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A Comparison between the Modified Homotopy Perturbation Method and Adomian Decomposition Method for Solving Nonlinear Heat Transfer Equations

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Abstract: In this study, we applied a new algorithm based on Homotopy Perturbation Method (HPM) to evaluate the temperature distribution of a straight rectangular fin with temperature dependent surface heat flux for all possible types of heat transfer. The local heat transfer coefficient is considered to vary with a power-law function of temperature. The time interval is divided into several subintervals and the HPM solutions are applied successively over these reduced time intervals. Comparisons between the 13-term Adomian decomposition solution and 6-term modified HPM solution are made. Comparison of the results obtained by modified HPM with that obtained by the Adomian Decomposition Method (ADM) reveals that the obtained modified HPM solution is quite accurate when only the six terms are used in the series expansion.

Key words: Homotopy-perturbation method, adomian decomposition method, nonlinear DEs, fins

INTRODUCTION

Fins are extensively used to enhance the heat transfer between a solid surface and its convective, radiative, or convective radiative surface (Kern and Kraus, 1972). Finned surfaces are widely used, for instance, for cooling electric transformers, the cylinders of air-craft engines, and other heat transfer equipment. Finned surfaces are widely used, for instance, for cooling electric transformers, the cylinders of air-craft engines, and other heat transfer equipment. The temperature distribution of a straight rectangular fin with a power-law temperature dependent surface heat flux can be determined by the solutions of a one-dimensional steady state heat conduction equation which, in dimensionless form, is given by Chang (2005):

$$\frac{d^2\theta}{dx^2} - N^2\theta^{n+1} = 0 \tag{1}$$

subject to the boundary conditions:

$$\frac{d\theta}{dx}(0) = 0,\ \theta(1) = 1 \tag{2}$$

where, the axial distance x is measured from the fin tip, θ is the temperature, and N is the convective-conductive parameter of the fin. The values of n vary in a wide range between 4 and 5 depending on the mode of boiling (Liaw and Yeh, 1994a,b). For example, the exponent n may take the respective values -4, -0.25, 0, 2 and 3, depending on whether the fin is subject to transition boiling, laminar film boiling or condensation, convection, nucleate boiling, and radiation into free space at zero absolute temperature.

The approximate analytical solution to 1-2 was presented by Chang (2005) using the analytic Adomian Decomposition Method (ADM). Sometimes it is a very intricate problem to calculate the so-called Adomian polynomials involved in ADM. Another analytic method which has been shown to be much simpler than the ADM is called the Homotopy-perturbation Method (HPM), first developed by He (1998, 1999, 2000, 2003, 2004, 2006a,b). We note that based on HPM, Ghorbani and Saberi-Nadjafi (2007) and Ghorbani (2009) were able to overcome the difficulty in ADM through the so-called He polynomials. HPM yields rapidly convergent series solutions (He, 2006a; El-Latif, 2005; Noor and Mohyud-Din, 2007). Recently, the applicability of HPM was extended to singular second-order differential equations (Chowdhury and Hashim, 2007a,b), nonlinear population dynamics models (Chowdhury and Hashim, 2007c), general time-independent Emden-Fowler equations (Chowdhury and Hashim, 2009), time-dependent Emden-Fowler type equations (Chowdhury and Hashim, 2007b), Klein-Gordon and sine-Gordon equations (Chowdhury and Hashim, 2009). Very recently, Chowdhury *et al.* (2008) and Hashim and Chowdhury (2008) were the first to successfully apply the Multistage Homotopy-perturbation Method (MHPM) to the chaotic Lorenz system and a class of system of ODEs.

In this study, we present a proper procedure based on HPM for solving analytically problem 1-2. In doing so, we corrected the work of Ganji (2006).

MODIFIED HOMOTOPY-PERTURBATION METHOD

Since the HPM is now standard and for brevity, the reader is referred to He (1998, 1999, 2000, 2003, 2004, 2006a,b) for basic ideas of HPM. To illustrate the basic ideas of the Modified Homotopy-perturbation Method (MHPM), we consider the following general nonlinear differential equation (He, 2006a, b):

$$A(y) - f(r) = 0, \quad r \in \Omega \tag{3}$$

with boundary conditions:

$$B(y, \partial y/\partial n) = 0 \quad r \in \Gamma \tag{4}$$

where, A is general differential operator, B is a boundary operator, f(r) is a known analytic function, and Γ is the boundary of the domain Ω.

The operator A generally divided into two parts L and N, where L is linear while N is nonlinear. Therefore, Eq. 4 can be written as follows:

$$L(y) + N(y) - f(r) = 0 \tag{5}$$

We construct a homotopy $y(r,p):\Omega \times [0, 1] \rightarrow \Re$ of Eq. 3 which satisfies:

$$H(Y,P) = (1-p)[L(Y)+L(Y_0)]+p[A(y)-f(r)] = 0 \quad p \in [0,1],\ r \in \Omega \tag{6}$$

which is equivalent to:

$$H(y,p) = L(y) - L(y_0) + pL(Y_0) + p[N(y) - f(r)] = 0 \tag{7}$$

where $p \in [0,1]$ is an embedding parameter and Y_0 is an initial approximation which satisfies boundary conditions. It follows from Eq. 6 and 7 that:

$$H(y,0) = L(y) - L(y_0) = 0 \text{ and } H(y,1) = A(y) - f(r) = 0 \tag{8}$$

Thus, the changing process of p from 0 to 1 is just that of $y(r,p)$ from $y_0(r)$ to $y_r(p)$. In topology this called deformation and $L(y) - L(0)$ and $A(y) - f(r)$ are called homotopic. Here the embedding parameter is introduced much more naturally, unaffected by artificial factors; further it can be considered as a small for $0 \leq p \leq 1$. So it is very natural to assume that the solution of Eq. 7 and 8 can be expressed as:

$$y(x) = u_0(x) + u_1(x)\, u_2(x) + \ldots.. \tag{9}$$

According to HPM, the approximate solution of Eq. 7 can be expressed as a series of the power of p, i.e.,

$$y = \lim_{p \to 1} y = u_0 + u_1 + u_2 + u_3 \cdots \tag{10}$$

Now, we apply the above procedure as an algorithm for approximation the dynamics response in a sequence of time intervals (time step) $[0,1),[0,t_2),[0,t_3),\ldots[t_{n-1}\ t_n)$ such that the initial condition in $[t_p,t_{p+1}]$ is taken to be the condition at t_p. For practical computations, a finite number of terms in the series:

$$\phi_i(x) = \sum_{k=0}^{i-1} \theta_k \tag{11}$$

are used in a time step procedure just outlined.

Application of modified HPM: Here, we apply an alternative approach of HPM to find approximate analytical solution to 1-2. To do so, we first construct a homotopy $y(r,p):\Omega \times [0,1] \rightarrow \Re$ which satisfies:

$$\frac{d^2\theta}{dx^2} - \frac{d^2 y_0}{dx^2} + p\left(\frac{d^2 y_0}{dx^2} - N^2\theta^{n+1}\right) = 0 \tag{12}$$

Suppose the solution of Eq. 1 has the form:

$$\theta(x) = u_0 p u_1(x) + p^2 u_2(x) + \ldots. \tag{13}$$

and let us choose the initial approximation as:

$$u_0(x) = y_0(x) = \theta(0) = c, \tag{14}$$

where c is to be determined.

Substituting 13 into 12 and equating the terms with identical powers of p, we get the following system of linear differential equations:

$$\frac{d^2u_1}{dx^2} + \frac{d^2y_0}{dx^2} - N^2 u_0^{n+1} = 0,$$
$$u_1(0) = 0, \frac{du_1}{dx}(0) = 0,$$
$$\frac{d^2u_2}{dx^2} - N^2(n+1)u_1 u_0^{n} = 0,$$
$$u_2(0) = 0, \frac{du_2}{dx}(0) = 0,$$
$$\frac{d^2u_3}{dx^2} - N^2(n+1)u_2 u_0^{n} - \frac{1}{2}N^2 n(n+1)u_1^2 u_0^{n-1} = 0,$$
$$u_3(0) = 0, \frac{du_3}{dx}(0) = 0,$$
$$\frac{d^2u_4}{dx^2} - N^2(n+1)[u_3 u_0^{n} + n u_0^{n-1} u_1 u_2 + \frac{1}{6}n(n-1)u_1^3 u_0^{n-2}] = 0,$$
$$u_4(0) = 0, \frac{du_4}{dx}(0) = 0$$

Solving the above equations, we have:

$$u_1(x) = \frac{1}{2}c^{n+1}N^2x^2,$$
$$u_2(x) = \frac{1}{24}c^{2n+1}(n+1)N^4x^4,$$
$$u_3(x) = \frac{1}{720}c^{3n+1}(n+1)(4n+1)N^6x^6,$$
$$u_4(x) = \frac{1}{40320}c^{4n+1}(n+1)(34n^2+5n+1)N^8x^8,$$
$$u_5(x) = \frac{1}{3628800}c^{5n+1}(n+1)(496n^3-66n^2+69n+1)N^{10}x^{10}$$

etc.

According to Eq. 13 and the assumption p = 1, the six-term approximate solution to (1) is:

$$\theta \simeq \frac{1}{2}c^{n+1}N^2x^2 + \frac{1}{24}c^{2n+1}(n+1)N^4x^4 + \frac{1}{720}c^{3n+1}(n+1)(4n+1)N^6x^6 + \frac{1}{40320}c^{4n+1}(n+1)(34n^2+5n+1)N^8x^8 + \frac{1}{3628800}c^{5n+1}(n+1)(496n^3-66n^2+69n+1)N^{10}x^{10} \quad (15)$$

The complete solution is obtained once the constant c is determined by imposing the second boundary condition given by Eq. 2. Note that the value of c must lie in the interval (0,1) to represent the temperature at the fin tip (Chang, 2005).

To carry out the iterations in every subinterval of equal length Δt $[0,t_1)$, $[0,t_2)$, $[0,t_3)$...$[t_{n-1}, t)$, we would need to know the values of the following initial conditions, $c = \theta(t^*)$.

In general, we do not have these information at our clearance except at the initial point $t^* = t_0 = 0$ but we can obtain these values following the MHPM. We note that the 6-term approximations of θ denoted as $\phi_6(x) = \sum_{i=0}^{5} y_{1,i}$. For practical computations, a finite number of terms in the series solution are used in a time step procedure just outlined.

RESULTS AND DISCUSSION

Now we consider the nonlinear Eq. 1. Taking the actual physiological data in Chang (2005) the 13-term ADM approximate solution for N = 1, n = 5 is:

$$\phi_{13} = 0.81620 + 0.14780x^2 + 0.02675x^4 + 0.00678x^6 + 0.00183x^8 + 0.00051x^{10} + 0.00014x^{12} + 0.00004x^{14} + 0.00001x^{16} + 0.000004x^{18} + 0.000001x^{20} + 0.0000003x^{22} + 0.0000001x^{24} \quad (16)$$

Incorporating the recursive algorithm 12-15, the 6-term approximate MHPM solution for N = 1, n = 5 is

$$\phi_6 = 0.81623 + 0.14785x^2 + 0.02678x^4 + 0.00679x^6 + 0.00183x^8 + 0.00051x^{10} \quad (17)$$

similar expression have been obtained for other values of:

$$-4 \le n \le 5$$

The MHPM algorithm is coded in computer algebra package Maple and the Maple environment variable Digits is set to 16 in all calculation done for the current problem. Obviously the accuracy of our present 6-term MHPM solution is verified by the 13-term ADM solution. First we note that the special case $N^2 = \varepsilon$ and n = 3 reduces problem 1-2 to that studied by Ganji (2006). Unfortunately, his Eq. 35 and 36 are in error, that is the correct solution for Y^2 (u^2 in our notation) should be $Y_2 = \frac{1}{6}\varepsilon^2(x^4 - 6x^2 + 5)$ and hence his Eq. 36 should be:

$$\theta = 1 + \frac{1}{2}\varepsilon(x^2 - 1) + \frac{1}{6}\varepsilon^2(x^4 - 6x^2 + 5)$$

Since this 3-term HPM solution is exactly the same as the solution by the classical perturbation method, (Ganji, 2006) (corrected) HPM solution is valid only for small ε. Ganji (2006) poor HPM solution is due the improperly chosen initial conditions $Y_0 = 1$, $Y_1 = 0$, $Y_2(1) = 0$ etc. for his linear system of ODEs 30-32. The proper initial conditions should instead be $Y_0(0) = c$, $Y_1(0) = Y_2(0) = 0$ etc.

In Fig. 1, we present the correct 6-term MHPM solution (15), the 3-term HPM solution given by (Ganji, 2006) and the 13-term ADM solution (Chang, 2005). Obviously the accuracy of our 6-term MHPM solution is verified by the 13-term ADM solution and the HPM solution obtained by Ganji (2006) formulation is totally incorrect.

Figure 2 shows the temperature profiles for several assigned values of n at N = 1 given by Eq. 17 on the time step h = 0.01. All these numerical results are in very good agreement with the 13-term ADM solutions. As indicated in Eq. 17, the temperature along the fin is expressed in an explicit function of position x. Thus the temperature profile can be easily obtained for any exponent value n. The characteristics of temperature profiles have been discussed by Liaw and Yeh (194a,b); and Dul'kin and Garas'ko (2002). The former used the hyper-geometric formulas to determine the profiles and the latter derived an inversed form for the temperature distribution along the fin, and then evaluated the profile via an iterative procedure. Chang (2005) used ADM to

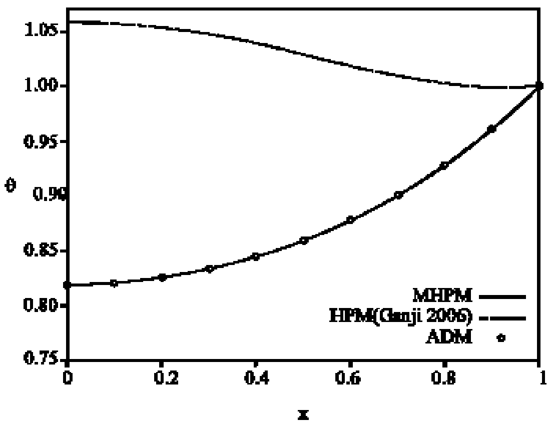

Fig. 1: Comparison between the correct 6-term MHPM solution, the 3-term HPM of Ganji (2006) and the 13-term ADM solution (Chang, 2005) for $N^2 = 0.7$ and n = 3

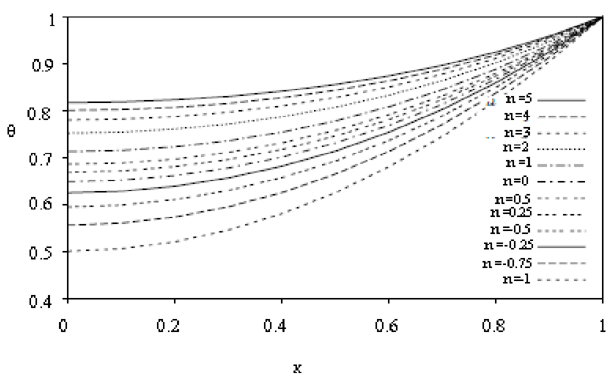

Fig. 2: The numerical solutions ϕ_6 at time step, h = 0.01 and N = 1

analyze the thermal characteristics of straight rectangular fin using 13 terms in the series expansion.

The present results are consistent with both of them while with more straightforward process and less computation and only 6 terms used in series expansion. To make a proper comparison, we first determine the accuracy of MHPM for the solution of Eq. 1 for time step h = 0.01. In Table 1, we compare 13 term ADM solutions between 6 terms MHPM solutions for time steps h = 0.01 when n = -0.5.

In Table 2 we present the absolute errors between 6 terms MHPM solutions at time steps h = 0.01 and 13 terms ADM solutions when n = -0.5. In Table 3, we compare 13-term ADM solutions between 6 terms MHPM solutions for time step h = 0.01 when n = -0.5. In Table 4 we present the absolute errors between 6 terms MHPM solutions at time step h = 0.01 and 13 terms ADM solutions when n = -0.5. On the time step h = 0.01 the 6-term MHPM solutions match with 13-term ADM solutions at least 4 decimal places. This suggests that the

Table 1: Comparison between 13-term ADM solution and 6-term MHPM solutions at time step h = 0.01 when n = -0.5

X	ϕ_{13}	ϕ_6
0.0	0.5944461335	0.5944515788
0.1	0.5983032326	0.5983086955
0.2	0.60989947616	0.6099049921
0.3	0.6293093861	0.6293149903
0.4	0.6566561362	0.6566618639
0.5	0.6921100992	0.6921159838
0.6	0.7358869771	0.7358930398
0.7	0.7882456461	0.7882518437
0.8	0.8494858433	0.8494918787
0.9	0.9199458118	0.9199505587
1.0	0.9999999999	0.9999999999

Table 2: Absolute errors between 13-term ADM, and 6-term MHPM at time step h = 0.01 when n = -0.5

X	$\lvert\phi_{13}-\phi_6\rvert$
0.0	5.445E-06
0.1	5.463E-06
0.2	5.516E-06
0.3	5.604E-06
0.4	5.728E-06
0.5	5.885E-06
0.6	6.063E-06
0.7	6.198E-06
0.8	6.035E-06
0.9	4.747E-06
1.0	1.000E-15

Table 3: Comparison between 13-term ADM solutions and 6-term MHPM solutions at time steps h = 0.01 when n = -0.5

X	ϕ_{13}	ϕ_6
0.0	0.8161488485	0.8162264350
0.1	0.8176292306	0.8177076630
0.2	0.8221028946	0.8221838988
0.3	0.8296699290	0.8297553375
0.4	0.8405060050	0.8405978392
0.5	0.8548771743	0.8549777252
0.6	0.8731640540	0.8732758154
0.7	0.8959000779	0.8960248088
0.8	0.9238322087	0.9239663936
0.9	0.9580195857	0.9581375940
1.0	1.000000000	1.000000000

Table 4: Absolute errors between 13-term ADM, and 6-term MHPM at time step h = 0.01 when n = -0.5

X	$\lvert\phi_{13}-\phi_6\rvert$
0.0	7.759E-05
0.1	7.843E-05
0.2	8.100E-05
0.3	8.541E-05
0.4	9.183E-05
0.5	1.006E-04
0.6	1.118E-04
0.7	1.247E-04
0.8	1.342E-04
0.9	1.180E-04
1.0	1.000E-14

present MHPM solutions using only 6 terms on the time step h = 0.01 are accurate enough when n = -0.5 and n = 5. Similar conclusions have been obtained for the other values of $-4 \le n \le 5$. Several cases with variations of parameters N and n are also tested and it can be conclude that the use of 6 terms in Eq. 17 is sufficient to yield accurate results. Obviously, the present method gives fast and accurate results instead of complicated numerical integration and iteration procedure.

CONCLUSION

In this study, the power-law fin-type problem was solved via a new algorithm of HPM. It is obvious that the new algorithm completely overcomes the shortcomings of parameter N. The solutions obtained are in convergent series form with easily computable terms. Comparison with the decomposition method shows that the homotopy-perturbation method is a promising tool for finding approximate analytical solutions to strongly nonlinear problems.

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