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Existence and Uniqueness Theorems of Higher Order Fractional Differential Equations

Ahmad Adawi
Department of Mathematics, Hashemite University, Zarqa, Jordan

Abstract: There was limited information on the analysis of differential equations of fractional order. Therefore, study on fractional differential equations is essential to understand the solution behavior of many applications in sciences. In this study, initial value problems are discussed for the fractional differential equations and various criteria on existence and uniqueness are obtained.

Key words: Fractional calculus, differential equations of fractional order, existence and uniqueness

INTRODUCTION

Fractional differential equations have been of great interest recently. In cause, in part to both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, chemistry, mechanics engineering, etc. For more details one can see from Hadid *et al.* (1996a), Diethelm and Ford (2002), Kilbas *et al.* (2006), Rabha and Momani (2007), Lin (2007), Kosmatov (2009) and Lakshmikantham and Vatsala (2008).

This study is concerned with the existence and uniqueness of solutions for initial value problems of fractional order of the form:

$$D^{m\alpha}z(t) = h(t, \begin{bmatrix} z \\ D^{\alpha}z \\ D^{2\alpha}z \\ \vdots \\ D^{(m-1)\alpha}z \end{bmatrix}), \qquad 0<\alpha\le 1 \tag{1}$$

with conditions:

$$\begin{bmatrix} D^{\alpha-1}z \\ D^{2\alpha-1}z \\ D^{3\alpha-1}z \\ \vdots \\ D^{m\alpha-1}z \end{bmatrix}(t_0) = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{m-1} \end{bmatrix} \tag{2}$$

Here, $z\in R^n$, $t\in I\subseteq R$, $t_0\in I\subseteq R$, $z_0, z_1, \ldots, z_{m-1}\in R^n$ and h is a function from $I\times R^{nm}$ to R^n

Upon substituting:

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} z \\ D^{\alpha}z \\ D^{2\alpha}z \\ \vdots \\ D^{(m-1)\alpha}z \end{bmatrix} \tag{3}$$

and

$$Y_0 = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{m-1} \end{bmatrix} \tag{4}$$

System (1)-(2) amounts to the system:

$$D^{\alpha}Y(t) = \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_m \\ h(t,Y) \end{bmatrix} = \begin{bmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_n \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} Y + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h(t,Y) \end{bmatrix}$$

$$= AY + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h(t,Y) \end{bmatrix} \tag{5}$$

with the condition:

$$D^{\alpha-1}Y(t_0) = Y_0 \tag{6}$$

where,

$$A = \begin{bmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_n \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{7}$$

and I_n is the identity matrix.
System Eq. 5 and 6 has the form:

$$D^{\alpha}x(t) = f(t, x), \quad 0<\alpha\le 1 \tag{8}$$

with the condition:

$$D^{\alpha-1}x(t_0) = x_0 \qquad (9)$$

The following existence and uniqueness theorems for system Eq. 8 and 9 were proved by Hadid (1995) and Hadid *et al*. (1996b).

Theorem 1 (Hadid, 1995): If (t, x) is continuous in a closed and bounded rectangular box:

$$D: 0<t_0<t<t_0+a, \quad \|X-(t-t_0)^{\alpha-1}x_0\| \le d$$

Then there exists at least one solution x(t) of Eq. 8-9 on $0<t_0<t\le t_0+\beta$ for some $\beta>0$.

Theorem 2 (Hadid *et al*., 1996b): The initial value problem Eq. 8-9 has a unique solution defined on the interval $0<t_0<t\le t_0+a$ if the function f(t, x) is continuous and bounded in the strip:

$$D: 0<t_0<t<t_0+a \quad \|x\|<\infty$$

And satisfies in this strip the Lipschitz condition:

$$\|f(t, x)-f(t, y)\| \le L\|x-y\|$$

for some positive constant L.

The above theorems will be used to investigate the results.

MAIN RESULTS

In this section we give existence and uniqueness results for the IVP (Eq. 1 and 2).

Theorem 3: If h(t, Y) is continuous in a closed and bounded rectangular box:

$$D: 0<t_0<t<t_0+c, \quad \|Y-(t-t_0)^{\alpha-1}Y_0\| \le d$$

where,

$$Y = \begin{bmatrix} z \\ D^{\alpha}z \\ D^{2\alpha}z \\ \vdots \\ D^{(m-1)\alpha}z \end{bmatrix}$$

and

$$Y_0 = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{m-1} \end{bmatrix}$$

Then there exists at least one solution z(t) of (1)-(2) on $0<t_0<t\le t_0+\beta$ for some $\beta>0$

Proof: By the substitution Eq. 3 and 4, system Eq. 1 and 2 is converted into system Eq. 5 and 6 which has the form of system Eq. 8 and 9. Now using Theorem 1 with:

$$f(t,y) = AY + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h(t,Y) \end{bmatrix}$$

Due to the continuity of the function h(t, Y) in D, the function f is continuous in D. By Theorem 1 system (5)-(6) has at least one solution Y(t) on $0<t_0<t\le t_0+B$ for some $\beta>0$ So system (1)-(2) has at least one solution z(t) on $0<t_0<t\le t_0+\beta$ for some $\beta>0$.

Theorem 4: If the function h(t,Y) is continuous and bounded in the strip:

$$D: 0<t_0<t<t_0+c, \quad \|Y-(t-t_0)^{\alpha-1}Y_0\| \le d$$

where,

$$Y = \begin{bmatrix} z \\ D^{\alpha}z \\ D^{2\alpha}z \\ \vdots \\ D^{(m-1)\alpha}z \end{bmatrix}$$

and

$$Y_0 = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{m-1} \end{bmatrix}$$

and satisfies in this strip the Lipschitz condition:

$$\|h(t, Y_1) - h(t, Y_2)\| \le L\|Y_1 - Y_2\|$$

for some positive constant L. Then the initial value problem (1)-(2) has a unique solution z(t) defined on the interval $0<t_0<t\le t_0+a$

Proof: Upon substituting Eq. 3 and 4, system Eq. 1 and 2 amounts to system Eq. 5 and 6. Common practice involves the function:

$$f(t,y) = AY + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h(t,Y) \end{bmatrix}$$

Since h is continuous and bounded in the strip $0<t_0<t\le t_0+a$, $\|Y\|<\infty$ the function f satisfies in this strip:

$$\begin{aligned}\|f(t,Y_1)-f(t,Y_2)\| &= \left\| A(Y_1-Y_2) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h(t,Y_1)-h(t,Y_2) \end{bmatrix} \right\| \\ &\le \|A(Y_1-Y_2)\| + \left\| \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h(t,Y_1)-h(t,Y_2) \end{bmatrix} \right\| \\ &\le \|A\|\|Y_1-Y_2\| + L\|Y_1-Y_2\| \\ &= (\|A\|+L)\|Y_1-Y_2\|\end{aligned}$$

for some positive number L. From Theorem 2, system Eq. 5 and 6 has a unique solution Y(t) on the interval $0<t_0<t\le t_0+a$ and so system Eq. 1 and 2 has a unique solution z(t) on this interval.

Theorem 5: If $A_0(t)$, $A_1(t)$, ..., A_{m-1} (t) are continuous and bounded n×n matrix functions on some interval $0<t_0<t\le t_0+a$ and g(t) is a continuous and bounded vector function on some interval, then the linear fractional differential equation:

$$\begin{aligned}D^{m\alpha}z(t) &= A_{m-1}(t)D^{(m-1)\alpha}z(t) + A_{m-2}(t)D^{(m-2)\alpha}z(t) + \cdots \\ &+ A_1(t)D^{\alpha}z(t) + A_0(t)z(t) + g(t),\end{aligned} \tag{10}$$

$$\begin{bmatrix} D^{\alpha-1}z \\ D^{2\alpha-1}z \\ D^{3\alpha-1}z \\ \vdots \\ D^{m\alpha-1}z \end{bmatrix}(t_0) = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{m-1} \end{bmatrix} \tag{11}$$

has a unique solution z(t) on the interval $0<t_0<t\le t_0+a$

Proof: By the substitutions Eq. 3 and 4, system Eq. 1 and 2 can be written in the form:

$$D^{\alpha}Y(t) = \begin{bmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_n \\ A_0(t) & A_1(t) & A_2(t) & \cdots & A_{m-1}(t) \end{bmatrix} Y + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{bmatrix} = B(t)Y + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{bmatrix}$$

with the condition:

$$D^{\alpha-1}Y(t_0) = Y_0$$

where,

$$B(t) = \begin{bmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_n \\ A_0(t) & A_1(t) & A_2(t) & \cdots & A_{m-1}(t) \end{bmatrix}$$

Let

$$h(t,Y) = B(t)Y + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{bmatrix}$$

From the definition of the function h it seems clear that:

$$\|h(t,Y_1)-h(t,Y_2)\| = \|B(t)(Y_1-Y_2)\| \le \|B(t)\|\|Y_1-Y_2\|$$

The desired result now follows from Theorem 2.

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