Partial Densities on the Rational Numbers

Norris Sookoo and Ashok Sahai
University of the West Indies, St. Augustine Trinidad, West Indies

Abstract: Conditions are obtained under which a partial density on a certain class of locally compact abelian groups can be extended to a density. These groups each consist of the integral multiples of a particular rational number, with the discrete topology. It is established that a collection of compatible measures on some of the quotient groups of one such group can be induced by a measure on a particular quotient group. This leads to the result that a partial density can be extended to a density when compatibility conditions are satisfied.

Key words: Partial density, extension to density

INTRODUCTION

Interest in probability measures on locally compact abelian groups has been sustained for a long period. Berg and Rubel (1969) studied the Banach algebra of densities on the compact quotients of a locally compact abelian group, where a density is a system of compatible measures on these compact quotients. They characterized groups \( G \) for which any density is induced by a measure on the semi-periodic compactification of \( G \).

Semigroups of probability measures have also received attention. Siebert (1981) considered semigroups of probability measures and obtained results about the generating functional.

Carnal and Feldman (1995) studied probability measures on a locally compact abelian group. They found conditions under which the identification of a probability measure given the absolute value of its Fourier transform is possible up to a shift and a central symmetry. Carnal and Feldman (1997 and 2000) continued their study of probability distributions which can be identified when the absolute value of their Fourier transforms are known. Mauro Del Muto and Talamanea (2006) considered a Markov process on a locally compact, noncompact, totally disconnected, metrizable abelian group \( G \) and showed that the process may be obtained as a limit of discrete processes on discrete quotient groups of \( G \).

Niederreiter and Sookoo (2000) obtained conditions under which a partial density on the group of integers can be extended to a density. Niederreiter and Sookoo (2002) also investigated conditions under which a partial density on a locally compact abelian group can be extended to a density. In this study, we obtain conditions for extending a partial density to a density when the LCA group is the group of integral multiples of a particular rational number, with the discrete topology.

DEFINITIONS AND NOTATIONS

Notation 1: Let \( \mathbb{Z} \) denote the set of integers and \( \mathbb{R} \) the set of real numbers.

Definition 1: Let \( \frac{p}{q} \) and \( \frac{r}{s} \) be rational numbers in lowest terms. \( \frac{p}{q} \) is called an \( r \)-divisor of \( \frac{r}{s} \) if \( \frac{r}{s} \cdot \frac{p}{q} \) is a whole number.

Definition 2: The \( r \)-greatest common divisor (RGCD) of \( \frac{p}{q} \) and \( \frac{r}{s} \) is the largest rational number \( \frac{h}{k} \cdot \frac{p}{q} \) and \( \frac{r}{s} \cdot \frac{h}{k} \) are whole numbers.

Definition 3: The \( r \)-least common multiple (RLCM) of two rational numbers \( \frac{p}{q} \) and \( \frac{r}{s} \) in lowest terms is the smallest rational number \( \frac{u}{v} \cdot \frac{p}{q} \) and \( \frac{r}{s} \cdot \frac{u}{v} \) are whole numbers.

Definition 4: \( \frac{p}{q} \) and \( \frac{r}{s} \) are called \( r \)-relatively prime if the greatest common divisor of \( p \) and \( r \) is 1 and the greatest common divisor of \( q \) and \( s \) is 1.

Notation 2: If \( p \in \{0, 2, 4, \ldots, \} \), then \( p \) denotes the coset \( \{\ldots, -1+p, p, 1+p, \ldots, \} \) of \( \mathbb{Z} \) in \( h\mathbb{Z}/\mathbb{Z} \).
A density (cf. Berg and Rubel (1969)) on an LCA group \( G \) consists of a system of measures on subgroups of \( G \) of compact index satisfying compatibility conditions.

**Notation 3:** For a suitable index set \( A \), \( \{H_\alpha | \alpha \in A\} \) denotes the set of all subgroups of \( G \) of compact index. \( \{G_\alpha | \alpha \in A\} \) denotes the set of compact quotients of \( G \), where

\[
G_\alpha = G / H_\alpha, \alpha \in A.
\]

**Definition 5:** Let \( D \) be a system of measures given by

\[
D = \{\mu_\alpha | \mu_\alpha \text{ is a probability measure on } G_\alpha, \alpha \in A\}
\]

\( D \) is called a density on \( G \) if the following condition is satisfied:

If \( \psi: G_\beta \rightarrow G_\gamma \) is the natural homomorphism from \( G_\beta \) to a quotient \( G_\gamma \) of \( G_\beta \), then for any Borel set \( B \) in \( G_\alpha \),

\[
\mu_\gamma (\psi^{-1}(B)) = \mu_\beta (B).
\]

Next we define a partial density (Niederreiter, 1975).

**Definition 6:** Let \( G \) be an LCA group and \( \{H_\alpha | \alpha \in A\} \) be the set of all subgroups of compact index of \( G \). For a subset \( B \) of \( A \), let

\[
P = \{\mu_\alpha | \mu_\alpha \text{ is a probability measure on } G_\alpha, \alpha \in B\} \text{ is a system of measures satisfying the following compatibility condition.}
\]

If \( H_\alpha \supseteq H_\beta \) and \( H_\alpha \supseteq H_\beta \) where \( \alpha \in A_1, \beta_1 \in B_1, \beta_2 \in B \), then \( \mu_\beta_1 \) and \( \mu_\beta_2 \) induce the same measure on \( G_\alpha \).

Then \( P \) is called a partial density on \( G \).

**RLCM and RGCD of Rationals**

The following results will be needed to establish the results on systems of measures.

**Theorem 1:** If \( \frac{h}{k} \) is the RGCD of \( \frac{P}{q} \) and \( \frac{r}{s} \), then \( h \) is the GCD of \( p \) and \( r \) and \( k \) is the LCM of \( q \) and \( s \).

**Proof:** Since \( \frac{h}{k} \) must be as large as possible, \( h \) must be as large as possible and \( k \) must be as small as possible, \( h \) has no common factor with \( k \), so it must go into \( p \). Similarly, \( h \) must go into \( r \). Hence \( h \) must be the GCD of \( p \) and \( r \). Also, \( q \) and \( s \) must be divisors of \( k \).

Hence \( k \) must be the LCM of \( q \) and \( s \).

**Theorem 2:** The RLCM of \( \frac{p}{q} \) and \( \frac{r}{s} \) is \( \frac{[\text{LCM of } p \text{ and } r]}{[\text{GCD of } q \text{ and } s]} \)

**Proof:** Let \( \text{RLCM of } \frac{p}{q} \) and \( \frac{r}{s} \) be \( \frac{u}{v} \) in lowest terms.

Then \( \frac{u}{v} = \frac{p}{q} \) times a whole number, that is, \( \frac{u}{v} \times \frac{q}{p} = \frac{u}{v} \) is a whole number. Hence \( p \) must be a divisor of \( u \) and \( v \) must be a divisor of \( q \). Similarly, \( r \) must be a divisor of \( u \) and \( v \) must be a divisor of \( s \). Since \( \frac{u}{v} \) must be as small as possible, \( u \) must be the LCM of \( p \) and \( r \) and \( v \) must be the GCD of \( q \) and \( s \).

**Corollary 1:** RLCM of

\[
\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n}
\]

is \( \frac{[\text{LCM of } p_1, p_2, \ldots, p_n]}{[\text{GCD of } q_1, q_2, \ldots, q_n]} \).

**Extension of Partial Densities on Rationals**

**Theorem 3:** If \( \frac{P}{q} \) and \( \frac{r}{s} \) are relatively prime, \( \mu_1 \) is a probability measure on \( \frac{1}{qs} Z / \frac{1}{q} Z \) and \( \mu_2 \) is a probability measure on \( \frac{1}{qs} Z / \frac{r}{s} Z \), then there exists a measure \( \mu \) on \( \frac{1}{qs} Z / \frac{r}{s} Z \) which induces \( \mu_1 \) and \( \mu_2 \).

**Proof:** Let \( \mu \) be the measure on

\[
\frac{1}{qs} \frac{1}{q} Z \times \frac{1}{qs} \frac{r}{s} Z
\]

given by \( \mu = \mu_1 \times \mu_2 \).

Then

\[
\mu (A \times \frac{1}{qs} \frac{r}{s} Z) = \mu_1 (A) \mu_2 (\frac{1}{qs} \frac{r}{s} Z) = \mu_1 (A), \forall A \in \frac{1}{qs} Z / \frac{p}{q} Z
\]
Similarly,
\[
\mu\left(\frac{1}{qs} Z \cdot \frac{P}{q} Z \right) = \mu_1\left(\frac{1}{qs} Z \cdot \frac{P}{q} Z \right) \mu_2(B) = \mu_2(B),
\forall B \in \frac{1}{qs} Z \cdot \frac{f}{s} Z.
\]

Hence \( \mu \) induces \( \mu_1 \) and \( \mu_2 \).

Let \( a \) and \( b \) be integers. Then
\[
X = a \mod p_{\text{qs}} \quad \text{and} \quad X = b \mod r \Leftrightarrow
\]
\[
X = a + vqr \quad \text{and} \quad X = b + vq\equiv
\]
where \( u \) and \( v \) are integers
\[
\Rightarrow X = r \mod p_{\text{qs}}, k \in Z \Leftrightarrow \frac{X}{qs} = \frac{r}{qs} + kpr
\]
\[
\Leftrightarrow \frac{X}{qs} = \frac{r}{qs} \mod p_{\text{qs}}
\]

Hence \( \frac{1}{qs} Z / p_{qs} Z \cdot \left(\frac{1}{qs} Z / p_{qs} Z \right) \times \left(\frac{1}{qs} Z / f_{qs} Z \right) \)

Hence \( \mu \) can be considered as a measure on \( \frac{1}{qs} Z / p_{qs} Z \) and it is compatible with \( \mu_1 \) and \( \mu_2 \).

Remark 1 In the previous theorem, \( \frac{1}{qs} \) is the RGCD of \( \frac{P}{q} \) and \( \frac{r}{s} \).

Theorem 4 Let \( \text{RGCD} \left(\frac{P}{q}, \frac{r}{s}\right) = d \). If \( \mu_1 \) is a measure on \( hZ / P_{q} Z \) and \( \mu_2 \) is a measure on \( hZ / f_{s} Z \), where \( h = q \) and \( s \) are divisors of \( P \) and \( f \), and \( \{\mu_1, \mu_2\} \) is a partial density on \( hZ \), then there exists a measure \( \mu \) on \( hZ / Z \) which induces \( \mu_1 \) and \( \mu_2 \), where \( I \) is the RLCM of \( \frac{P}{q} \) and \( \frac{r}{s} \).

\textbf{Proof:} We can reduce the problem to d/h simpler ones. In each case, we will have two r-relatively prime numbers, as in the previous theorem. The equations in which the measures of the cosets \( \{ih, ih + d, ih + 2d, ..., ih + (1 - d)\}, I = 0, h, 2h, ..., d, h \) occur do not involve the measures of any other cosets. We show this as follows:

Let \( \mu_1 \) take the values \( X_0, X_1, ..., X_{\frac{P}{q} - 1} \) and \( \mu_2 \) the values \( Y_0, Y_1, ..., Y_{\frac{r}{h} - 1} \). We must establish the existence of a measure \( \mu \) on \( hZ / Z \) such that
\[
\mu(\frac{ih}{q}) + \mu(\frac{P}{q} + \frac{ih}{q}) + \mu(\frac{2P}{q} + \frac{ih}{q}) + ... + \mu(\frac{1}{q} + \frac{ih}{q})
\]
\[
= x_0, i = 0, 1, 2, ..., \frac{P}{q} - 1.
\]

and
\[
\mu(\frac{jh}{h}) + \mu(\frac{r}{s} + \frac{jh}{h}) + \mu(\frac{2r}{s} + \frac{jh}{h}) + ... + \nu(\frac{1}{s} + \frac{jh}{h})
\]
\[
= \nu_j, j = 0, 1, 2, ..., \left(\frac{r}{hs} - 1\right)
\]

Since \( \mu_1 \) and \( \mu_2 \) are compatible,
\[
x_0 + x_1 + x_2 + ... + x_{\frac{P}{q} - 1} = x_0 + x_1 + x_2 + ... + x_{\frac{r}{h} - 1} = \alpha.
\]

The set \( S_0 \) of equations in which \( \mu(0), \mu(\frac{d}{d}), \mu(\frac{2d}{d}), ..., \mu(\frac{1-d}{d}) \) occur does not involve measures of any other cosets of \( hZ / Z \), so they can be solved independently.

If \( \alpha = 0 \), let \( \mu(i) = 0 \) for \( I \in \{0, d, 2d, ..., I-d\} \).

If \( \alpha > 0 \), consider the following problem:

Let \( \lambda_1 \) be a measure on \( hZ / P_{q} Z \cdot \frac{1}{q} \) taking the values
\[
\frac{x_0}{\alpha}, \frac{x_1}{\alpha}, \frac{x_2}{\alpha}, ..., \frac{x_{\frac{P}{q} - 1}}{\alpha}
\]

and let \( \lambda_2 \) be a measure on \( hZ / f_{s} Z \cdot \frac{1}{s} \) taking the values
\[
\frac{y_0}{\alpha}, \frac{y_1}{\alpha}, \frac{y_2}{\alpha}, ..., \frac{y_{\frac{r}{h} - 1}}{\alpha}
\]

832
\( \lambda_1 \) and \( \lambda_2 \) are probability measures, \( \frac{ph}{qd} \) and \( \frac{rh}{sd} \) are \( r \)-relatively prime and also \( h \) is the RGCD of \( \frac{ph}{qd} \) and \( \frac{rh}{sd} \).

From the previous theorem, there exists a measure \( \lambda \) on \( hZ/\mathbb{Z} \) such that \( \lambda \) induces \( \lambda_1 \) and \( \lambda_2 \). Therefore, if we multiply each equation in \( S_1 \) by \( \frac{1}{\alpha} \) and replace \( \mu(\id) \) by \( \lambda(h); i = 0, 1, 2, \ldots, \frac{1}{d} - 1 \), then the new set of equations obtained has at least one solution such that \( \lambda(ih) \geq 0; i = 0, 1, 2, \ldots, \frac{1}{d} - 1 \). Hence \( S_1 \) has solutions for which \( \mu(\id) \geq 0; i = 0, 1, 2, \ldots, \frac{1}{d} - 1 \). We can show, in a similar way, that the set \( S_2 \) of equations in which \( \mu(\id), \mu(\id + d) \), \( \mu(\id + 2d), \ldots, \mu(\id + (d-1)d) \) occur (where \( j \) is a fixed, arbitrary element of \( \mathbb{Z}/h \) has nonnegative solutions. Hence the original set of equations has nonnegative solutions and so there exists a measure \( \mu \) which induces \( \mu_1 \) and \( \mu_2 \).

**Theorem 5** Let \( \{\mu_{\id}; i = 1, 2, \ldots, k\} \) be a partial density on \( hZ \), where \( \mu_{\id} \) is a measure on \( hZ_1 \), and let \( R_1 \) be the RLCM of \( P_1, P_2, P_3, P_4 \). If \( R_1 \) and \( \frac{\mu_{\id}}{q_i} \) for at least one of \( \frac{P_1}{q_i}, \frac{P_2}{q_i}, \frac{P_3}{q_i}, \frac{P_4}{q_i} \), exists a measure on \( hZ/R_1Z \) which induces \( \mu_{\id} \) and \( \mu_{\id} \), from the previous theorem.

Proof: There exists a measure \( v_1 \) on \( hZ/TZ \) which induces \( \mu_{\id} \) and \( \mu_{\id} \), from the previous theorem.

Let \( T = \text{RGCD}(R_1, P_1) \). Then \( T \) is an \( r \)-divisor of at least one of \( P_1, P_2 \). Assume that \( T \) is an \( r \)-divisor of \( P_1 \). Since \( \{\mu_{\id}; i = 1, 2, \ldots, k\} \) is a partial density, \( \mu_{\id} \) and \( \mu_{\id} \) induce the same measure, \( \omega \) on \( hZ/TZ \) and since \( v_1 \) induces \( \mu_{\id} \), \( v_2 \) and \( \mu_{\id} \) also induce the measure \( \omega \) on \( hZ/TZ \). Hence \( v_1 \) and \( \mu_{\id} \) comprise a partial density on \( hZ/TZ \). From the previous theorem, there exists a measure \( v_1 \) on \( hZ/R_1Z \) which induces \( v_2 \) and \( \mu_{\id} \), since the RLCM of \( R_1 \) and \( P_1 \) is the RLCM \( \frac{P_1}{q_1}, \frac{P_2}{q_1}, \frac{P_3}{q_1}, \frac{P_4}{q_1} \). Clearly, \( v_1 \) also induces \( \mu_{\id} \) and \( \mu_{\id} \). If we assume that \( T \) is an \( r \)-divisor of \( P_1 \), we reach the same conclusion. Now \( \text{RGCD}(R_1, P_1) \) is a divisor of at least one of \( \frac{P_1}{q_1}, \frac{P_2}{q_1}, \frac{P_3}{q_1}, \frac{P_4}{q_1} \). As above, we can show that there exists a measure \( v_1 \) on \( hZ/R_1Z \) which induces \( \mu_{\id}, \mu_{\id}, \mu_{\id} \) and \( \mu_{\id} \). Continuing like this, we can prove the existence of a measure \( v_k \) on \( hZ/R_kZ \) which induces \( \mu_{\id}, i \in \{1, 2, \ldots, k\} \).

Remark 2 If \( \mu_i \) is a measure on \( Z/4Z \) and \( \mu_4 \) is a measure on \( Z/6Z \), then for compatibility it is necessary that:

\[
\mu_i(1+6Z) + \mu_i(3+6Z) + \mu_i(5+6Z) = \mu_i(1+4Z) + \mu_i(3+4Z)
\]

since

\[
(1+6Z) \cup (3+6Z) \cup (5+6Z) = (1+4Z) \cup (3+4Z)
\]

as we illustrate on the following portion of the real line.

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*, ** and *** represent \((1+6Z), (3+6Z)\) and \((5+6Z)\), respectively and # and ## represent \((1+4Z)\) and \((3+4Z)\), respectively.

**Theorem 6:** Let \( \{\mu_{\id}; i \in B\} \) be a partial density on \( hZ \) such that \( \text{RGCD}(R_1, P_{\id}) \) is a divisor of at least one of \( \frac{P_1}{q_i} \), for every \( a \in B \). Then \( P \) can be extended to a density on \( hZ \).
Proof: Let \( \mathbb{N}_{\frac{B}{q}} \) be the set of continuous, real-valued functions on \( hZ \) with period \( \frac{P_i}{q_i} \) for each \( i \in B \). Also, let

\[
M = \left\{ \sum_{i=1}^{n} k_i f_i \mid f_i \in \mathbb{N}_{\frac{B_i}{q_i}}, k_i \in \mathbb{R}, i = 1, 2, \ldots, n \right\}.
\]

Define \( L \) on \( M \) as follows:

\[
L(f) = \sum_{i=1}^{n} \int_{Z/\mathbb{Z}} k_i f_i d\mu_{\frac{B_i}{q_i}}
\]

where

\[
f = \sum_{i=1}^{n} k_i f_i, f_i \in \mathbb{N}_{\frac{B_i}{q_i}}, k_i \in \mathbb{R}
\]

and \( \mu_{\frac{B_i}{q_i}} \) is the measure on \( hZ/\frac{P_i}{q_i} Z \) in \( P_i \); \( i \in \{1, 2, \ldots, n\} \).

Let \( g \in M \) and \( g \geq 0 \). Then \( g = g_1 + g_2 + \ldots + g_a \), where \( a \in B \) and \( g \) has period \( \frac{P_i}{q_i}, i \in \{1, 2, \ldots, a\} \).

From the previous theorem, there exists a probability measure \( \mu \) on \( hZ/\mathbb{Z} \) which induces \( \mu_{\frac{B_i}{q_i}} \); \( i = 1, 2, \ldots, a \).

Hence

\[
L(g) = \sum_{i=1}^{n} \int_{Z/\mathbb{Z}} g d\mu_{\frac{B_i}{q_i}} = \sum_{i=1}^{n} \int_{Z/\mathbb{Z}} g_i d\mu = \int_{Z/\mathbb{Z}} g d\mu \geq 0,
\]

since \( g \geq 0 \).

Hence \( L \) is positive. Theorem 3.3 of (Carnal and Felman, 2000) implies that \( P \) can be extended to a density on \( Z \).

REFERENCES