Probabilistic Aspects of Lagrange 3D-Interpolational

Kamal Al-Dawoud
Department of Mathematics and Statistics, Mutah University, 61710 Mutah, Jordan

Abstract: In this study, the problem of polynomial 3D interpolation on finite elements is studied and probabilistic aspects of finite-element approximation on three-dimensional models is presented. The theorems for new probabilistic properties of basis functions are proved.

Key words: Geometrical probability, Lagrange polynomial, finite-element method, tetrahedron, regular hexahedron, three-linear interpolation

INTRODUCTION

An application of geometrical probability (Kamal Al-Dawoud and Khomchenko, 2007) for constructing polynomials basic functions essentially simplifies problems of approximation in finite-elements method (Norrie and de Vries, 1978; Oden, 1972). In this paper the probabilistic aspects of finite-element approximation on three-dimensional models are presented. Special attention is given to a simplex (a tetrahedron, 4 units) and multiplex (a cube, 8 units). Usually nodal parameters are more favorably to choose on vertexes of an element, as vertexes are the general for more number of elements, than units on edges or lateral sides. Such choice reduces the general number of central parameters of system elements and reduces the size of a global matrix of the linear algebraic equations system (Norrie and de Vries, 1978).

Simplex models (Oden, 1972) are concerned with using linear polynomials in finite-elements. They were among the first used in (FEM) in 1956 (Turner, Clough, Martin and Toppi, in 1957 (Synga), in 1962 (Gallagher, Padlog and Bijlhaard). Finite-elements in the form of a cube have quickly won popularity in three-dimensional problems, where one cube took the same volume, as 6 tetrahedrons. Let's notice, that irregular splitting of the area into tetrahedrons is difficult for carrying out even with the help of a Computer (Strang and Fix, 1973). Three-linear approximation on a cubic element for the first time was used in 1963. (Melosh), then in 1966 (Key), in 1967 (Zienkiewicz and Cheung), in 1969 (Oden).

Kolmogorov's model of random wanderings on a three-dimensional grid allows schematizing random wanderings with random start and absorbing units in vertexes of a finite-element. Computer experiments give the basis to assume, that the transitive probabilities have properties of stability and are independent of the form of the trajectory and the number of steps. The establishment of the specified properties allows ignoring a history of random wanderings and stimulates searches of the simplified single-step scheme, which appreciably accelerate calculations in Monte Carlo methods. The economical schemes of random transitions are the result of minimizing the number of steps. In this study, it is theoretically proved that the transitive probability invariant concerning the form of a route depends only on coordinates of a starting-point and vertex of an element (finish-point). In the optimum scheme, the particle for one step on a start-line route reaches vertex of an element. Interpolations function of three arguments is a mathematical expectation of nodal values. From the mechanical point of view, transitive probabilities show, how to distribute a single mass on vertexes of an element, which barycentric appeared in a reselected point.

FORMULATION AND SOLUTION

On Fig. 1, the three-dimensional simplex-a tetrahedron with 4 nodes is represented. This element has equipment with 4 basic functions. In research problems of scalar fields in each node, there is one degree of freedom (for example, temperature). Traditional algebraic procedure of designing polynomial a interpolation is reduced to the definition of 4 parameters $a$, in a general view polynomial:

$$P (x, y, z) = a_0 + a_x x + a_y y + a_z z$$

The source information contains 16 numbers: coordinates of nodes $p_k (k = 1, 4)$ and nodal temperatures $f_k (k = 1, 4)$. For determining $a$ using systems of linear algebraic equations $4 \times 4$, where $k$-th equation of system is given by:

$$a_0 + a_1 x_k + a_2 y_k + a_3 z_k = f_k, k = 1, 4$$

The system (2) has the unique solution as its determinant $\Delta$ is not zero:
Property (5) represents special interest, as it has a precise probabilistic sense. To each nodal value of function \( f_k \) is matched a corresponding probability \( N_k \). Thus, we can write the law of distribution of probabilities for the function of a random point \( M(x, y, z) \):

\[
\begin{array}{c|c|c|c|c|c}
\text{f} & N_1 & N_2 & N_3 & N_4 \\
\hline
p_i & N_i & N_i & N_i & N_i \\
\end{array}
\]

Now it is clear that interpolation polynomials value (4) in any point of a simplex is determined by the formula of expectation. Feature of the resulted table, where selective values are fixed and a random factor is present at the second row. Functions of a random point \( N_k(x, y, z) \) are interpreted as transitive probabilities of a wandering particle, from a random point \( M(x, y, z) \), to vertexes of a tetrahedron \( \Pi_k \).

On Fig. 1, arrows are shown the routes of random transitions. Thus, in a tetrahedron the single-step 4-routing scheme of random transitions with random start and absorption in vertexes is realized. In terms of Monte Carlo method, formula (4) is the average compensation for an output of particles in vertex. It means construction of interpolation polynomials is reduced to the definition of transitive probabilities. On a simplex \( N_k \) are easily defined geometrically through relations of volumes of two tetrahedrons with the general side. For example:

\[
N_k(x, y, z) = \frac{\Delta_k}{\Delta}, \quad k = 1, 4
\]

The determinant \( \Delta_k \) produced from a determinant (3) replacement in k-th row of coordinates of vertex \( \Pi_k \) by coordinates of the current point \( M(x, y, z) \). It is easy to notice, that basis Lagrange \( \{N_k\} \) will consist of barycentric coordinates of a three-dimensional simplex, which have the following properties:

\[
0 \leq N_k \leq 1, \sum_{k=1}^{4} N_k = 1, \quad N_k(x, y, z) = \delta_{ik}, \quad (5)
\]

Where:
\( \delta_{ik} = \text{Kronecker's symbol} \)
\[ N_i(x, y, z) = \frac{1}{8} (1-x)(1-y)(1-z) \]  

(9)

Other functions \( N_i \) are defined similarly or from \( N_i \) using consecutive transformation of parallel route carried on 2 units along one of coordinate directions (Fig. 2). Properties (5) are easy to check up, as in this model.

A lot of interesting properties of posteriori transitive probabilities \( \eta_i/n \) are found out in computer experiments with random wanderings in multiplex on nodes of an orthogonal spaces grid. Here \( n \)-the general number of particles, starting from control node, \( \eta_i \)- (number of the particles absorbing vertex \( \Pi_i \)). In the experiments, the Kolmogorov’s classical model with a 6-routing pattern and equally probability transitions on each step is used. An output of a particle on side multiplex wanderings turns into two-dimensional, at an output on an edge-in one-dimensional and come to an end in one of two vertexes \( \Pi_{10} \), belonging to the given edge. First of all it is necessary to specify convergence in probability:

\[ \eta_i/n \rightarrow N_i \text{ at } n \rightarrow \infty \]

Experiments have confirmed independence of transitive probability of the form of a route and number of steps from start to finish. There are no bases to doubt about the result of experiments. However we shall try to prove the following theorem theoretically using probabilistic representations.

**Theorem 1**: For a particle, starting from any point of \( M \) multiplex, the probability of absorbing vertex \( \Pi_i \) is invariantly concerning the form of a trajectory and also coincides with corresponding function \( N_i \)-three-linear interpolation.

**Proof**: On Fig. 2, different routes from a point \( M(x, y, z) \) in vertex \( \Pi_i (-1; -1; -1) \) are demonstrated. Each broken line consists of three straight-line segments, parallel to coordinate axes. On any route, the particle for 3 steps reaches vertex \( \Pi_i \). On the first step, the particle goes to one of three edges containing vertex \( \Pi_i \). On the second step, the particle goes to one of two edges containing vertex \( \Pi_i \), on the third step, the particle is absorbed by vertex \( \Pi_i \). For the particles absorbed by other vertexes, the situation is similar. To prove this theorem, it is enough to consider one of six possible routes from \( M \) to \( \Pi_{10} \), for example:

\[ M \rightarrow A_i \rightarrow B_i \rightarrow \Pi_{10} \]

The probability of transition \( M \rightarrow A_i \) is defined geometrically and is:
The integral test of harmonic, suggested by Privalov, is the mean-value integral by element:

\[ \frac{1}{8} \iint N_8(x, y, z) dxdydz = N_8(0, 0, 0) = \frac{1}{8}. \]  

(10)

Simple consideration shows, that \( N_8(x, y, z) \) satisfies Laplace equation and the rules mean-value. Notice, that in formula (10) multiplier 1/8 before triple integral is a density of uniform distribution of a random point in multiplex. Therefore, Privalov's test gives expectation of function of a random point. This result has exactly probability meaning and is formulated as the following theorem.

Theorem 3: Expectation of transitive probability \( N_8(x, y, z) \) on all random trajectories in multiplex is equal to probability of transition of a particle from barycentric to vertex.

For proof it is enough to refer to formula (10).

Remark: Surprisingly, the function \( N_8(x, y, z) \), containing members of the second and third degree, supposes exact integration using a simplified approached formula with a unique node in barycentric of an element.

**CONCLUSION**

New probability properties of basic functions Lagrange 3D-interpolation are established. It stimulates attempts to distribute probability approaches on polynomials of the higher orders in one-dimensional, two-dimensional and three-dimensional finite elements. Special interest is represented with penal routes with negative transitive probabilities. Such generalization of models of random wanderings will need correct and grounded formulations.

**REFERENCES**