Journal of
Applied Sciences

ISSN 1812-5654
Operational Tau Approximation for Neutral Delay Differential Systems

J. Sedighi Hafshejani, S. Karimi Varani and J. Esmaily
Department of Mathematics, Islamic Azad University, Shahrekord Branch, P.O. Box 166, Shahrekord, Iran

Abstract: Neutral Delay Differential Systems (NDDSs) arise in many areas of various mathematical modeling. Infectious diseases, population dynamics, physiological and pharmaceutical kinetics and chemical kinetics, the navigational control of ships and mechanical systems, chemical process simulation and optimal control are the main field concerning with NDDS. The purpose of this study was to present an extension of the algebraic formulation of the Operational Tau Method (OTM) for the numerical solution of NDDSs. The proposed method converts the delay parts of the desired NDDS to some operational matrices. Then the NDDS reduces to a set of algebraic equations. Some orthogonal bases including shift Chebyshev and shifted Legendre polynomials are used to decrease the volume of computations. Two illustrative linear and nonlinear experiments are included to show the high accuracy and efficiency of the proposed method.

Key words: Spectral methods, tau method, delay differential equations, neutral delay differential systems

INTRODUCTION

Spectral methods and spectral modelings provide a computational approach which achieved substantial popularity in the last three decades (Gharibyan et al., 2011; Taiwo and Abubakar, 2011; Tasci, 2003). Tau method is one of the most important spectral methods which is extensively applied for numerical solution of many problems. This method was invented by Lanczos (1938) for solving Ordinary Differential Equations (ODEs) and then the expansion of the method were done for many different problems such as Partial Differential Equations (PDEs) (Matos et al., 2004; Kong and Wu, 2009; Doha and Abdel-Elhameed, 2005), Integral Equations (IEs) (El-Daou and Khajah, 1997), Integro-Differential Equations (IDEs) (Pour-Mahmoud et al., 2005) and etc. (Rahimi-Ardabili and Shahmorad, 2007; Garcia-Olivares, 2002; Parand and Razaghi, 2004).

A time delay phenomenon is encountered in a wide variety of scientific and engineering applications, including circuit analysis, computer-aided design, real-time simulation of mechanical systems, chemical process simulation, optimal control, population dynamics and vibrating masses attached to an elastic bar (Hale and Verduyn Lunel, 1993; Taiwo and Odetunde, 2010; Rao et al., 2011; Varani and Aminataei, 2009, 2010; Kolmanovskii and Myshkis, 1992; Salamon, 1984).

In this study, we are interested in solving NDDSs with an operational approach of the Tau method. Because in the Tau method, we are dealing with a system of equations wherein the matrix of unknown coefficients is sparse and can be easily invertible. Also, the delay parts appearing in the equation are replaced by their operational Tau representation. Then, we obtain a system of algebraic equations wherein its solution is easy.

Operational tau method: In this section, we state some preliminaries and notations used in this study.

For any integrable functions \( \Psi (x) \) and \( \phi (x) \) on \((a, b)\), we define the scalar product \( < > \) by:

\[
< \psi (x), \phi (x) > := \int_a^b \psi (x) \phi (x) \omega (x) dx,
\]

where \( \| \psi \|_\omega := \int_a^b \psi (x) \omega (x) dx \) and \( \omega (x) \) is a weight function. Let \( L^2_{\omega} [a, b] \) be the space of all functions \( f : [a, b] \rightarrow \mathbb{R} \) with \( \| f \|_\omega < \infty \).

The main idea of the method is to seek a polynomial to approximate \( u (x) \in L^2_{\omega} [a, b] \). Let \( \phi_i \in (\mathcal{D}(\omega)) \) be a set of arbitrary orthogonal polynomial bases defined by a lower triangular matrix \( \Phi \) and \( X^t = [1, x, x^2, \ldots]^T \).

Lemma 1: Suppose that \( u (x) \) is a polynomial as \( u(x) = \sum_{i=1}^n u_i x^i = u X_i \), then we have:

\[
D^r \psi (x) = \frac{d^r}{dx^r} \psi (x) = t M^r X_i, \quad r = 0, 1, 2, \ldots
\]

(1)

\[
x^s u(x) = u N^s X_i, \quad s = 0, 1, 2, \ldots
\]

(2)

Corresponding Author: J. Sedighi Hafshejani, Department of Mathematics, Islamic Azad University, Shahrekord Branch, P.O. Box 166, Shahrekord, Iran Tel: 00989131858762
and:
\[
\int_{t_0}^{t} u(t) dt = uPX_t - uPX_{t_0},
\]
(3)
where \( u = [u_0, u_1, \ldots, u_n, \ldots]^T \), \( x_t = [1, a, a^2, \ldots]^T \), \( a \in \mathbb{R} \) and \( M, N \) and \( P \) are infinite matrices with only nonzero elements:
\[
M_{n+1} = 1 + 1, N_{n+1} = 1, P_{n+1} = \frac{1}{i + 1}, i = 0, 1, 2, \ldots
\]

**Proof:** (Liu and Pan, 1999).
Let us consider:
\[
u(x) = \sum_{j=0}^{\infty} \phi_j(x) = u_0 X_0,
\]
(4)
To be an orthogonal series expansion of the exact solution where \( u = \{u_j\}_{j=0}^{\infty} \) is a vector of unknown coefficients, \( \Phi X_0 \) is an orthogonal basis for polynomials in \( \mathbb{R} \).
In the Tau method, the aim is to convert the linear and nonlinear terms to an algebraic system using some operational matrices. Therefore, we state the following lemma.

**Lemma 2:** Let \( X_0 = [1, x, x^2, \ldots]^T \), \( u = [u_0, u_1, u_2, \ldots] \) be infinite vectors and \( \Phi = [\phi_0, \phi_1, \phi_2, \ldots] \), \( \phi_i \) are infinite columns of matrix \( \Phi \). Then, we have:
\[
X_0 u \Phi X_0 = U \Phi X_0
\]
(5)
where, \( U \) is an upper triangular matrix as:
\[
U_{ij} = \sum_{k=0}^{\infty} \phi_k(x) \phi_{k+i}(x), \quad j \geq i, i, j = 0, 1, \ldots
\]
(6)
In addition, if we suppose that \( u(x) = u \Phi X_0 \) represents a polynomial, then for any positive integer \( p \), the relation:
\[
u^p(x) = u \Phi U^{p-1} X_0
\]
(7)
is valid.

**Proof:** We have:
\[
X_0 u \Phi = [1, x, x^2, \ldots]^T [u_0 u_1 u_2 \ldots] = \left[\begin{array}{cccc}
u_0 & \nu_0 \Phi & \nu_0 \Phi^2 & \cdots \\
u_1 \Phi & \nu_1 \Phi^2 & \nu_1 \Phi^3 & \cdots \\
u_2 \Phi^2 & \nu_2 \Phi^3 & \nu_2 \Phi^4 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{array}\right]
\]
Therefore:
\[
X_0 u \Phi X_0 = \left[\begin{array}{cccc}
u_0 & \nu_0 \Phi & \nu_0 \Phi^2 & \cdots \\
u_1 \Phi & \nu_1 \Phi^2 & \nu_1 \Phi^3 & \cdots \\
u_2 \Phi^2 & \nu_2 \Phi^3 & \nu_2 \Phi^4 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{array}\right]
\]
\[
= \left[\begin{array}{cccc}
u_0 & \nu_0 \Phi & \nu_0 \Phi^2 & \cdots \\
0 & \nu_1 \Phi & \nu_1 \Phi^2 & \cdots \\
0 & 0 & \nu_2 \Phi^2 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{array}\right]
\]
If we call the last upper triangular coefficient matrix as \( U \), then we have:
\[
U_{ij} = \sum_{k=0}^{\infty} \phi_k(x) \phi_{k+i}(x), \quad j \geq i, i, j = 0, 1, \ldots
\]
(6)
Now, in order to prove Eq. 7, we apply induction. For \( p = 1 \), it is obvious that \( u(x) = u \Phi X_0 \). For \( p = 2 \) we rewrite \( u^2(x) = u \Phi X_0 u \Phi X_0 = u \Phi (X_0 u \Phi X_0) \) and using Eq. 5, we have:
\[
u^2(x) = u \Phi u X_0
\]
Therefore, Eq. 7 is hold for \( p = 2 \). Now, suppose that Eq. 7 is hold for \( p = k \), then we must prove that the relation is valid for \( s = k + 1 \). Thus:
\[
u^{k+1}(x) = u^{k+1}(x) = (u \Phi U^{k-1} X_0) = u \Phi U^{k-1} (X_0 u \Phi X_0) = u \Phi U^k X_0
\]
(6)
So, Eq. 7 is proved.

**Application on NDDS:** Let us consider the following NDDS:
\[
U(x) = A(x)U(x) + B(x)U(\alpha(x)) + C(x)U(\beta(x)) + F(x), \quad a \leq x \leq b,
\]
(8)
\[
U(x) = \Phi(x), \quad x \leq a,
\]
Where:
\[
U(x) = [u_1(x), u_2(x), \ldots, u_n(x)]^T, \quad U_1(x) \in C, \quad k = 1, 2, \ldots, m
\]
is the state vector and:
\[
U(\alpha(x)) = [u_1(\alpha_1(x)), u_2(\alpha_2(x)), \ldots, u_n(\alpha_n(x))]^T,
\]
\[
U(\beta(x)) = [u_1(\beta_1(x)), u_2(\beta_2(x)), \ldots, u_n(\beta_n(x))]^T
\]
(10)
Such that \((\alpha_i(x) < b)_{i,n}^m\) and \((\beta_i(x) < b)_{i,n}^m\) are delay functions; \(A(x), B(x)\) and \(C(x)\) are \(m\)-dimensional matrices which their entries are complex functions of \(x\). Also:

\[
\Psi(x) = [\psi_1(x), \psi_2(x), ..., \psi_m(x)]^T, \quad \psi_i(x) \in C, k=1,2, ..., m. \tag{11}
\]

\[
F(x) = [f_1(x), f_2(x), ..., f_m(x)]^T, \quad f_i(x) \in C, k=1,2, ..., m, \tag{12}
\]

represent the initial vector function and known vector function, respectively.

Now the aim is to write \(u_i(\alpha_i(x))\) and \(\hat{u}_i(\beta_i(x))\) \(i = 0, 1, ..., m\), in operational forms. Using Eq. 4, we have:

\[
u_i(\alpha_i(x)) - u \Phi X_{n}^{\alpha_i(x)} i = 0, 1, ..., m \tag{13}
\]

We know that \(X_{n} = [1, x, x^2, ..., x^m]^T\), therefore:

\[
X_{n}^{\alpha_i(x)} = [1, \alpha_i(x), \alpha_i^2(x), ..., \alpha_i^m(x)]^T
\]

By approximating each power of \(\alpha_i(x)\) as \(\alpha_i(x) = \sum_{\eta=0}^{m} \eta_x \rho \alpha^x\), we obtain:

\[
X_{n}^{\alpha_i(x)} = \left[ \begin{array}{cccc} 1 & 1 & \cdots & 0 \\
\alpha_i & \alpha_1 & \cdots & \alpha_i \\
\alpha_i^2 & \alpha_2 & \cdots & \alpha_i^2 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_i^m & \alpha_m & \cdots & \alpha_i^m 
\end{array} \right] \left[ \begin{array}{c} 1 \\
0 \\
0 \\
\vdots \\
0 
\end{array} \right] = \left[ \begin{array}{c} 1 \\
\alpha_i \\
\alpha_i^2 \\
\vdots \\
\alpha_i^m 
\end{array} \right] = X_{n}^{\alpha_i(x)} \tag{14}
\]

If \(\Pi_j\) be the last coefficient matrix then we have:

\[
X_{n}^{\alpha_i(x)} - \Pi_j \cdot X_{n} \tag{15}
\]

Substituting Eq. 14 in Eq. 13 we get:

\[
u_i(\alpha_i(x)) = u \Phi \Pi_j \cdot X_{n} \quad i = 0, 1, ..., m \tag{16}
\]

Also, from Eq. 1 and 4, it is obvious that:

\[
u_i(x) = u \phi \Xi_{l} X_{n} \quad i = 0, 1, ..., m \tag{17}
\]

In the same manner from Eq. 13 to 15, there exist the coefficient matrices \(\Delta_j\) such that:

\[
u_i(\beta_i(x)) = u \phi \Delta_j \cdot X_{n} \quad i = 0, 1, ..., m \tag{18}
\]

In next step we desire to approximate each elements of matrices \(A(x), B(x)\) and \(C(x)\) in operational forms. Since, each elements of \(A(x), B(x)\) and \(C(x)\) are smooth functions therefore we can approximate them as follows:

\[
A_j(x) = \sum_{i=0}^{m} a_i x^i, \quad B_j(x) = \sum_{i=0}^{m} b_i x^i, \quad C_j(x) = \sum_{i=0}^{m} c_i x^i \quad i,j = 0,1,...,m \tag{19}
\]

Substituting above equations in Eq. 8 and using Eq. 2, we obtain:

\[
A(x)U(x) - \left[ \sum_{i=0}^{m} a_i u_i(x) \right] \left[ \sum_{i=0}^{m} a_i u_i(x) \right]^T \tag{20}
\]

\[
B(x) = \left[ \sum_{i=0}^{m} b_i u_i(x) \right] \left[ \sum_{i=0}^{m} b_i u_i(x) \right]^T \tag{21}
\]

\[
C(x) = \left[ \sum_{i=0}^{m} c_i u_i(x) \right] \left[ \sum_{i=0}^{m} c_i u_i(x) \right]^T \tag{22}
\]

In the same way, we have:

\[
B(x) U(x) = \hat{B} X_{n}, \quad \hat{B}_i = \sum_{j=0}^{m} a_i u_i, \phi \Pi \cdot N^i \tag{23}
\]

The vectors \(U(x)\) and \(F(x)\) also can be considered as:

\[
u(x) = \hat{u}(x), \quad \hat{u}_i = u_i \tag{24}
\]

\[
F(x) = \hat{F} X_{n}, \quad \hat{F}_i = f_i \tag{25}
\]

Thus Eq. 8 is replaced by the following algebraic system:

\[
u \phi X_{n} = \hat{A} \phi \phi X_{n} + \hat{B} \phi \phi X_{n} + \hat{C} \phi \phi X_{n} + \hat{F} \phi \phi X_{n} \tag{26}
\]

So, the residual matrix \(R(x)\) of Eq. 8, can be written as:

\[
R(x) = \left[ \hat{U} - \hat{A} \phi \phi - \hat{B} \phi \phi - \hat{C} \phi \phi - \hat{F} \phi \phi \right] \phi X_{n} = \hat{R} \phi X_{n} \tag{27}
\]

Where:

\[
\hat{R} = \left[ \hat{U} - \hat{A} \phi \phi - \hat{B} \phi \phi - \hat{C} \phi \phi - \hat{F} \phi \phi \right] \phi \phi X_{n} \tag{28}
\]
Now, we set the residual matrix $R = 0$ or we use the following inner products:

$$ \langle R, \phi_i(x) \phi_k(x) \rangle = 0, \ i, k = 0, 1, 2, \ldots \tag{26} $$

For supplementary conditions of Eq. 8 we have:

$$ u(a) = u(x) = \psi(a) \tag{27} $$

Therefore, imposing supplementary conditions and setting $R = 0$, a system of algebraic equations is obtained. Since, somewhere we require finite terms of approximation, then we must truncate the series solution to finite number of terms. This is the so-called operational Tau method which is applicable for finite, infinite, regular and irregular domains.

**Some shifted orthogonal polynomials:** We have considered OTM based on arbitrary orthogonal polynomials. Orthogonal functions can be used to obtain a good approximation for transcendental functions. Since shifted Chebyshev and Legendre polynomials are more applicable orthogonal functions for a wide range of problems therefore, we consider them, briefly.

**Shifted chebyshev polynomials:** The Chebyshev polynomials are defined on $[-1, 1]$ as:

\[
\begin{bmatrix}
T_0(x) = 1, & T_1(x) = x, \\
T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), & n = 1, 2, 3, \ldots
\end{bmatrix}
\tag{28}
\]

or:

\[
\left[ T_0(x), T_1(x), T_2(x), \ldots \right] = TX_x, \quad \text{where} \quad T = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
\vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

and shifted Chebyshev polynomials are defined as:

\[
\begin{bmatrix}
T_0^*(x) = 1, & T_1^*(x) = \frac{2x - (b + a)}{b - a}, & x \in [a, b], \\
T_n^*(x) = 2 \frac{2x - (b + a)}{b - a} T_{n-1}^*(x) - T_{n-2}^*(x), & n = 1, 2, 3, \ldots
\end{bmatrix}
\tag{29}
\]

**Lemma 3:** Suppose that $T$ and $T^*$ are coefficient matrices of Chebyshev polynomials $\{T(x) \mid x \in [-1, 1], i = 0, 1, 2, \ldots\}$ and shifted Chebyshev polynomials $\{T^*(x) \mid x \in [a, b], i = 0, 1, 2, \ldots\}$, respectively. Hence, we have:

$$ T^* = TQ $$

Where:

$$ Q_{ij} = \begin{cases}
1 & i \geq j, i, j = 0, 1, 2, \ldots \\
0 & i < j
\end{cases} $$

with $v = 2/b - a$ and $w = a + b/a - b$.

**Proof:** Definition of $T$ states that:

\[
\begin{bmatrix}
T_0(x) \\
T_1(x) \\
T_2(x) \\
\vdots
\end{bmatrix} = T \begin{bmatrix}
x \\
x^2 \\
x^3 \\
\vdots
\end{bmatrix} \Rightarrow \begin{bmatrix}
T_0^*(x) \\
T_1^*(x) \\
T_2^*(x) \\
\vdots
\end{bmatrix} = T \begin{bmatrix}
\frac{1}{v} x^0 + \frac{w}{v^2} x^2 \\
\frac{2x - (b + a)}{b - a} \\
\frac{2x - (b + a)}{b - a} x^2 - \frac{2x - (b + a)}{b - a} x^0 \\
\vdots
\end{bmatrix}
\]

We know that $(vx + w)x = \sum_{i=0}^{n} \binom{n}{i} v^i w^{n-i} x^i$, thus:

\[
\begin{bmatrix}
1 \\
\frac{v}{vx + w} \\
\frac{v^2}{(vx + w)^2} \\
\frac{v^3}{(vx + w)^3} \\
\vdots
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots \n v & w & 0 & 0 & \cdots \\
\frac{v^2}{vx + w} & \frac{2w}{vx + w} & \frac{w^2}{(vx + w)^2} & 0 & \cdots \\
\frac{v^3}{(vx + w)^2} & \frac{3vw}{(vx + w)^2} & \frac{3vw^2}{(vx + w)^3} & \frac{w^3}{(vx + w)^3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

If we let $Q$ to be the last coefficient matrix, then:

\[
\left[ T_0^*(x), T_1^*(x), T_2^*(x), \ldots \right] = TQX_x, \quad \text{where} \quad Q_{ij} = \begin{cases}
1 & i \geq j, i, j = 0, 1, 2, \ldots \\
0 & i < j
\end{cases}
\]

so

$$ T^* = TQ $$

Therefore, the lemma is valid.

**Shifted legendre polynomials:** The Legendre polynomials on $[-1, 1]$ are defined as:

\[
\begin{bmatrix}
P_0(x) = 1, & P_1(x) = x, \\
P_i(x) = \left(2 - \frac{1}{i^2}\right) x P_{i-1}(x) - \left(1 - \frac{1}{i^2}\right) P_{i-2}(x), & i = 2, 3, 4, \ldots
\end{bmatrix}
\tag{30}
\]
and we define shifted Legendre polynomials as:
\[
P^*(x) = \frac{1}{(1-x^2)^{1/2}} P_i(x), \quad i = 2, 3, 4, \ldots
\]
\[
P^*_i(x) = \frac{1}{(1-x^2)^{1/2}} P_i^2(x) - \left(1 - \frac{1}{i} \right) P_i^2(x), \quad i = 2, 3, 4, \ldots
\]
Compared with the exact solution. Results are given in Table 1 and 2 for \( u_1(x) \), \( u_2(x) \) and \( u_3(x) \), respectively.

From the numerical results in Table 1 and 2, it is easy to conclude that obtained results by OTM are in good agreement with the exact solution. Also, during the running of programs we find out the run time of OTM is 0.952 sec. Therefore, the algorithm of OTM is fast.

**Experiment 2:** Consider the following nonlinear NDDS (Vanani and Aminataei, 2009):
\[
u_1(x) = \frac{u_1u_2^2}{2} - \frac{u_2}{2} + \cos(x)u_2 \sin(x) + \frac{1}{2} u_1^2 \sin^2(x) - \frac{1}{2} u_2^2 \sin^2(x), \quad 0 \leq x \leq 1,
\]
\[u_2(x) = u_2(\sqrt{x}) + u_0(1) + u_0(\sin(\sqrt{x})), \quad 0 \leq x \leq 1,
\]
\[u_3(x) = u_3(\cos(x)) - \frac{1}{2} u_1^2 - \frac{1}{2} u_2^2 - u_3(x), \quad 0 \leq x \leq 1,
\]
\[u_4(x) = e^{2u_1}, \quad x \leq 0,
\]
\[u_5(x) = e^{1}, \quad x \leq 0,
\]
\[u_6(x) = e^2, \quad x \leq 0.
\]
The exact solution is:
\[u_1(x) = e^{x}, \quad u_2(x) = e^x, \quad u_3(x) = \frac{x}{2}, \quad u_4(x) = e^x, \quad u_5(x) = e^{2}, \quad u_6(x) = e^{2}.
\]

We have solved this experiment by OTM for \( n = 20 \) with shifted Chebyshev and Legendre bases. Results are given in Table 3 and 4 for \( u_1(x) \), \( u_2(x) \), \( u_3(x) \) and \( u_4(x) \), respectively. Numerical results in Tables 3 and 4 illustrate a good agreement between OTM solutions and exact solutions. In this experiment, the run time of OTM is 1.607 sec. Again, we can conclude that OTM is a fast method.

**Table 1: Exact and approximate solutions of \( u_1(x), u_2(x) \) and \( u_3(x) \) of experiment 1**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( u_1(x) )</th>
<th>( u_2(x) )</th>
<th>( u_3(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.0000000000</td>
<td>1.0000000000</td>
<td>1.0000000000</td>
</tr>
<tr>
<td>1.0</td>
<td>0.9999999722</td>
<td>0.9999999722</td>
<td>0.9999999722</td>
</tr>
<tr>
<td>0.2</td>
<td>0.9999982293</td>
<td>0.9999982293</td>
<td>0.9999982293</td>
</tr>
<tr>
<td>0.3</td>
<td>0.9999795316</td>
<td>0.9999795316</td>
<td>0.9999795316</td>
</tr>
<tr>
<td>0.4</td>
<td>0.9998803417</td>
<td>0.9998803417</td>
<td>0.9998803417</td>
</tr>
<tr>
<td>0.5</td>
<td>0.999576811</td>
<td>0.999576811</td>
<td>0.999576811</td>
</tr>
<tr>
<td>0.6</td>
<td>0.980756264</td>
<td>0.980756264</td>
<td>0.980756264</td>
</tr>
<tr>
<td>0.7</td>
<td>0.968951697</td>
<td>0.968951697</td>
<td>0.968951697</td>
</tr>
<tr>
<td>0.8</td>
<td>0.951932556</td>
<td>0.951932556</td>
<td>0.951932556</td>
</tr>
<tr>
<td>0.9</td>
<td>0.946796226</td>
<td>0.946796226</td>
<td>0.946796226</td>
</tr>
<tr>
<td>1.0</td>
<td>0.9751818168</td>
<td>0.9751818168</td>
<td>0.9751818172</td>
</tr>
</tbody>
</table>

**Table 2: Exact and approximate solutions of \( u_1(x), u_2(x) \) and \( u_3(x) \) of experiment 2**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( u_1(x) )</th>
<th>( u_2(x) )</th>
<th>( u_3(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.0000000000</td>
<td>1.0000000000</td>
<td>1.0000000000</td>
</tr>
<tr>
<td>1.0</td>
<td>0.9999999722</td>
<td>0.9999999722</td>
<td>0.9999999722</td>
</tr>
<tr>
<td>0.2</td>
<td>0.9999982293</td>
<td>0.9999982293</td>
<td>0.9999982293</td>
</tr>
<tr>
<td>0.3</td>
<td>0.9999795316</td>
<td>0.9999795316</td>
<td>0.9999795316</td>
</tr>
<tr>
<td>0.4</td>
<td>0.9998803417</td>
<td>0.9998803417</td>
<td>0.9998803417</td>
</tr>
<tr>
<td>0.5</td>
<td>0.999576811</td>
<td>0.999576811</td>
<td>0.999576811</td>
</tr>
<tr>
<td>0.6</td>
<td>0.980756264</td>
<td>0.980756264</td>
<td>0.980756264</td>
</tr>
<tr>
<td>0.7</td>
<td>0.968951697</td>
<td>0.968951697</td>
<td>0.968951697</td>
</tr>
<tr>
<td>0.8</td>
<td>0.951932556</td>
<td>0.951932556</td>
<td>0.951932556</td>
</tr>
<tr>
<td>0.9</td>
<td>0.946796226</td>
<td>0.946796226</td>
<td>0.946796226</td>
</tr>
<tr>
<td>1.0</td>
<td>0.9751818168</td>
<td>0.9751818168</td>
<td>0.9751818172</td>
</tr>
<tr>
<td>Table 2: Exact and approximate solution of $u_1(x)$ of experiment 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-----------------</td>
<td>-----------------</td>
<td>-----------------</td>
<td>-----------------</td>
</tr>
<tr>
<td>$x$</td>
<td>$u_{Exp}$</td>
<td>$u_{ap}$</td>
<td>$u_{Exact}$</td>
</tr>
<tr>
<td>0.0</td>
<td>1.0000000000</td>
<td>1.0000000000</td>
<td>1.0000000000</td>
</tr>
<tr>
<td>0.1</td>
<td>1.0105170918</td>
<td>1.0105170918</td>
<td>1.0105170918</td>
</tr>
<tr>
<td>0.2</td>
<td>1.2214027581</td>
<td>1.2214027581</td>
<td>1.2214027581</td>
</tr>
<tr>
<td>0.3</td>
<td>1.3498588075</td>
<td>1.3498588075</td>
<td>1.3498588075</td>
</tr>
<tr>
<td>0.4</td>
<td>1.4918256097</td>
<td>1.4918256097</td>
<td>1.4918256097</td>
</tr>
<tr>
<td>0.5</td>
<td>1.6487072704</td>
<td>1.6487072704</td>
<td>1.6487072704</td>
</tr>
<tr>
<td>0.6</td>
<td>1.8221188003</td>
<td>1.8221188003</td>
<td>1.8221188003</td>
</tr>
<tr>
<td>0.7</td>
<td>2.013752074</td>
<td>2.013752074</td>
<td>2.013752074</td>
</tr>
<tr>
<td>0.8</td>
<td>2.2255409284</td>
<td>2.2255409284</td>
<td>2.2255409284</td>
</tr>
<tr>
<td>0.9</td>
<td>2.4596031109</td>
<td>2.4596031109</td>
<td>2.4596031109</td>
</tr>
<tr>
<td>1.0</td>
<td>2.7182818270</td>
<td>2.7182818270</td>
<td>2.7182818270</td>
</tr>
</tbody>
</table>

| Table 3: Exact and approximate solution of $u_2(x)$ and $u_3(x)$ of experiment 2 |
|-----------------|-----------------|-----------------|-----------------|
| $x$             | $u_{Exp}$        | $u_{ap}$        | $u_{Exact}$     |
| 0.0             | 1.0000000000    | 1.0000000000    | 1.0000000000    |
| 0.1             | 1.1049886802    | 1.1049886802    | 1.1049886802    |
| 0.2             | 1.2197785358    | 1.2197785358    | 1.2197785358    |
| 0.3             | 1.3438252434    | 1.3438252434    | 1.3438252434    |
| 0.4             | 1.4762194604    | 1.4762194604    | 1.4762194604    |
| 0.5             | 1.6151462959    | 1.6151462959    | 1.6151462959    |
| 0.6             | 1.7588188451    | 1.7588188451    | 1.7588188451    |
| 0.7             | 1.9040965336    | 1.9040965336    | 1.9040965336    |
| 0.8             | 2.0490068491    | 2.0490068491    | 2.0490068491    |
| 0.9             | 2.1887419108    | 2.1887419108    | 2.1887419108    |
| 1.0             | 2.317768223     | 2.317768223     | 2.317768223     |

| Table 4: Exact and approximate solution of $u_4(x)$ and $u_5(x)$ of experiment 2 |
|-----------------|-----------------|-----------------|-----------------|
| $x$             | $u_{Exp}$        | $u_{ap}$        | $u_{Exact}$     |
| 0.0             | 1.0000000000    | 1.0000000000    | 1.0000000000    |
| 0.1             | 1.0517091879    | 1.0517091879    | 1.0517091879    |
| 0.2             | 1.1051709180    | 1.1051709180    | 1.1051709180    |
| 0.3             | 1.1618345246    | 1.1618345246    | 1.1618345246    |
| 0.4             | 1.2214027581    | 1.2214027581    | 1.2214027581    |
| 0.5             | 1.2802451464    | 1.2802451464    | 1.2802451464    |
| 0.6             | 1.3498588073    | 1.3498588073    | 1.3498588073    |
| 0.7             | 1.4196758488    | 1.4196758488    | 1.4196758488    |
| 0.8             | 1.4918256097    | 1.4918256097    | 1.4918256097    |
| 0.9             | 1.5683128152    | 1.5683128152    | 1.5683128152    |
| 1.0             | 1.6487072704    | 1.6487072704    | 1.6487072704    |

CONCLUSION

In the present study, OTM is proposed for solving NDDSs. Reducing the NDDSs to algebraic equations is the first characteristic of the proposed method. The main idea of the proposed method is to convert the NDDS including linear and nonlinear terms to an algebraic system to simplify the computations. Arbitrary orthogonal polynomial bases were applied as basis functions to reduce the volume of computations. Furthermore, this method yields the desired accuracy only in a few terms in a series form of the exact solution. All of these advantages of the OTM to solve nonlinear problems assert the method as a convenient, reliable and powerful tool.

REFERENCES