Probability Distribution of m Binary n-tuples

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Abstract: In coding theory, a string of ones and zeros is called a word. When these digits are transmitted and received electronically, errors may occur and any digit may be misidentified, thus giving rise to a string of errors, one for each digit. For the case where m binary words of length n are transmitted and received, we obtain the probability distribution of the error tuple, which shows the number of words having i errors, as i varies from 1 to n. Related probabilities are obtained and also the probability and moment generating functions are derived.

Key words: Weight tuple, generating function, probability function

INTRODUCTION

Discrete distributions have been widely studied, often in connection with models that arise in reliability experiments. Samaniego and Jones (1981) studied maximum likelihood estimation for a class of multinomial distributions that arise in reliability experiments in which components have different probabilities of failure. Binomial signals in noise are encountered in diverse areas. Woodward (1953) discussed a communication system for which the observed output is the sum of independent Bernoulli variables. Study of discrete signal detection includes the work of Greenstein (1974) on block coding of binary signals.

Studies of almost binomial distributions abound. Shah (1966) obtained moment estimates for truncated binomial distributions and Thompson (2002) considered a variation of the binomial distribution in which the parameter n can be any positive number. This new distribution is useful in approximation situations.

Balakrishnan (1997) has reviewed many recent applications of combinatorial methods in Probability and Statistics. The multinomial distribution (Johnson et al., 1997) is a well known distribution which generalizes the binomial distribution. We use Combinatorial methods to study another generalization of the binomial distribution. We consider an n-tuple of zeros and ones to be a sequence of indicators of n Bernoulli trials. That is, suppose that a sequence of n zeros and ones, called a word in coding theory, is transmitted electronically such that the probability of getting a non-zero error in a received digit is a constant, p. The sequence of n zeros and ones representing the errors is called an error vector in coding theory.

The distributions we study can also be utilized in investigating the reliability of systems. If a system is made up of n identical components connected in parallel and the probability that a component will fail in a given time period is a constant p, then the number x of components that will fail in that period has binomial distribution. For m systems of this type, the first result is equal to the probability that x systems will experience i failures. Thus if a system is made up of m subsystems like the above and this larger system functions properly only if k of the m subsystems function properly, we can calculate the relevant probability using the distribution obtained in this study. For the subsystems, the probability that each component will function may vary from subsystem to subsystem and result covers this case.

DEFINITIONS AND NOTATIONS

The following concepts are taken from Coding Theory.

Definition 1: Let c be an n-tuple of zeros and ones. c is called a word of length n.

Definition 2: The weight of c, denoted by w(c), is the number of ones in c.

We assume that when words are transmitted and received, the probability that an error in a digit occurs is p, the probability that no error occurs is q and p+q = 1. An error occurs if 1 is transmitted and identified as 0, or 0 is transmitted and identified as 1.

Definition 3: Suppose that m consecutive words of length n are transmitted. The m words will be called an mn-block of words. The associated m error vectors of length n will be called an mn-block of error vectors.

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**Definition 4:** The weight-tuple associated with an m*n block of error vectors is \( Y = (Y_0, Y_1, ..., Y_n) \), where \( Y_i \) is the number of error vectors of weight \( i \) in the m*n block, \( (i = 0, 1, ..., n) \).

**Definition 5:** The joint probability mass function of \( Y \) is

\[
P_Y(y) = P(Y = y) = P(Y_0 = y_0, Y_1 = y_1, ..., Y_n = y_n)
\]

where \( y = (y_0, y_1, ..., y_n) \).

**Definition 6:** The probability generating function (Johnson et al., 1997) of \( Y \) is

\[
G_Y(n, m, s) = \sum_{y_0, y_1, ..., y_n} P(Y = y) s_0^{y_0}s_1^{y_1}...s_n^{y_n}
\]

subject to the condition \( \sum_{i=0}^{n} y_i = m \).

**Definition 7:** The moment generating function (Johnson et al., 1997) of \( Y \) is

\[
\phi_Y(t) = E[e^{b_0Y_0+b_1Y_1+...+b_nY_n}]
\]

**Theorem 1:** If an m*n-block is received, then

\[
P[Y = (y_0, y_1, ..., y_n)] = \frac{m!}{y_1!y_2!...y_n!} \left( \frac{n}{1} \right)^{y_0} \left( \frac{n}{2} \right)^{y_1} ... \left( \frac{n}{n} \right)^{y_n} p^y(1-p)^{m-y}
\]

where \( y = y_0 + 2y_1 + ... + ny_n \).

**Proof:** Suppose that an m*n-block is received. The block of associated error vectors can be represented by an m*n matrix of zeroes and ones. Consider such a matrix where each of the first \( y_0 \) rows is of the form 1 0 0 ... 0, each of the next \( y_1 \) rows is of the form 1 1 0 0 ... 0, each of the next \( y_2 \) rows is of the form 1 1 1 0 0 ... 0 and so on, with the last \( y_n \) rows having the form 1 1 1 ... 1.

The probability that the received m*n-block of error vectors has the above form is

\[
m! \left[ p^y(1-p)^{m-y} \right]^y = p^y(1-p)^{m-y}
\]

where \( y = y_0 + 2y_1 + ... + ny_n \).

The number of different m*n-blocks with the first \( y_i \) error vectors having weight \( 0 \), the next \( y_i \) error vectors having weight 1, the next \( y_i \) error vectors having weight 2, etc. \( \binom{n}{0}^{y_0} \binom{n}{1}^{y_1} ... \binom{n}{n}^{y_n} \). Also, the number of different m*n-blocks with the \( y_i \) error vectors of weight \( i \), \( (i = 1, 2, ..., n) \) in fixed positions is the same. The number of different ways that the \( m \) error vectors in an m*n-block can be arranged so that the \( y_i \) error vectors words of weight \( i \) \( (i = 1, 2, ..., n) \) occupy the same position is \( \frac{m!}{y_1!y_2!...y_n!} \).

**Theorem 2:** The probability generating function of \( Y \) is

\[
q^m s_0 + \binom{n}{1} q s_1 + \binom{n}{2} q^2 s_2 + ... + \binom{n}{n} q^n s_n \]

subject to the condition \( y_0 + y_1 + ... + y_n = m \).

**Proof:** The probability generating function is

\[
\sum_{y_0, y_1, ..., y_n} P[Y = (y_0, y_1, ..., y_n)] q^{y_0} s_1^{y_1} ... s_n^{y_n}
\]

subject to the condition \( \sum_{i=0}^{n} y_i = m \), where \( s = y_0 + 2y_1 + ... + ny_n \).

\[
= q^m \left[ s_0 + \binom{n}{1} q s_1 + \binom{n}{2} q^2 s_2 + ... + \binom{n}{n} q^n s_n \right]^{m}
\]

**Theorem 3:** The moment generating function of \( Y \) is

\[
\phi_Y(t) = q^m \left[ e^{t_0s_0} e^{t_1s_1} e^{t_2s_2} ... e^{t_ns_n} \right]^{n}
\]

**Proof:** \( \phi_Y(t) = E[e^{t_0Y_0+t_1Y_1+...+t_nY_n}] \)

\[
\sum_{y_0, y_1, ..., y_n} e^{t_0y_0+t_1y_1+...+t_ny_n} \frac{m!}{y_1!y_2!...y_n!} \left( \frac{n}{1} \right)^{y_0} \left( \frac{n}{2} \right)^{y_1} ... \left( \frac{n}{n} \right)^{y_n} q^{y_0} p^{y_1} q^{y_2} p^{y_3} q^{y_4} ...
\]

(where \( \delta \) is the same as in the previous theorem.)
\[ P[Y_i = y_i] = \binom{n}{y_i} \left( \frac{m}{n} \right)^{y_i} p^{y_i} q^{n-y_i} \]

**Proof:**

\[ P[Y_i = y_i] = \sum_{y_1+y_2+\ldots+y_m = n} \frac{m!}{y_1!y_2!\ldots y_m!} \left( \frac{n}{m} \right)^{y_1}\left( \frac{n}{m-1} \right)^{y_2}\ldots \left( \frac{n}{m-y_m+1} \right)^{y_m} p^{y_1} q^{n-y_1-y_2-\ldots-y_m} \]

\[ = \binom{n}{i} \frac{1}{i!} \frac{1}{i!} \frac{1}{i!} \ldots \frac{1}{i!} \left( \frac{n}{m} \right)^{y_i}\left( \frac{n}{m-1} \right)^{y_i+1}\ldots \left( \frac{n}{m-y_i+1} \right)^{y_i} p^{y_1} q^{n-y_1-y_2-\ldots-y_m} \]

\[ = \binom{n}{i} \frac{1}{i!} \frac{1}{i!} \frac{1}{i!} \ldots \frac{1}{i!} \left( \frac{n}{m} \right)^{y_i}\left( \frac{n}{m-1} \right)^{y_i} \ldots \left( \frac{n}{m-y_i} \right)^{y_i} p^{y_1} q^{n-y_1-y_2-\ldots-y_m} \]

**DIFFERENT ERROR PROBABILITIES FOR DIFFERENT WORDS**

Suppose that \( m \) words, each of length \( n \), are transmitted such that when the \( i \)th word is being transmitted and received, \( P \) [a digit is incorrectly identified] = \( p_i \), and \( P \) [a digit is correctly identified] = \( q_i \), where \( q_i, \ i = 1, 2, \ldots, m \). We define \( Y \) as before and set \( \tau = \frac{p_i}{q_i} \).

**Theorem 5:**

\[ P[Y] = \left( y_1, y_2, \ldots, y_m \right) = \prod_{i=1}^{m} \frac{p_i^{y_i} q_i^{n-y_i}}{y_1! y_2! \ldots y_m!} \]

where the summation is taken over all permutations \((b_1, b_2, \ldots, b_m)\) of \( y_1, y_2, \ldots, y_m \) for \( y_1, y_2, \ldots, y_m \) integers.

**Proof:**

Let us consider an outcome in which the first \( y_0 \) error vectors are of weight zero, the next \( y_1 \) error vectors are of weight one, the next \( y_2 \) error vectors are of weight two, \( \ldots \), the last \( y_m \) error vectors are of weight \( n \).

Probability of such an outcome

\[ = \prod_{i=0}^{y_0} p_i^{y_i} \prod_{i=1}^{n} q_i^{n-y_i} \]

\[ \times \prod_{i=1}^{y_1} (\prod_{j=1}^{y_0} p_j^{y_j}) \prod_{i=2}^{n} q_i^{n-y_i} \]

\[ \times \prod_{i=2}^{y_2} (\prod_{j=2}^{y_1} p_j^{y_j}) \prod_{i=3}^{n} q_i^{n-y_i} \]

\[ \ldots \]

\[ \times \prod_{i=n}^{y_n} (\prod_{j=n-1}^{y_{n-1}} p_j^{y_j}) \prod_{i=n+1}^{n} q_i^{n-y_i} \]

Every possible outcome with the desired weight distribution will have a probability equal to the above with some rearrangement of the powers. Hence

\[ P(Y) = \sum_{b_1, b_2, \ldots, b_n} \prod_{i=0}^{y_0} p_i^{y_i} \prod_{i=1}^{n} q_i^{n-y_i} \]

where the condition on the \( b_i \)'s are as stated in the theorem.

**Theorem 6:** The probability generating function of \( Y \) is

\[ G_Y(t, s) = \sum_{y_1, y_2, \ldots, y_m} \prod_{i=1}^{m} t_i^{y_i} s_i^{n-y_i} \]

\[ G_Y(t, s) = \prod_{i=1}^{m} \frac{p_i t_i}{q_i} \]

\[ = \sum_{y_1, y_2, \ldots, y_m} \prod_{i=1}^{m} \frac{p_i^{y_i} q_i^{n-y_i}}{y_1! y_2! \ldots y_m!} \]

\[ = \prod_{i=1}^{m} \frac{p_i^{y_i} q_i^{n-y_i}}{y_1! y_2! \ldots y_m!} \]

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where \( Q = (q_1, q_2, ..., q_n)^p \)

**Proof:**

\[
G = \sum_{y_0, y_1, ..., y_n} P [ Y = (y_0, y_1, ..., y_n) ] s_0^{y_0} s_1^{y_1} ... s_n^{y_n} \\
= \sum_{y_0, y_1, ..., y_n} Q \{ y \} \sum_{i_0, i_0, ..., i_n} r_0^{i_0} r_1^{i_1} ... r_n^{i_n} s_0^{y_0} s_1^{y_1} ... s_n^{y_n} \\
\]

where the inner summation is taken over all permutations 
(b_1, b_2, ..., b_m) of y_0 zeros, y_1 ones, y_2 twos, ..., y_n n's.

\[
= Q(r_0^{i_0} s_0 + r_1^{i_1} s_1 + ... + r_n^{i_n} s_n) \\
\times (r_0^{i_0} s_0 + r_1^{i_1} s_1 + ... + r_n^{i_n} s_n) \\
\times ...
\times (r_0^{i_0} s_0 + r_1^{i_1} s_1 + ... + r_n^{i_n} s_n)
\]

because in multiplying out the above product, each time we choose \( s_n \), the power of the corresponding \( r \) will be i. Hence the coefficient of \( s_0^{y_0} s_1^{y_1} ... s_n^{y_n} \) will be \( Q \{ y \} \cdot r_0^{i_0} r_1^{i_1} ... r_n^{i_n} \) where \( b_1, b_2, ..., b_m \) is as described above.

**REFERENCES**


