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Stability of the Cubic Functional Equation in Menger Probabilistic Normed Spaces

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Abstract: In this study, the stability of the cubic functional equation: $f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)$ in the setting of Menger probabilistic normed spaces is proved.

Key words: Hyers-Ulam-Rassias stability, cubic mappings, generalized normed space, Banach space, normed space

INTRODUCTION

The functional equation:

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)$$

is said to be the cubic functional equation. Skof (1983) by proving that if $f$ is a mapping from a normed space $X$ into a Banach space $Y$ satisfying:

$$||f(x+y)+f(x-y)-2f(x)-2f(y)||\leq \varepsilon$$

for some $\varepsilon>0$, then there is a unique quadratic function $g:X\rightarrow Y$ such that:

$$||f(x)-g(x)||\leq \frac{\varepsilon}{2}$$


In the sequel, the usual terminology, notations and conventions of the theory of random normed spaces shall be adopted, as by Schweizer and Sklar (1983). Throughout this paper, the space of all probability distribution functions (briefly, d.f.’s) is denoted by:

$$\Delta = \{f: R \cup \{-\infty, +\infty\} \rightarrow [0,1]: F(0) = 0 \text{ and } F(\infty) = 1\}$$

where $F$ is left continuous and non decreasing on $R$. Also the subset is the set:

$$D' = \{f(\Delta^+): I'(\Delta^+) = 1\}$$

where, $I'(\Delta)$ denotes the left limit of the function $f$ at the point $x$, $I'(\Delta) = \lim_{x \rightarrow -} f(t)$. The space $\Delta^+$ is partially ordered by the usual point-wise ordering of functions, i.e., $D \leq G$ if and only if $F(t) \leq G(t)$ for all $t$ in $R$. The maximal element for $\Delta^+$ in this order is the d.f. given by:

$$\tau_0(t) = \begin{cases} 
0, & \text{if } t \leq 0 \\
1, & \text{if } t > 0 
\end{cases}$$

Definition 1: A mapping $T:[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if $T$ satisfies the following conditions:

- $T$ is commutative and associative
- $T$ is continuous
- $T(a, 1) = a$ for all $a \in [0, 1]$
- $T(a, b) = T(c, d)$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0, 1]$

Two typical examples of continuous t-norm are $T(a, b) = \min(a, b)$ and $T(a, b) = \min(a, b)$

Now t-norms are recursively defined by $T' = T$ and

$$T' \left( x_1, \ldots, x_n \right) = T \left( T' \left( x_1, \ldots, x_{n-1} \right), x_n \right)$$

for $n \geq 2$ and all $x \in [0, 1]$, for all $I \in \{1, 2, \ldots, n+1\}$

The t-norm $T$ is Hadzic type if for given $e \in (0, 1)$ there is $\delta \in (0, 1)$ such that:

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\[ T^*(\delta,\ldots,\delta) > 1 - \varepsilon, \quad \text{for } n \in \mathbb{N} \]

A typical example of such a t-norm is \( T(a, b) = \min(a, b) \). Recall that if \( T \) is a t-norm and \( \{X_0\} \) is a given sequence of numbers in \([0, 1]\), \( T_{\infty} \), is defined recursively by:

\[ T_{\infty} X_i = s_i \]

and

\[ T_{\infty} X_i = T(T_{\infty} X_i, X_i) \text{ for } n \geq 2 \]

is defined as

\[ \lim_{n \to \infty} T_{\infty} X_i \]

**Definition 2:** A Menger Probabilistic normed space (briefly, Menger PN-space) is a triple \((X, \mu, T)\), where \( X \) is a nonempty set, \( T \) is a continuous t-norm and \( \mu \) is a mapping from \( X \) into \( D^* \) such that, the following conditions hold:

- (PN1) \( \mu_x \) for all \( t > 0 \) if and only if \( x = 0 \)
- (PN2) \( \mu_{x+y}(t) = \text{min}(\mu_x(t), \mu_y(t)) \) for all \( x, y \in X \) and \( t > 0 \)
- (PN3) \( \mu_{xy}(t) = \mu_x(t) \mu_y(t) \) for all \( x, y \in X, t > 0 \)
- (PN4) \( \mu_{xy}(t+s) \geq T(\mu_x(t), \mu_y(s)) \) for all \( x, y, z \in X \) and \( t, s > 0 \)

Clearly every Menger PN-space is a probabilistic metric space having a metrizable uniformity on \( X \) if \( \text{sup}_{x \in X} T(x, a) = 1 \).

**Definition 3:** Let \((X, \mu, T)\) be a Menger PN-space:

- A sequence \( \{x_n\} \) in \( X \) is said to be convergent to \( x \) in \( X \) if, for every \( t > 0 \) and \( \varepsilon > 0 \), there exists positive integer \( N \) such that \( \mu_{x_n}(t) > 1 - \varepsilon \) whenever \( n \geq N \)
- A sequence \( \{x_n\} \) in \( X \) is called Cauchy sequence if, for every \( t > 0 \) and \( \varepsilon > 0 \), there exists positive integer \( N \) such that \( \mu_{x_n}(t) > 1 - \varepsilon \) whenever \( n, m \geq N \)
- A Menger PN-space \((X, \mu, T)\) is said to be complete if and only if every Cauchy sequence in \( X \) is convergent to a point in \( X \)

**Theorem 1:** If \((X, \mu, T)\) is a Menger PN-space and \( \{x_n\} \) is a sequence such that \( x_n \to x \) then \( \lim_{n \to \infty} \mu_{x_n}(t) = \mu_x(t) \)

In this study, the stability of the quadratic functional equation in the setting of Menger probabilistic normed spaces is established.

**MAIN RESULTS**

**Definition 4:** Let \( X, Y \) be vector spaces. The functional equation \( f: X \to Y \) defined by:

\[ f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \quad (1) \]

is called cubic functional equation.

**Theorem 2:** Let \((X, \nu, R)\) be Menger PN space and \((Y, \mu, T)\) be a complete Menger PN-space. If \( f: X \to Y \) be a mapping such that:

\[ \mu_{\nu(\xi(t)+\xi(t))} \geq \xi_{\nu(\xi(t))} \geq \xi_{\mu(\xi(t))} \quad (2) \]

for \( t > 0 \) in which \( \xi: X^2 \to D^* \) and

\[ \lim_{t \to \infty} \mu_{\nu(\xi(t))} = 1 \quad (3) \]

Then there exists a unique quadratic mapping \( Q: X \to Y \) such that:

\[ \mu_{\nu(\xi(t))} \geq \xi_{\nu(\xi(t))} \geq \xi_{\mu(\xi(t))} \quad (4) \]

**Proof:** Putting \( y = 0 \) in Eq. 2, then:

\[ \mu_{\nu(\xi(t))} \geq \xi_{\nu(\xi(t))} \geq \xi_{\mu(\xi(t))} \quad (5) \]

Replacing \( x \) by \( 2x \) in Eq. 5, then:

\[ \mu_{\nu(\xi(2t))} \geq \xi_{\nu(2t)} \quad (6) \]

Triangular inequality implies that:

\[ \mu_{\nu(\xi(t))} \geq T(\xi_{\nu(\xi(t))}, \xi_{\nu(\xi(t))}) \quad (7) \]

Thus:

\[ \mu_{\nu(\xi(t))} \geq T(\xi_{\nu(\xi(t))}, \xi_{\nu(\xi(t))}) \quad (8) \]

Replacing \( x \) by \( 4x \) in Eq. 5 and triangular inequality implies that:
To prove Eq. 4, take the limit as \( n \to \infty \) in Eq. 10.
To prove the uniqueness of the quadratic function \( Q \) subject to Eq. 4, assume that there exists a quadratic function \( Q' \) which satisfies Eq. 4. Obviously,

\[
Q(2^n x) = 2^n Q(x) \quad \text{and} \quad Q(2^n x) = 2^n Q'(x) \quad \text{for all} \ x \in X \ \text{and} \ n \in \mathbb{N}
\]

Hence, it follows from Eq. 4 that:

\[
\mu_{q(2^n x)} \preceq \mu_{q(2^n x)}(t) \geq \mu_{q(2^n x)}(2^{n+1} t)
\]

\[
\geq T \left( \mu_{q(2^n x)}(2^n t), \mu_{q(2^n x)}(2^n t), \mu_{q(2^n x)}(2^n t) \right)
\]

for all \( x \in X \). By letting \( n \to \infty \) in Eq. 4, implies that the uniqueness of \( Q \).

**REFERENCES**


