilon(-\infty, -T_2)$.

$$u(x,t) = \ln(-\frac{1}{2}\phi_+ + \frac{3}{2}\phi_+ \tanh^2(\omega_3(x-ct))) \quad (32)$$

Especially, when $\alpha = \beta = -1$, the special DBM equation has the unbounded wave solution for $x - ct\in(-T_2, T_2)$

$$u(x,t) = \ln\left[\frac{1}{2} - \frac{3}{2}\tanh^2\left(\frac{1}{2}\sqrt{\frac{3}{c^2-1}}(x-ct)\right)\right] \quad (33)$$

where

$$T_2 = 2\sqrt{\frac{c^2-1}{3}}\tanh^{-1}(\frac{1}{\sqrt{3}}).$$

For system (16), when $\alpha < 0$, $\beta > 0$, $c^2 - 1 < 0$ (Fig. 1 (4)), we have

$$\phi_+ = \left(-\frac{\beta}{\alpha}\right)^{\frac{1}{3}}, \ h_1 = \frac{-3\alpha}{c^2-1}\phi_+^2.$$

Corresponding to $H_1(\phi_+, 0) = h_1$ defined by (17), system (16) has two orbits connecting the saddle $A_+(\phi_+, 0)$. Two orbits have the same algebraic equation for $\alpha <$, $\beta > 0$, $c^2 - 1 < 0$

$$y^2 = \frac{\alpha}{c^2-1}\phi(\phi + \frac{1}{2}\phi_+)(\phi - \phi_+)^2 \quad (34)$$

Thus, from (34), we obtain the parametric representations of the arch orbit for $x - ct\in(-T_3, T_3)$, $x - ct\in(T_3, \infty)$ and $x - ct\in(-\infty, -T_3)$ as follows:

$$\phi(\xi) = \frac{\phi_+}{3\tanh^2(\omega_4\xi) - 2} \quad (35)$$

where

$$\omega_4 = \sqrt{\frac{3\alpha\phi_+^2}{8(c^2-1)}}, \ T_3 = \frac{1}{\omega_4}\tanh^{-1}(\sqrt{\frac{2}{3}}).$$

Clearly, we have that $\phi(T_3) = \infty$, $\phi(\pm\infty) = 0$.

It follows that Eq. (15) has two unbounded wave solutions with the parametric representations for $x - ct\in(T_3, \infty)$ and $x - ct\in(-\infty, -T_3)$ as follows:

$$u(x,t) = \ln\phi(x-ct) = \ln\left(\frac{\phi_+}{3\tanh^2(\omega_4(x-ct)) - 2}\right) \quad (36)$$

For system (19), when $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$ (Fig. 1 (9)), we have

$$\phi_+ = \left(-\frac{\beta}{\alpha}\right)^{\frac{1}{4}}, h_1 = \frac{-2\alpha}{c^2-1}\phi_+^2.$$

Corresponding to $H_1(\phi_+, 0) = h_1$ defined by (20), system (19) has two orbits connecting the node points $S_\pm$ and the saddle $A_\pm(\phi_\pm, 0)$, respectively. Four orbits have the same algebraic equation for $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$

$$y^2 = \frac{\alpha}{c^2-1}(\phi^2 - \phi_+^2)^2 \quad (37)$$

Thus, from (37), we obtain the parametric representations of the arch orbit for $\xi\in(0, \infty)$ and $\xi\in(-\infty, 0)$ as follows:

$$\phi(\xi) = \pm\phi_+ \tanh(\omega_5\xi) \quad (38)$$

where

$$\omega_5 = \phi_+\sqrt{\frac{\alpha}{c^2-1}}.$$

Clearly, we have that $\phi(0) = 0$, $\phi(\pm\infty) = \pm\phi_+$.

It follows that Eq. (18) has two unbounded wave solutions with the parametric representations for $x - ct\in(0, \infty)$ and $x - ct\in(-\infty, 0)$

$$u(x,t) = \ln\phi(x-ct) = \ln(\pm\phi_+ \tanh(\omega_5(x-ct))) \quad (39)$$

Uncountable infinite many exact explicit unbounded wave solutions: For system (13), when $\alpha < 0$, $\beta > 0$, $c^2 - 1 < 0$ or $\alpha < 0$, $\beta > 0$, $c^2 - 1 > 0$ (Fig. 1 (7), (8)), we have

$$\phi_+ = \left(-\frac{\beta}{\alpha}\right)^{\frac{1}{3}}, \ h_1 = \frac{-3\alpha}{c^2-1}\phi_+^2.$$

Corresponding to $H_1(\phi_+, 0) = h_1$, $h\in(-\infty, 0)$ defined by (14), system (13) has two families of arch orbits connecting two node points $S_\pm$ which lie in the left (or right) side of the straight line $\phi = 0$, respectively. These orbits have the algebraic equation for $\alpha < 0$, $\beta > 0$, $c^2 - 1 < 0$

$$y^2 = \phi^2 h + \frac{2\alpha}{c^2-1}\phi^3 - \frac{\beta}{c^2-1} = \frac{2\alpha}{c^2-1}(\phi-\phi_g)(\phi-\phi_M)(\phi-\phi_m) \quad (40)$$

and for $\alpha < 0$, $\beta > 0$, $c^2 - 1 < 0$

$$y^2 = \phi^2 h + \frac{2\alpha}{c^2-1}\phi^3 - \frac{\beta}{c^2-1} = -\frac{2\alpha}{c^2-1}(\phi_M - \phi)(\phi - \phi_m)(\phi - \phi_l) \quad (41)$$

By using (40), we obtain the parametric representations for the right arch orbits of (13), for $\alpha < 0$, $\beta > 0$, $c^2 - 1 < 0$ as follows:

$$\phi(\xi) = \frac{\phi_g - \phi_M \mathrm{sn}^2(\Omega_1\xi, k_1)}{\mathrm{cn}^2(\Omega_1\xi, k_1)}, \xi \in (-P_1, P_1) \quad (42)$$

where sn (x, k) is the Jacobin elliptic functions with the modulo k and

$$\Omega_1 = \sqrt{\frac{-\alpha(\phi_g - \phi_m)}{2(c^2-1)}}, k_1^2 = \frac{\phi_M - \phi_m}{\phi_g - \phi_m},$$

$$P_1 = \frac{1}{\Omega_1}\mathrm{sn}^{-1}\left(\sqrt{\frac{\phi_g}{\phi_M}}, k_1\right), \phi(P_1) = 0$$

By using (41), we obtain the parametric representations for the right arch orbits of (13) for $\alpha < 0$, $\beta < 0$, $c^2 - 1 > 0$

$$\phi(\xi) = \phi_M - (\phi_M - \phi_m)\mathrm{sn}^2(\Omega_2\xi, k_2), \xi \in (-P_2, P_2) \quad (43)$$

where

$$\Omega_2 = \sqrt{\frac{\alpha(\phi_M - \phi_l)}{2(c^2-1)}}, k_2^2 = \frac{\phi_M - \phi_m}{\phi_M - \phi_l},$$

$$P_2 = \frac{1}{\Omega_2}\mathrm{sn}^{-1}\left(\sqrt{\frac{\phi_M}{\phi_M - \phi_m}}, k_2\right), \phi(P_2) = 0$$

Thus, (42) and (43) give rise to the following uncountable infinite many exact explicit unbounded wave solutions of (12) for $\alpha < 0$, $\beta > 0$, $c^2 - 1 > 0$

$$u(x,t) = \ln\phi(x-ct) = \ln\frac{\phi_g - \phi_M \mathrm{sn}^2(\Omega_1(x-ct), k_1)}{\mathrm{cn}^2(\Omega_1(x-ct), k_1)}, \quad (44)$$
$$x - ct \in (-P_1, P_1)$$

and for $\alpha < 0$, $\beta < 0$, $c^2 - 1 > 0$

$$u(x,t) = \ln\phi(x-ct) = \ln(\phi_M - (\phi_M - \phi_m)\mathrm{sn}^2(\Omega_2(x-ct), k_2)),$$
$$x - ct \in (-P_2, P_2) \quad (45)$$

For system (19), when $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$ (Fig. 1 (9)), we have

$$\phi_+ = \left(-\frac{\beta}{\alpha}\right)^{\frac{1}{4}}, h_1 = \frac{-2\alpha}{c^2-1}\phi_+^2.$$

Corresponding to $H_1(\phi, 0) = h$, $h \in (-\infty, 0)$ defined by (20), system (19) has two families of arch orbits connecting two node points $S_\pm$ which lie in the left (or right) side of the straight line $\phi = 0$, respectively. These orbits have the algebraic equation for

$$y^2 = \frac{\alpha}{c^2-1}\left(\phi^4 + \frac{h(c^2-1)\phi^2 - \beta}{\alpha}\right) = \frac{\alpha}{c^2-1}(\phi - e_1)(\phi - e_2)$$
$$(\phi - e_3)(\phi - e_4) \quad (46)$$

By using (46), we obtain the parametric representations for the right arch orbits of (19) for $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$

$$\phi(\xi) = \frac{e_1(e_2 - e_4) - e_2(e_1 - e_4)\mathrm{sn}^2(\Omega_3\xi, k_3)}{e_2 - e_4 - (e_1 - e_4)\mathrm{sn}^2(\Omega_3\xi, k_3)}, \xi \in (-P_3, P_3) \quad (47)$$

where

$$\Omega_3 = \sqrt{\frac{\alpha(e_1 - e_3)(e_2 - e_4)}{4(c^2-1)}}, k_3^2 = \frac{(e_1 - e_2)(e_2 - e_4)}{(e_1 - e_3)(e_2 - e_4)},$$

$$P_3 = \frac{1}{\Omega_3}\mathrm{sn}^{-1}\left(\sqrt{\frac{e_1(e_2 - e_4)}{e_2(e_1 - e_4)}}, k_3\right), \phi(P_3) = 0$$

Thus, (47) give rise to the following uncountable infinite many exact explicit unbounded wave solutions of (18) for $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$

$$u(x,t) = \ln\frac{e_1(e_2 - e_4) - e_2(e_1 - e_4)\mathrm{sn}^2(\Omega_3(x-ct), k_3)}{e_2 - e_4 - (e_1 - e_4)\mathrm{sn}^2(\Omega_3(x-ct), k_3)}, \quad (48)$$
$$x - ct \in (-P_3, P_3)$$

Uncountable infinite many exact explicit periodic wave solutions: For n = 1, m = 1, $\alpha\beta < 0$, $\beta(c^2 - 1) > 0$ corresponding to $H_1(\phi, y) = h$, $h \in (h_1, \infty)$,

$$(h_1 = \frac{4\sqrt{-\alpha\beta}}{c^2-1})$$

defined by (11), system (10) have a family of periodic solutions enclosing the center $(\phi_+, 0)$ which lie in the right side of the straight line $\phi = 0$, these orbits determine uncountable infinite many periodic wave solutions of (9) (Fig. 1 (1)).

$$y^2 = \frac{2\alpha}{c^2-1}\phi\left(\phi^2 + \frac{h(c^2-1)}{2\alpha}\phi - \frac{\beta}{\alpha}\right) = \frac{2\alpha}{c^2-1}\phi(\phi-\phi_M)\left(\phi+\phi_M+\frac{h(c^2-1)}{2\alpha}\right) \quad (49)$$

Thus, from (49), we obtain the parametric representations of the arch orbit for $\xi\epsilon(0, P_4)$ as follows:

$$\phi(\xi) = \phi_M cn^2(\omega_1\xi, k_1) \quad (50)$$

where

$$\omega_1 = \sqrt{\frac{-h(c^2-1)-4\alpha\phi_M}{4(c^2-1)}}, k_1^2 = \frac{2\alpha\phi_M}{h(c^2-1)+4\alpha\phi_M},$$

$$P_4 = \frac{F(k,\frac{\pi}{2})}{\omega_1}.$$

Clearly, we have that $\phi(0) = \phi_M$, $\phi(P_4) = 0$.

Thus, Eq. (12) has uncountable infinite many periodic wave solutions,

$$u(x,t) = \ln\phi(x-ct) = \ln(\phi_M cn^2(\omega_1(x-ct),k_1))), \quad x-ct \in (0,P_4) \quad (51)$$

For n = 2, m = 1, $\alpha\beta < 0$, $\beta(c^2-1) > 0$ corresponding to $H(\phi, y) = h$, $h\epsilon(h_1, \infty)$,

$$(h_1 = \frac{4\sqrt{-\alpha\beta}}{c^2-1})$$

defined by (14), system (13) have a family of periodic solutions enclosing the center $(\phi_+, 0)$ which lie in the right side of the straight line $\phi = 0$, these orbits determine uncountable infinite many periodic wave solutions of (12) (Fig. 1 (17)).

$$y^2 = \frac{2\alpha}{c^2-1}\left(\phi^3 + \frac{h(c^2-1)}{2\alpha}\phi^2 - \frac{\beta}{2\alpha}\right) = \frac{2\alpha}{c^2-1}(\phi-\phi_g)(\phi-\phi_M)(\phi-\phi_m) \quad (52)$$

Thus, from (52), we obtain the parametric representations of the arch orbit for $\xi\epsilon(-P_5, P_5)$ as follows:

$$\phi(\xi) = \phi_g - (\phi_g - \phi_M)sn^2(\omega_2\xi, k_2) \quad (53)$$

where

$$\omega_2 = \sqrt{\frac{-\alpha(\phi_g-\phi_m)}{2(c^2-1)}}, k_2^2 = \frac{\phi_g-\phi_M}{\phi_g-\phi_m},$$

$$P_5 = \frac{1}{\omega_2}sn^{-1}\left(\sqrt{\frac{\phi_g}{\phi_g-\phi_M}}, k_2\right).$$

Clearly, we have that $\phi(P_5) = 0$

Thus, Eq. (12) has uncountable infinite many periodic wave solutions,

$$u(x,t) = \ln\phi(x-ct) = \ln\left[\phi_g - (\phi_g-\phi_M)sn^2(\omega_2(x-ct),k_2)\right], \quad x-ct \in (-P_5,P_5) \quad (54)$$

For n = 1, m = 2, $\alpha\beta < 0$, $\beta(c^2-1) > 0$ corresponding to $H(\phi, y) = h$, $h\epsilon(h_1, \infty)$,

$$(h_1 = \frac{-3\alpha}{c^2-1}\phi_+)$$

defined by (17), system (16) have a family of periodic solutions enclosing the center $(\phi_+, 0)$ which lie in the right side of the straight line $\phi = 0$, these orbits determine uncountable infinite many periodic wave solutions of (15) (Fig. 1 (3)).

$$y^2 = \frac{\alpha}{c^2-1}\phi\left(\phi^3 + \frac{h(c^2-1)\phi^2 - 2\beta}{\alpha}\right) \quad (55)$$

Thus, we obtain the parametric representations of the arch orbit for $\xi\epsilon(-P_6, P_6)$ as follows:

$$\frac{1}{\phi(\xi)} = \phi_M - (\phi_M - \phi_m)sn^2(\omega_3\xi, k_3) \quad (56)$$

where

$$\omega_3 = \sqrt{\frac{\beta(\phi_M-\phi_g)}{2(c^2-1)}},\ k_3^2 = \frac{\phi_M-\phi_m}{\phi_M-\phi_g},$$

$$P_6 = \frac{1}{\omega_3}sn^{-1}\left(\sqrt{\frac{\phi_M}{\phi_M-\phi_m}}, k_3\right).$$

Clearly, we have that $\phi(P_6) = \infty$

Thus, Eq. (15) have uncountable infinite many periodic wave solutions,

$$u(x,t) = \ln\phi(x-ct) = -\ln\left[\phi_M - (\phi_M-\phi_m)sn^2(\omega_3(x-ct),k_3)\right], \quad x-ct \in (-P_6,P_6) \quad (57)$$

EXISTENCE OF UNBOUNDED WAVE SOLUTIONS AND PERIODIC WAVE SOLUTIONS OF (1)

Here, by using the phase portraits we show the existence of unbounded wave and periodic wave solutions of (1) for any integer m = 2k +1, m = 2k. We have mentioned that we are only interesting the positive solutions of $\phi(\xi)$, because of u(x, t) = In ϕ(x − ct).

We see from Fig. 1(23) or Fig. 1(16) that corresponding to a branch of the curves H(ϕ, y) = h, or h$\in(-\infty, \infty_1)$ or given by (8), in the right side of the (ϕ, y)-phase plane, there exist uncountable infinity many bounded solutions of $\phi(\xi)$ (but, $\phi'(\xi)$ are unbounded). These solutions approach to ϕ = 0 as $\xi \to \pm\infty$. These $\phi(\xi)$ are breaking solutions of (4) near. ϕ = 0 Similarly, some solution families in Fig. 1 (14), (18), (20), (22) or (12) have the same dynamical behavior. We use Fig. 2 (2-1)-(2-7) to show these wave profiles (k, l $\in Z^+$).

From the above discussion, we have the following conclusions.

Theorem 1

(i) Suppose that m = 2k, n = 2*l* + 1, $\alpha\beta > 0$, $\beta(c^2 - 1) > 0$, h$\in(h_1, \infty)$. Then, Eq. (1) has a family of uncountable infinity many periodic wave solutions which correspond to a branch of the curves H(ϕ, y) = h given by (8) in the right side of the (ϕ, y)-phase plane (Fig. 1 (13)).

(ii) Suppose that m = 2k − 1, n = 2*l*, $\alpha\beta < 0$, $\beta(c^2 - 1) > 0$, h$\in(h_1, \infty)$. Then, Eq. (1) has a family of uncountable infinity many periodic wave solutions which corresponds to a branch of the curves H(ϕ, y) = h given by (8) in the right side of the (ϕ, y)-phase plane (Fig. 1 (17)).

(iii) Suppose that Then, Eq. (1) has a family of uncountable infinity many periodic wave solutions which corresponds to a branch of the curves H(ϕ, y) = h given by (8) in the right side of the (ϕ, y)-plane (Fig. 1 (21)).

Theorem 2

(i) Suppose that m = 2k − 1, n = 2*l* − 1, $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$, h$\in(-\infty, h_1)$. Then, Eq. (1) has a family of uncountable infinity many unbounded wave solutions which corresponds to a branch of the curves H(ϕ, y) = h given by (8) in the right side of the (ϕ, y)-phase plane (Fig. 1 (12)).

(ii) Suppose that m = 2k, n = 2*l* + 1, $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$, h$\in(-\infty, h_1)$. Then, Eq. (1) has a family of uncountable infinity many unbounded wave solutions which corresponds to a branch of the curves H(ϕ, y) = h given by (8) in the right side of the phase (ϕ, y)-plane (Fig. 1 (14)).

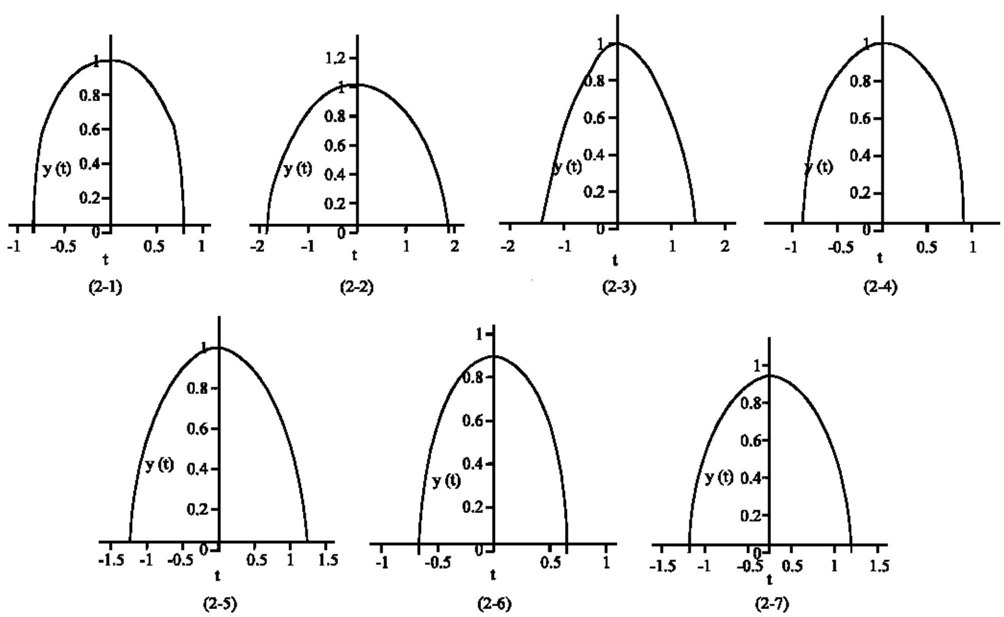

Fig. 2: The wave profiles of bounded solutions of (2-1) m = 2k − 1, n = 2*l* + 1, $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$, (2-2) m = 2k, n = 2*l* + 1, $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$, (2-3) m = 2k, n = 2*l* + 1, $\alpha\beta > 0$, $\beta(c^2 - 1) < 0$, (2-4) m = 2k − 1, n = 2*l* + 1, $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$, (2-5) m = 2k − 1, n = 2*l*, $\alpha\beta > 0$, $\beta(c^2 - 1) < 0$, (2-6) m = 2k, n = 2(l + 1), $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$, (2-7) m = 2k, n = 2(l + 1), $\alpha\beta > 0$, $\beta(c^2 - 1) > 0$

(iii) Suppose that $m = 2k - 1$, $n = 2l$, $\alpha\beta < 0$, $\beta(c^2 - 1) < 0$, $h \in (-\infty, h_1)$. Then, Eq. (1) has a family of uncountable infinity many unbounded wave solutions which corresponds to a branch of the curves $H(\phi, y) = h$ given by (8) in the right side of the (ϕ, y)-phase plane (Fig. 1 (18)).

(iv) Suppose that $m = 2k$, $n = 2l$, $\alpha\beta < 0$, $\beta (c^2 - 1) > 0$, $h \in (-\infty, h_1)$. Then, Eq. (1) has a family of uncountable infinity many unbounded wave solutions which corresponds to a branch of the curves $H(\phi, y) = h$ given by (8) in the right side of the (ϕ, y)-phase plane (Fig. 1 (22)).

(v) Suppose that $m = 2k - 1$, $n = 2l + 1$, $\alpha\beta > 0$, $\beta(c^2 - 1) < 0$, $h \in (-\infty, +\infty)$. Then, Eq. (1) has a family of uncountable infinity many unbounded wave solutions which corresponds to a branch of the curves $H(\phi, y) = h$ given by (8) in the right side of the (ϕ, y)-plane (Fig. 1 (16)).

(vi) Suppose that $m = 2k - 1$, $n = 2l$, $\alpha\beta > 0$, $\beta(c^2 - 1) < 0$, $h \in (-\infty, +\infty)$ Then, Eq. (1) has a family of uncountable infinity many unbounded wave solutions which corresponds to a branch of the curves $H(\phi, y) = h$ given by (8) in the right side of the (ϕ, y)-phase plane (Fig. 1 (20)).

(vii) Suppose that $m = 2k$, $n = 2l$, $\alpha\beta > 0$, $\beta(c^2 - 1) < 0$, $h \in (-\infty, +\infty)$. Then, Eq. (1) has a family of uncountable infinity many unbounded wave solutions whichcorresponds to a branch of the curves $H(\phi, y) = h$ given by (8) in the right side of the (ϕ, y)-phase plane (Fig. 1 (23)).

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