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A Low-cost Numerical Algorithm for the Solution of Nonlinear Delay Boundary Integral Equations

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Abstract: Telescoping Decomposition Method (TDM) as a new modification of the well-known Adomian Decomposition Method (ADM) for solving Delay Boundaries Integral Equations (DBIEs) is presented. The proposed method yields an iterative algorithm to obtain the numerical and analytical solutions of DBIEs including linear and nonlinear terms. The main characteristic of the proposed method is to avoid calculating the Adomian polynomials and yields a simple algorithm. In the obtained algorithm, some orthogonal polynomials are effectively implemented to achieve better approximation for the nonhomogeneous and nonlinear terms that leads to facilitate the computational work. Some illustrative linear and nonlinear experiments are given to show the capability and validity of the proposed algorithm.

Key words: Delay integral equations, Adomian decomposition method, telescoping decomposition method

INTRODUCTION

A time delay phenomenon is encountered in a wide variety of scientific and engineering applications, such as physics and physical models (Brunner, 1994; Cahlon and Schimidt, 1997; Alnasr, 2004), biomathematics and biological models (Baker and Derakhshan, 1993; Hu, 1999; Precup, 1995), population growth, infectious diseases and epidemics (Canada and Zertiti, 1994) and the influence of noise (Ashwin *et al*., 2001; Brunner and Hu, 2005; Vanani *et al*., 2011a; Vanani and Aminataei, 2009) etc. In this study, we consider the following DBIE:

$$u(x)=f(x)+\lambda\int_{x-\sigma(x)}^{x-\tau(x)}G(x,t,u(t))dt, \quad x,t\in[a,b] \tag{1}$$

where, f and g are given smooth functions, $\tau(x)$ and $\sigma(x)$ represent the delay functions such that $a\le x-\sigma(x),\ x-\tau(x)\le b$. Also λ is a constant.

We are interested in solving Eq. 1 using a new modification of ADM. The ADM was first introduced by Adomian (1986, 1988). In recent years a large amount of literatures developed concerning ADM (Adomian, 1994; Rach *et al*., 1992; Adomian *et al*., 1995; Wazwaz, 1997; Adomian and Rach, 1992) and the related modification to investigate various scientific models (Hosseini, 2006; Wazwaz, 1999a, b; Wazwaz, 2000; Wazwaz, 2002; Vanani *et al*., 2011b).

Calculating the Adomian polynomials is the main part of the ADM. Many researchers have discussed this issue and presented different approaches for calculating the Adomian polynomials (Wazwaz and El-Sayed, 2001).

The most popular one is the formula obtained by Adomian (1994) and Adomian (1988) as:

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}f\left(\sum_{i=0}^{\infty}\lambda^i u_j\right)_{\lambda=0} \tag{2}$$

where, A_n denotes the Adomian polynomial of degree n:

$$u=\sum_{i=0}^{\infty}u_i$$

is the exact solution of the problem and f(u) is the nonlinear term in the equation. It is worth noting that calculating the Admian polynomials is difficult for large n and formula (2) can not be applied if f is a function of more than one variable, such as f = f(u, u'). Also, the ADM is shown to be divergent for certain problems (Hosseini and Nasabzadeh, 2006). Therefore, we desire to overcome these problem using TDM.

APPLICATION OF TDM ON DBIES

The structure of TDM is as follows:

Let, the problem (1) is given. Therefore, we consider its solution of the form:

$$u(x)=\sum_{n=0}^{\infty}u_n(x),$$

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where $u_n(x)$ has to be determined sequentially upon the following algorithm:

$$\begin{cases} u_0(x) = f(x), \\ u_1(x) = \lambda\int_{x-\sigma(x)}^{x-\tau(x)} G(x,t,u_0(t))\,dt, \\ u_2(x) = \lambda\int_{x-\sigma(x)}^{x-\tau(x)} G\left(x,t,\sum_{k=0}^{1}u_k(t)\right)dt - \lambda\int_{x-\sigma(x)}^{x-\tau(x)} G(x,t,u_0(t))\,dt, \\ u_3(x) = \lambda\int_{x-\sigma(x)}^{x-\tau(x)} G\left(x,t,\sum_{k=0}^{2}u_k(t)\right)dt - \lambda\int_{x-\sigma(x)}^{x-\tau(x)} G\left(x,t,\sum_{k=0}^{1}u_k(t)\right)dt, \\ u_4(x) = \lambda\int_{x-\sigma(x)}^{x-\tau(x)} G\left(x,t,\sum_{k=0}^{3}u_k(t)\right)dt - \lambda\int_{x-\sigma(x)}^{x-\tau(x)} G\left(x,t,\sum_{k=0}^{2}u_k(t)\right)dt, \\ \vdots \\ u_n(x) = \lambda\int_{x-\sigma(x)}^{x-\tau(x)} G\left(x,t,\sum_{k=0}^{n-1}u_k(t)\right)dt - \lambda\int_{x-\sigma(x)}^{x-\tau(x)} G\left(x,t,\sum_{k=0}^{n-2}u_k(t)\right)dt \end{cases}$$

Adding the above equations, we obtain:

$$u(x) = \sum_{i=0}^{n} u_i(x) = u_0 + \lambda\int_{x-\sigma(x)}^{x-\tau(x)} G\left(x,t,\sum_{k=0}^{n-1}u_k(t)\right)dt$$

This is the so-called Telescoping Decomposition Method which is applicable for finite, infinite, regular and irregular domains. It is noticeable that the convergency of this algorithm has been proven by Al-Refai *et al.* (2008).

Although, this algorithm provides an approximate solution for a wide class of nonlinear problems in terms of convergent series with easily computable components, some times it is difficult to calculate the complicated integrals in each iteration. To overcome this problem, we use orthogonal series such as Chebyshev and Legendre polynomials to obtain the operational and computational forms of each iteration $u_i(x)$, i = 0, 1,..., n. This idea improves the method and decreases the volume of computations. We extend the aforementioned idea as follows:

Let us suppose that $\psi(x)$ and $\phi(x)$ are integrable functions on [a, b], we define the inner product $<,>$ by:

$$<\psi(x),\varphi(x)>_w = \int_a^b \psi(x)\varphi(x)\omega(x)dx$$

where, $\|\psi\|_w^2 = <\psi)(x), \psi(x)>_\omega$ and $\omega(x)$ is a weight function. Let $L^2_\omega[a, b]$ be the space of all functions f: [a, b], with $\|\psi\|_w^2<\infty$.

The main object is to seek an orthogonal series expansion of the exact solution of i-th step $u_i(x)\in L^2_\omega[a, b]$, i = 0, 1,...

Suppose that:

$$u_i(x) = \sum_{j=0}^{n} u_{ij}\varphi_j(x),\ \ u_{i,j} = \int_a^b u_i(x)\varphi_j(x)\omega(x)dx,\ j = 0,1,\ldots,n \quad (3)$$

or,

$$u_i(x) = u_i\Phi X_x,\ \ u_i = [u_{i0}, u_{i0}, \ldots, u_{in}] \quad (4)$$

to be an orthogonal series expansion of the exact solution of i-th step. Also $\{\phi_j(x)\}_{j=0}^{n} = \Phi X_x$ is a set of arbitrary orthogonal polynomial bases defined by a lower triangular matrix Φ, $X_x = [1, x, x^2, \ldots, x^n]^T$.

It is obvious that the functions $x-\sigma(x)$, $x-\tau(x)$ and $G(x, t, u(t))$ can be written as:

$$x - \sigma(x) = a\Phi X_x,\ \ a = [a_0, a_1, \ldots, a_n] \quad (5)$$

$$x - \tau(x) = b\Phi X_x,\ \ b = [b_0, b_1, \ldots, b_n] \quad (6)$$

and

$$G(x,t,u(t)) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{m=0}^{\infty} g_{rsm}x^r t^s u^m(t) \quad (7)$$

It is easily can be shown that there exist a matrix G such that $G(x, t, u(t)) = G\Phi X_t$. Therefore, each step is expressed as a series expansion based on orthogonal polynomials and calculating the integrals of each step is done, easily.

Let us suppose that:

$$u_i(x) = \sum_{k=0}^{\infty} u_{i,k}\varphi_k(x) = u_i\Phi X_x,\ \ i = 0,1,\ldots,n \quad (8)$$

where, $:\phi_k(x)\}_{k=0}^{\infty} = \Phi X_x$ is a set of arbitrary orthogonal polynomial bases defined by a lower triangular matrix Φ, $u_i = [u_{i0}, u_{i1}, u_{i2}, \ldots]$ and $X_x = [1, x, x^2, \ldots]^T$. Therefore, we have:

$$\sum_{i=0}^{n} u_i(x) = \sum_{i=0}^{n}\sum_{k=0}^{\infty} u_{i,k}\varphi_k(x) = \sum_{i=0}^{n} u_i\Phi X_x = u\Phi X_x,\ \ u = \sum_{i=0}^{n} u_i \quad (9)$$

Thus, the approximate series of the exact solution of Eq. 1 is considered as:

$$\hat{u}_n(x) = u\Phi X_x$$

This modification decreases the volume of computations and runtime of the algorithm of the method due to the orthogonality of the polynomials using in the aforementioned procedure.

SOME ORTHOGONAL POLYNOMIALS

Orthogonal functions can be used to obtain a good approximation for transcendental functions. Since, Chebyshev and Legendre polynomials are more applicable orthogonal functions for a wide range of problems therefore we consider them, briefly.

Chebyshev polynomials: The Chebyshev polynomials are defined on [-1, 1] as:

$$\begin{cases} T_0(x)=1, T_1(x)=x, \\ T_{i+1}(x)=2xT_i(x)-T_{i-1}(x), \quad i=1,2,3,\ldots, \end{cases} \tag{10}$$

or,

$$[T_0(x), T_1(x), T_2(x), \ldots]^T = TX_x, \quad \text{where} \quad T=\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ -1 & 0 & 2 & 0 & \cdots \\ 0 & -3 & 0 & 4 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and shifted Chebyshev polynomials are defined as:

$$\begin{cases} T_0^*(x)=1, T_1^*(x)=\dfrac{2x-(b+a)}{b-a}, \quad x\in[a,b], \\ T_{i+1}^*(x)=2(\dfrac{2x-(b+a)}{b-a})T_i^*(x)-T_{i-1}^*(x), \quad i=1,2,3,\ldots. \end{cases} \tag{11}$$

Legendre polynomials: The Legendre polynomials on [-1, 1] are defined as:

$$\begin{cases} P_0(x)=1, P_1(x)=x, \\ P_i(x)=(2-\dfrac{1}{i})xP_{i-1}(x)-(1-\dfrac{1}{i})P_{i-2}(x), \quad i=2,3,4,\ldots, \end{cases} \tag{12}$$

or,

$$[P_0(x), P_1(x), P_2(x), \ldots]^T = PX_x, \quad \text{where} \quad P=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ -\frac{1}{2} & 0 & \frac{3}{2} & 0 & 0 & \cdots \\ 0 & -\frac{3}{2} & 0 & \frac{5}{2} & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and shifted Legendre polynomials are defined as:

$$\begin{cases} P_0^*(x)=1, P_1^*(x)=\dfrac{2x-(b+a)}{b-a}, \quad x\in[a,b], \\ P_i^*(x)=(2-\dfrac{1}{i})(\dfrac{2x-(b+a)}{b-a})P_{i-1}^*(x)-(1-\dfrac{1}{i})P_{i-2}^*(x), \quad i=2,3,4,\ldots. \end{cases} \tag{13}$$

ILLUSTRATIVE NUMERICAL EXPERIMENTS

Here, we consider four test problems corresponding to the DBIE (1) to demonstrate the efficiency of the proposed method. In all experiments, we use orthogonal polynomials to decrease the volume of computations. The computations associated with these experiments were performed in Maple 13 on a PC, CPU 2.4 GHz.

Experiment 1: Consider the following DBIE:

$$u(x)=f(x)+\int_{x^3}^{x^2} x\sin(t)u(t)dt, \quad 0\le x\le 1$$

where:

$$f(x)=e^x+\frac{1}{2}xe^{x^3}\left[\sin\left(x^3\right)-\cos\left(x^3\right)\right]+\frac{1}{2}xe^{x^2}\left[\cos\left(x^2\right)-\sin\left(x^2\right)\right]$$

The exact solution is $u(x) = e^x$ and the delay functions are $\sigma(x) = x-x^3$ and $\tau(x) = x-x^2$.

We have solved this experiment using TDM with shifted Legendre polynomials and n = 10. The sequence of approximate solution is obtained as follows:

$$\begin{aligned} &u_0(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}-\frac{59x^5}{20}+\frac{x^6}{720}+\frac{841x^7}{5040} \\ &+\frac{x^8}{40320}-\frac{30239x^9}{362880}+\frac{1209601x^{10}}{3628800}+O(x^{11}), \\ &u_1(x)=\frac{x^5}{2}-\frac{x^7}{6}+\frac{x^9}{12}-\frac{x^{10}}{3}+O(x^{11}), \\ &u_2(x)=0+O(x^{11}), \\ &u_3(x)=0+O(x^{11}), \\ &\vdots \\ &u_{10}(x)=0+O(x^{11}) \end{aligned}$$

Thus, we obtain:

$$u(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots+\frac{x^{10}}{10!}+O(x^{11})$$

Hence, we get:

$$u(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots+\frac{x^n}{n!}+O(x^{n+1})$$

This has the closed form $u(x) = e^x$ which is the exact solution of the problem.

Also, we test the runtime of the algorithm of the method for different n. For n = 5, 10, 15 and 20 the runtimes are obtained as 0.109, 0.250, 0.437 and 0.733 in sec, respectively.

Experiment 2: Consider the following DBIE:

$$u(x) = f(x) + \int_{\frac{x}{10}}^{\frac{x}{2}} \sin(x+t)u(t)dt, \quad 0 \le x \le 1$$

where:

$$f(x) = \sin(x) - \frac{1}{5}x\cos(x) - \frac{1}{4}\sin\left(\frac{6}{5}x\right) + \frac{1}{4}\sin(2x)$$

The exact solution is u(x) = sin(x). Also, delay functions are as σ(x) = x-x/10 and τ(x) = x-x/2.

We have solved this problem using TDM with shifted Chebyshev polynomials and n = 10. The sequence of approximate solution is obtained as follows:

$$u_0(x) = x - \frac{41}{125}x^3 + \frac{2882}{46875}x^5 - \frac{1199383}{196875000}x^7 + \frac{20493173}{59062500000}x^9 + O(x^{11}),$$

$$u_1(x) = \frac{121}{750}x^3 - \frac{1771067}{31250000}x^5 + \frac{142831779343}{19687500000000}x^7 - \frac{471816000468251}{885937500000000000}x^9 + O(x^{11}),$$

$$u_2(x) = \frac{330451}{93750000}x^5 - \frac{6764470407403}{4921875000000000}x^7 + \frac{2861393244400571491}{1476562500000000000000}x^9 + O(x^{11}),$$

$$u_3(x) = \frac{21512690551}{1640625000000000}x^7 - \frac{20133234332705350711}{369140625000000000000000}x^9 + O(x^{11}),$$

$$u_4(x) = \frac{34138725274977961}{369140625000000000000000}x^9 + O(x^{11})$$

$$u_5(x) = 0 + O(x^{11})$$

$$\vdots$$

$$u_{10}(x) = 0 + O(x^{11}),$$

Thus, we obtain:

$$u(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + O(x^{11})$$

Therefore, we conclude that:

$$u(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + O(x^{2n+3})$$

This convergent series solution has the closed form u(x) = sinx which is the exact solution of the problem.

The runtime of the algorithm of the method for n = 4, 8 and 12 are obtained as 0.094, 0.921 and 3.760 in sec, respectively.

Experiment 3: Consider the following nonlinear DBIE:

$$u(x) = e^{-x} - \frac{1}{5}e^{-\frac{5}{6}x} + \frac{1}{5}e^{-\frac{5}{3}x} + \int_{\frac{x}{6}}^{\frac{x}{3}} u^5(t)dt, \quad 0 \le x \le 1$$

with the exact solution u(x) = e^{-x}. Delay functions are σ(x) = x-x/6 and τ(x) = x-x/3.

We have solved this problem using OTM with shifted Legendre polynomials and n = 5. The sequence of approximate solution is obtained as follows:

$$u_0(x) = 1 - \frac{7}{6}x + \frac{17}{24}x^2 - \frac{391}{1296}x^3 + \frac{1057}{10368}x^4 - \frac{27151}{933120}x^5 + O(x^6),$$

$$u_1(x) = \frac{1}{6}x - \frac{35}{144}x^2 + \frac{8645}{46656}x^3 - \frac{219775}{2239488}x^4 + \frac{9787537}{241864704}x^5 + O(x^6),$$

$$u_2(x) = \frac{5}{144}x^2 - \frac{1715}{31104}x^3 + \frac{3677525}{80621568}x^4 - \frac{454658369}{17414258688}x^5 + O(x^6),$$

$$u_3(x) = \frac{455}{93312}x^3 - \frac{446075}{53747712}x^4 + \frac{4595118251}{626913312768}x^5 + O(x^6),$$

$$u_4(x) = \frac{86975}{161243136}x^4 - \frac{407719781}{417942208512}x^5 + O(x^6)$$

$$u_5(x) = \frac{63781601}{1253826625536}x^5 + O(x^6).$$

Thus, we obtain:

$$u(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^6)$$

Hence, we conclude that:

$$u(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots + (-1)^n \frac{x^n}{n!} + O(x^{n+1})$$

This has the closed form u(x0 = e^{-x} which is the exact solution of the problem. The runtime of the algorithm of the method for n = 5, 10,15 and 20 are obtained as 0.015, 0.063, 0.109 and 0.203 in sec, respectively.

Experiment 4: Consider the following nonlinear DBIE:

$$u(x) = \cos(x) + \frac{1}{3}\cos^3\left(\frac{x}{2}\right) - \frac{1}{3}\cos^3\left(x^3\right) - \int_{\frac{x}{2}}^{x^3} \sin(t)u^2(t)dt, \quad 0 \le x \le 1$$

The exact solution is u(x) = cosx and the delay functions are $\sigma(x) = x\text{-}x/2$ and $\tau(x) = x\text{-}x^3$.

We have solved this problem using OTM with shifted Chebyshev polynomials and n = 9. The sequence of approximate solution is obtained as follows:

$$u_0(x) = 1 - \frac{5}{8}x^2 + \frac{23}{384}x^4 + \frac{4583}{9216}x^6 \frac{803}{10321920}x^8 + O(x^{10}),$$
$$u_1(x) = \frac{1}{8}x^2 - \frac{17}{768}x^4 - \frac{91811}{184320}x^6 + \frac{66449}{165150720}x^8 + O(x^{10}),$$
$$u_2(x) = \frac{1}{256}x^4 - \frac{31}{49152}x^6 - \frac{9401}{20971520}x^8 + O(x^{10}),$$
$$u_3(x) = \frac{1}{16384}x^6 - \frac{383}{50331648}x^8 + O(x^{10}),$$
$$u_4(x) = \frac{9}{16777216}x^8 + O(x^{10}),$$
$$u_5(x) = 0 + O(x^{10}).$$
$$\vdots$$
$$u_8(x) = 0 + O(x^{10})$$

Thus, we obtain:

$$u(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + O(x^{10})$$

Hence, we conclude that:

$$u(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + O(x^{2n+2})$$

This has the closed form u(x) = cosx(x), which is the exact solution of the problem.

For n = 5, 10, 15 and 20 the runtime of the method are obtained as 0.140, 0.624, 1.201 and 2.262 in sec, respectively.

CONCLUSION

In this study, the TDM was made applicable to DBIEs. TDM provides the solution of the problem without calculating Adomian's polynomials which is an important advantage over the Adomian decomposition method. Also, the orthogonality of the polynomials using in the proposed method reduced the volume computations of the resolvent algorithm. This modification considerably is capable for solving a wide range class of linear and nonlinear equations. This purpose was satisfied by solving some linear and nonlinear experiments. Furthermore, this method yields the desired accuracy only in a few terms in a series form of the exact solution. The method is also quite straightforward to write computer code. These facts illustrate the TDM as a fast, reliable, valid and powerful tool for solving DBIEs.

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