To the Ballistic Dimer Resonance in the Propagation of Mechanical Waves in One Dimensional Linear Lattice

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Abstract: The propagation of mechanical waves in one-dimensional Kronig-Penney random media is examined in presence of short-range binary correlation in disorder. A chain of harmonic oscillators is connected to a rigid foundation by linear springs while a string submitted to a uniform tension relates masses to each other. Using the transfer matrix formalism, the nature of the propagating modes is numerically investigated with the study of the transmission coefficient and the localization length. An interesting manner to construct the dimer unit cell is presented settling down the ballistic regime. The presence of such a particular dimer resonance provides the Bloch extended vibration modes in contradiction with the general conclusion of the localization phenomena in one-dimensional disordered system.

Key words: Anderson localization, random dimer model, delocalization, Bloch vibration modes, ballistic regime, acoustics, mechanical waves

INTRODUCTION

According to the universal conclusion of the scaling theory, it is well understood that in One-Dimensional (1D) disordered systems, all the elementary excitations are localized in the Anderson sense. The destructive quantum interferences appear to be the dominant mechanism in localizing the electronic eigenstates (Anderson, 1958; Abrahams et al., 1979; Sheng, 1979; Anderson et al., 1980; Ya Azbel, 1983). It is clear that the disorder precludes the presence of long-range propagation. However, relevant theoretical approaches have successfully examined different ways in delocalizing the eigenstates indicating that disorder can act also in creative fashion (Économou et al., 1988; Sanchez et al., 1994). In particular a challenging scenario has been put forwards in binary alloys to suppress localization allowing the propagation of waves: Namely correlation in disorder. Originally introduced by Dunlap et al. (1990), the Random Dimer Model (RDM) has been applied to various domains: Conducting polymers (Wu et al., 1991a, b), semiconductor disordered superlattices (Díez et al., 1995; Díez et al., 1996; Bentata et al., 2002) pointing out the existence of truly extended states supported by experimental evidences (Bellani et al., 1998; Kuhl et al., 2000). The key idea is that the RDM within a short length correlation restores the tunnel effect and then the necessary condition for delocalising the particle. However all this matter holds only for the quantum case since the competition between destructive interference and tunnel effect is the major cause leading to the localization or delocalisation of the electronic eigenstates.

Although extensive interest has been devoted to the electronic Random Dimer Model (RDM) case (Huang et al., 2001; Sedrakyan, 2004; Zhang et al., 2006), very few has been done in the context of mechanical waves. Indeed such classical ones are good candidates to illustrate the model-engineering structures for a better observation of the Anderson Localization in 1D systems (He et al., 1986; Richoux et al., 1999; Albuquerque et al., 2005; Sigalas et al., 2005). This prompts us to examine the effect of the binary correlated disorder via the random dimer model on the propagation of classical waves since the 1D mechanical systems illustrate perfectly the analogy between electron-wave and classical-wave. In fact, the propagating medium is constituted by a large string

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having negligible mass submitted to a uniform tension. A mass is linked at each regular lattice point to a ground foundation by a spring forming a linear oscillator unit cell. In this situation, the wave field consists of transverse amplitudes along the string. It could be noticed that such system may be easily experimented. In a perfect ordered system all the unit cells are identical while for a disordered system the variables mass and/or spring are random. Here we introduce the dimer effect by assuming a concentration of two successive identical cells at random through a host lattice of identical cells. Typically it corresponds to a random Kronig-Penney (K-P) binary alloy with dimer. It is expected that such system restores the existence of extended modes in the conventional sense (Sanchez et al., 1994).

**Theoretical model:** A semi-infinite tight string with homogeneous density $\rho$ is submitted to a uniform tension $T_0$. The string is formed by a large number of subsystems at each lattice discrete points $x_n = n d$, $d$ denoting the lattice spacing. Each subsystem is a harmonic oscillator for which the mass $M$ is connected to a grounding rigid foundation by a spring having a linear stiffness $K_0$ (Fig. 1).

In a perfect analogy with the electronic K-P model (Sanchez et al., 1994), namely:

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x)}{\partial x^2} + \sum_{n} \lambda_n \delta(x-x_n)$$

we are interested by the propagation of transverse wave in the vertical plane. The transverse displacement $y$ at the longitudinal coordinate $x$ is solution of the equation of motion:

$$\frac{\partial^2 y(x)}{\partial x^2} + k^2 y(x) = \sum_{n} \lambda_n \delta(x-x_n) y(x)$$

with

$$k = \frac{\omega}{v_0} \quad \text{and} \quad v_0 = \sqrt{\frac{K_0}{\rho}}$$

$k$ and $v_0$ denoting the wave vector and the wave (or sound) velocity through the whole system, respectively while $\omega$ is the incident frequency.

The term $\lambda_n$ related to the vibration mode at each delta peak is given by:

$$\lambda_n = \frac{1}{K_0} \left( K_0 - M_0 \omega_n^2 \right) = \frac{K_0}{T_0} \left( 1 - \frac{\omega_n^2}{\Omega_n^2} \right)$$

where

$$\Omega_n^2 = \frac{k_n}{M_n}$$

Physically $\Omega_n$ defines the free frequency of the $n$th subsystem while the parameter $\lambda_n = \lambda_n(\omega)$ has the meaning of effective potential delta peak strength since it depends on both $\omega$ and the media unit cell intrinsic parameters.

In this study the mechanical randomness may be introduced in different ways: Disorder in mass and/or stiffness, referred to the cellular disorder. Thus in this description, the masses are statistically independent variables given by a common probability distribution.

In the region $nd < x < (n + 1) d$, the solution of Eq. 2 is a superposition of forward and backward scattering waves:

$$y(x) = A_n \exp(i k x) + B_n \exp(-i k x)$$

$A_n$ and $B_n$ denote the amplitude coefficients in the $n$-th region. Introducing the reflection the transmission amplitudes $r_n$ and $t_n$ of the system, $y(x)$ satisfies the limit conditions:

$$y(x) = \begin{cases} 
\exp(i k x) + r_n \exp(-i k x) & x < 0 \\
\exp(i k x) & x > L 
\end{cases}$$

where $L = Nd$ is the system size. The amplitudes $A_n$ and $B_n$ through the initial and final amplitudes can be linearly expressed using boundary conditions giving out the total transfer matrix $M(L, 0)$ of the system.
\begin{align*}
\begin{pmatrix}
A_N \\
B_N = 0
\end{pmatrix} = M(L, 0) \begin{pmatrix}
A_0 \\
B_0 = 1
\end{pmatrix} & \quad (8)
\end{align*}

\begin{align*}
M(L, 0) = \frac{1}{2ik_0} \begin{pmatrix}
-ik_0 & -1 \\
-ik_0 & 1
\end{pmatrix} S(L, 0) \begin{pmatrix}
1 & 1 \\
-ik_0 & -ik_0
\end{pmatrix} & \quad (9)
\end{align*}

where \( S(L, 0) \) refers to the total scattering matrix.

This allows one to determine the transmission coefficient, which is the fundamental physical quantity of interest:

\begin{align*}
T = |M_{21}|^2 = \frac{4}{(S_{11} + S_{22})^2 + \left( \frac{S_{21}}{k} - S_{22} \right)^2} & \quad (10)
\end{align*}

Once the transmission coefficient is known, the nature of the propagating modes may be characterized by the reduced Lyapunov coefficient given by the ratio:

\begin{align*}
\frac{L}{\xi} = -\frac{1}{2} \log T & \quad (11)
\end{align*}

where \( \xi \) represents the localization length.

**RESULTS AND DISCUSSION**

Here, we first deal with the ordered case as a reference with the aim to show how the deterministic principal physical magnitudes can be achieved. Disorder is then considered by means uncorrelated random binary mass distribution. Finally, correlated case is examined with an interesting manner to construct the conventional dimer unit cell.

The propagating wire media is characterized with a mass density \( \rho = 5 \text{ kg m}^{-1} \) submitted to a tension \( T_0 = 10 \text{N} \) with the oscillator unit cell length \( d = 5.10^{-2} \). The above parameters have been chosen according to the available data in the literature (Richoux et al., 1999). We have studied rather large systems up to \( N = 1000 \) unit cells, i.e., 50 m length, to get convincing observation of the mechanical Anderson localization. All statistical averaged procedures are taken with a satisfactory convergence limit.

**Ordered case:** The system is constituted by identical oscillator cells formed by a mass \( M_0 = 0.100 \text{ kg} \) with a free frequency \( \Omega_0 = 40 \text{ rd s}^{-1} \). The corresponding analytical K-P formula \( a(\omega) \) provides the principal frequency \( \bar{\omega}_0 \approx 37.62 \text{ rd s}^{-1} \) for which \( a(\bar{\omega}_0) = 0 \) and the band edges \( \omega_{\text{low}} = 21 \text{ rd s}^{-1} \) and \( \omega_{\text{upper}} = 21 \text{ rd s}^{-1} \) by the condition \( |a(\omega)| = 1 \) as shown in Fig. 2. The symbol \( || \) denotes the absolute value.

The nature of the vibration modes can also be characterized with the reduced Lyapunov coefficient as depicted in Fig. 2. The criterion indicating the existence of forbidden or allowed bands may be simply formulated through the ratio

\begin{align*}
\frac{L}{\xi} = 1 & \quad (12)
\end{align*}

One has to note the existence of a divergence in the spatial extent \( \xi \) at the singular mode \( \Omega_0 \). This feature occurs only for the first allowed band and has not quantum equivalent in the ordered case (Sanchez et al., 1994). Indeed as the frequency \( \omega \) reaches the free frequency \( \Omega_0 \), the reduced Lyapunov coefficient behaves like:

\begin{align*}
\frac{L}{\xi} \approx \left| \omega - \Omega_0 \right|^v & \quad (13)
\end{align*}

with the critical exponent \( v = 1.99 \pm 0.06 \).

Moreover one has to note the existence of a spectacular phenomenon in \( T(L) \) at the free frequency \( \Omega_0 \). Indeed the transmission coefficient reaches it maximum unity value, independently from the system length since the effective potential profile vanishes at \( \Omega_0 \).

Consequently, the wave propagates as freely as possible along an effective empty wire, i.e., \( \lambda_s(\Omega_0)|_{\omega = \Omega_0} = 0 \), settling down a very interesting ballistic regime.

Fig. 2: Reduced Lyapunov coefficient and the absolute Kronig-Penney function for the perfectly ordered host lattice \( (M_0 = 0.100 \text{ kg}, \Omega_0 = 40 \text{ rd s}^{-1}, \bar{\omega}_0 = 43.20 \text{ rd s}^{-1}) \)
In order to appreciate more deeply the spatial coherence of a diffusive propagating wave, the variation of transmission coefficient $T(N)$ versus the length of the system size $L$ up to 2500 is also depicted in Fig. 3 for an allowed frequency, i.e., $\omega = 35$ rd s$^{-1}$. Indeed, the envelope function is constituted by a periodic superposition of forward and backward waves with uniform amplitudes. In the corresponding deterministic ordered limit, this feature is a signature of the obvious Bloch extended nature of the allowed modes. Furthermore, the envelope amplitude and its period depend strongly on the frequency of the allowed vibration mode. Stationary waves occur whenever the system length becomes a multiple of its half incident wavelength:

$$L = \frac{n\lambda(\omega)}{2}$$ (14)

as well known.

**Binary disordered case:** In this section, the effects of the binary disorder on the nature of the vibration modes are examined. Binary disorder is introduced by assuming that the host and impurity masses $M_a$ and $M_b$, respectively are statistically independent random variables given by a binary alloy distribution:

$$P(M) = c_a \delta(M - M_a) + (1 - c_a) \delta(M - M_b)$$ (15)

c_a = 1 - c_a and $c_a$ represent their corresponding concentrations.

As did in the previous ordered case, the impurity oscillator unit cell will be characterized with the free and principal frequencies, i.e., $\Omega_a$ and $\bar{\Omega}_a$, respectively. With this in mind, the corresponding averaged frequency responses $\langle T(\omega) \rangle$ and $\langle \frac{1}{\xi(\omega)} \rangle$ are investigated in Fig. 4 and 5. The uncorrelated disorder is especially considered for the comparison with the electronic case (Sanchez et al., 1994).

**The uncorrelated case:** The propagating properties are described within a statistical procedure. A satisfactory convergence for $10^7$ random samples has been checked up getting with an appropriate tolerance the averaged principal physical magnitude $\langle T(\omega) \rangle$. This statement is necessary to determine with enough accuracy all the other quantities of interest such the size dependence of the transmission coefficient.

![Fig. 3: Transmission coefficient T(N) versus system length for different allowed frequencies at the perfectly ordered case ($\omega = \Omega_a$ and $\omega = 35.00$ rd s$^{-1}$)](image)

![Fig. 4: Averaged transmission frequency response $\langle T(\omega) \rangle$ versus frequency for binary correlated (black color) and uncorrelated (grey color) cases with the parameters: Host lattice: $M_a = 0.100$ kg, $\Omega_a = 40.00$ rd s$^{-1}$, $\bar{\Omega}_a = 43.60$ rd s$^{-1}$. Impurity sub lattice: $M_b = 0.060$ kg, $\Omega_b = 10.75$ rd s$^{-1}$, $\bar{\Omega}_b = 40.00$ rd s$^{-1}$, $c_a = 0.4$)](image)

In this part, we deal with the oscillator binary disorder when the impurity and host unit cells are built with different masses $M_a$, $M_b$ and different springs $k_a$, $k_b$ leading to different free frequencies $\Omega_a$, $\Omega_b$. For instance, the impurity unit cell is characterized with the $\Omega_b = 10.75$ rd s$^{-1}$ and $M_b = 0.060$ kg unit cell while the host lattice is defined with the unit cell $\Omega_a = 40.00$ rd s$^{-1}$ and $M_a = 0.100$ kg.
is the standard K-P formula yielding the frequency spectrum corresponding to the delta peak strength $\lambda_{\omega}$.

This statement may be understood from an analytical consideration using the on site description given in Eq. 14 and 15. The corresponding translating matrices associated to the host and impurity unit cells are defined by:

$$ R_{\Lambda} (\omega) = \begin{pmatrix} 2a_{\Lambda}(\omega) & -1 \\ 1 & 0 \end{pmatrix} \quad (19) $$

and

$$ R_{\beta} (\omega) = \begin{pmatrix} 2a_{\beta}(\omega) & -1 \\ 1 & 0 \end{pmatrix} \quad (20) $$

In particular, at $\omega_{c}$, the two corresponding K-P formulas cross over, i.e., $a_{\Lambda}(\omega_{c}) = a_{\beta}(\omega_{c})$. Hence, the resulting matrix elements become identical and consequently, $R_{\Lambda} (\omega_{c})$ and $R_{\beta} (\omega_{c})$ commute. From a phenomenological point of view, the incident propagating wave does not distinguish the host unit cells from impurity ones since they present the same local translating attitudes. The propagating wire is felt as a perfect ordered lattice with constant effective delta peak strength $\lambda_{c} = \lambda_{\Lambda}(\omega_{c}) = \lambda_{\beta}(\omega_{c})$. Such a characteristic frequency $\omega_{c}$ referred as the commute frequency, can be determined analytically, from the conditions:

$$ R_{\Lambda} (\omega_{c}) = R_{\beta}(\omega_{c}) \quad (21) $$

$$ a_{\Lambda}(\omega_{c}) = a_{\beta}(\omega_{c}) \Rightarrow \lambda_{\Lambda}(\omega_{c}) = \lambda_{\beta}(\omega_{c}) \quad (22) $$

$$ \omega_{c}^{2} = \frac{M_{\Lambda} \Omega_{\Lambda}^{2}}{M_{\beta} - M_{\beta}} \quad (23) $$

At this resonant vibrating mode $\omega_{c} = 61.86 \, \text{rd} \, \text{s}^{-1}$, the two indiscernible unit cells, present an additive behaviour, giving rise to deterministic features. This conclusion is in agreement with the correlated electronic case reported by Hilke et al. (1997), Hakobyan et al. (2000), Bentata et al. (2001) and Bentata (2005) where the origin of such resonance is related to the commuting properties of the binary individual unit cells. We have also to notice that the existence of the set of extended states in a mini band is due to the non-abrupt character of the corresponding transition since the recursive matrix elements get still very close together i.e., $a_{\Lambda}(\omega_{c}) = a_{\beta}(\omega_{c})$ around the discrete resonance $\omega_{c}$. 

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Fig. 5: Reduced Lyapunov frequency response $<\frac{L}{\xi}(\omega)>$ versus frequency for binary correlated (black color) and uncorrelated (grey color) cases with the parameters: Host lattice: $M_{\Lambda} = 0.100$ kg, $\Omega_{\Lambda} = 40.00 \, \text{rd} \, \text{s}^{-1}$ and $\bar{\omega}_{\Lambda} = 43.20 \, \text{rd} \, \text{s}^{-1}$. Impurity sub lattice: $M_{\beta} = 0.060$ kg, $\Omega_{\beta} = 10.75 \, \text{rd} \, \text{s}^{-1}$, $\bar{\omega}_{\beta} = 40.00 \, \text{rd} \, \text{s}^{-1}$, $c_{i} = 0.4$

The commute resonance $\omega_{c}$: In contrast with the quantum electronic model (Sanchez et al., 1994), the classical uncorrelated delta peak binary disorder presents constructive effects around the frequency $\omega_{c} = 10^{-3}$ with the existence of a set of extremely delocalized states:

$$ \frac{L}{\xi(\omega_{c})} \approx 10^{-3} \quad \text{and} \quad T(\omega_{c}) = 1.0 \quad (16) $$

From an analytical point of view and taking the advantage of the $\delta$-function limit, the wave propagation equation can also be achieved within the Poincaré map representation (Bellisard et al., 1982), which in turns enables one to relate the displacements at successive lattice points. Indeed defining $y_{n} = y(x = n'd)$, Eq. 1 may be exactly transformed into a recursive site description:

$$ \begin{pmatrix} y_{n+1} \\ y_{n} \end{pmatrix} = R_{\alpha}(\omega) \begin{pmatrix} y_{n} \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 2a_{\alpha}(\omega) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_{n} \\ y_{n-1} \end{pmatrix} \quad (17) $$

where $R_{\alpha}(\omega)$ is the translating matrix and

$$ a_{\alpha}(\omega) = \cos(kd) + \frac{\lambda_{\alpha}(kd)}{2k} \quad (18) $$
The binary correlated case. A short-range correlation in such binary disorder is considered by introducing at random the impurity oscillators by pairs without any aggregates. As well known from the electronic case, the conventional dimer resonance can be realized with the total transparency of the dimer unit cells within the host allowed band. In other words, the dimer resonance happens when the principal impurity frequency $\omega_0$ belongs to the host allowed band.

The ballistic dimer resonance $\Omega_b$: The new interesting case we examine in our study concerns the conventional dimer effects when the impurity principal frequency coincides with the host free frequency:

$$\omega_b = \Omega_A$$

For the above parameters, this condition is verified. So the corresponding averaged responses $< T(\omega) >$ and $< \frac{L}{\xi(\omega)} >$ of such correlated case is reported in Fig. 4 and 5.

The commute resonance still survives at $\omega_c$ while the dimer resonance appears at the free frequency $\Omega_A$, with more singular and larger localization length, i.e., $\xi(\Omega_A)|_{\omega = \omega_c} = 5.10^6 \xi(\Omega_0)|_{\omega = \omega_c}$. The two different resonances $\Omega_b$ and $\omega_c = 61.86 \text{ rd s}^{-1}$ appear simultaneously within the same allowed band providing attractive various ways to propagate incident waves. Moreover the dimer resonance $\Omega_A$ presents a challenging advantage since it considerably improves the magnitude of the localization length, i.e., $\xi(\Omega_A) = 1000 \xi(\omega_c)$.

The nature of the resonant vibration modes is studied with the description of the transmission coefficient $T(N)$ versus the system length. As depicted in Fig. 6, the presence of periodic envelope function with uniform amplitude similar to an allowed vibration in the ordered case justifies the extended Bloch wave character at the commute resonance $\omega_c$. Indeed, the incident wave does not distinguish the host unit cells from the separately impurity one within the dimer unit cells.

In the other hand, the dimer resonance at the free frequency is similar to the ballistic resonance in ordered case since $T(N)$ gets the unity transmission value independently from the system length. In fact with the conventional dimer statements at $\Omega_0$, the dimer unit cells becomes totally transparent as well as all of the host unit cells where the effective potential profile vanishes over whole of the lattice sites. Consequently the incident wave does not discern again host elements from the dimer unit cells within a deterministic transparent propagating media.

Fig. 6: Averaged Transmission Coefficient $T(N)$ versus system length for the correlated case at $\omega = \Omega_A$ and $\omega = \omega_c = 61.86 \text{ rd s}^{-1}$

In this case, the dimer resonance seems to be equivalent to the commute one suggesting the existence of Bloch extended vibration modes at the free frequency $\Omega_A$. This controversy finding contradicts the general conclusion in the transport properties in one dimensional short range correlated disorder (Huang et al., 1997; Hilke et al., 1998).

CONCLUSION

With the aim to observe the phenomenological aspects of the Anderson localization, the propagation of mechanical waves in random media has been studied by using an analogy with the well known electronic disordered Kronig-Penney model. In this description, two particular frequencies characterize the corresponding ordered case: The principal frequency vanishes the Kronig-Penney analytical equation, i.e., $a(\omega) = 0$ while at the free frequency $\Omega_A$ the ballistic regime settles down i.e., $\lambda_n(\omega) = 0$ Singular behavior happens around the free frequency $\Omega_b$ since the spatial extent length diverges pointing out the Bloch wave functions.

Dimer unit cells can be constructed with a conventional manner or with a new interesting way that enhances the transport properties and ensures the deterministic potential profile even in presence of short range correlated binary disorder: At the ballistic dimer resonance, the Bloch extended vibration modes can be restored in controversies with the general belief in one-dimensional disordered systems.

To conclude, we have studied for the first time numerical results describing the ballistic dimer resonance in the random dimer effect within classical mechanic situation. At this stage, such model presents the main
advantage to be checked experimentally within a rather simple method to conceive ballistic mechanical filters with high quality responses.

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