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Characterization of Class of Super Lattice Measurable Sets

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Abstract: This study is an investigation on super lattice measurable sets. It characterizes super lattice, super lattice measurable set, elementary lattice, monotone class and establishes that the union, intersection, difference of two super lattice measurable sets is a super lattice measurable set. Also it ascertains that the class of elementary lattice is closed under union, intersection and difference. Finally, it confirms that the product of lattice σ- algebra is the smallest monotone class contains all elementary lattices.

Key words: Lattice, measure, σ-algebra

INTRODUCTION

Gabor (1964) has introduced the concept of product lattice. In the recent past (Royden, 1981) has made an effort on the concept of function lattice. Tanaka (2009) has established a Hahn Decomposition Theorem of Signed Lattice Measure and introduced the concept of lattice σ-algebra σ(L). Recently, Anil Kumar *et al.* (2011) made a categorization of Class of Measurable Borel Lattices. Also, Anil Kumar *et al.* (2011) made an investigation on Lattice Boolean Valued Measurable functions and defined the concepts of lattice measurable space, lattice measurable set, σ-lattice and δ-lattice.

In this study we establish the general frame work for the study of the characterization of super lattice measurable sets. Here some concepts in measure theory can be generalized by means of a lattice σ-algebra σ(L). We establish super lattice, super lattice measurable set, elementary lattice, monotone class. We studied the characterization of super lattice measurable sets.

In this study, we establish union, intersection, difference of two super lattice measurable sets. Also we establish that class of elementary lattice is closed under union, intersection and difference. Finally we confirm that the product lattice σ-algebra is the smallest monotone class contains all elementary lattices.

PRELIMINARIES

Here, we shall briefly review the well-known facts about lattice theory specified by Birkhoff (1967).

(L, ∧, ∨) is called a lattice if it is enclosed under operations ∧ and ∨ and satisfies, for any elements x, y, z, in L:

- (L1) commutative law: $x\wedge y = y\wedge x$ and $x\vee y = y\vee x$
- (L2) associative law: $x\wedge (y\wedge z) = (x\wedge y)\wedge z$ and $x\vee (y\vee z) = (x\vee y)\vee z$
- (L3) absorption law: $x\vee (y\wedge x) = x$ and $\wedge (y\vee x) = x$. Hereafter, the lattice (L, ∧, ∨) will often be written as L for simplicity. A lattice (L, ∧, ∨) is called distributive if, for any x, y, z, in L
- (L4) distributive law holds: $x\vee (y\wedge z) = (x\vee y)\wedge (x\vee z)$ and $x\wedge (y\vee z) = (x\wedge y)\vee (x\wedge z)$

A lattice L is called complete if, for any subset A of L, L contains the supremum ∨ A and the infimum ∧ A. If L is complete, then L itself includes the maximum and minimum elements which are often denoted by 1 and 0 or I and O, respectively.

A distributive lattice is called a Boolean lattice if for any element x in L, there exists a unique complement x^c such that:

- (L5) the law of excluded middle: $x\vee x^c = 1$
- (L6) the law of non-contradiction: $x\wedge x^c = 0$

Let L be a lattice and €: L→L be an operator. Then € is called a lattice complement in L if the following conditions are satisfied.

- (L5) and (L6): $\forall x\in L$, $x\vee x^c = 1$ and $x\wedge x^c = 0$
- (L7) the law of contrapositive: $\forall x, y\in L$, $x<y$ implies $x^c > y^c$
- (L8) the law of double negation: $\forall x\in L$, $(x^c)^c = x$

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Throughout this paper, we consider lattices as complete lattices which obey (L1), (L8) except for (L6) the law of non-contradiction. Unless otherwise stated, X is the entire set and L is a lattice of any subsets of X.

Definition 1: If a lattice L satisfies the following conditions, then it is called a lattice σ-Algebra:

- $\forall$ h∈, h^c L
- If $h_n \in L$ for n = 1, 2, 3....., then $\bigvee_{n=1}^{\infty}$ hn∈L

We denote σ(L), as the lattice σ-Algebra generated by L.

Note 1: By definition 1, it is clear that σ(L) is closed under finite unions and finite intersections.

Definition 2: Let σ(L) be a lattice σ-algebra of sub sets of a set X. A function μ: σ(L)>[0, ∞] is called a positive lattice measure defined on σ(L) if:

- $\mu(\phi)=0$
- $\mu(\bigvee_{n=1}^{\infty} A_n)=\sum_{n=1}^{\infty}\mu(A_n)$

where, $\{A_n\}$ is a disjoint countable collection of members of σ(L) and μ(A)<∞ for at least one A∈ σ(L).

Definition 3: The ordered pair (X, σ(L)) is said to be lattice measurable space.

Definition 4: A lattice A is said to be lattice measurable set if A belongs to σ(L).

Definition 5: A function lattice is a collection L^1 of extended real valued functions defined on a lattice L^1 with respect to usual partial ordering on functions. That is if f, $g \in L^1$ then $f \vee g \in L^1$, $f \vee g \in L^1$.

Definition 6: If f and g are extended real valued lattice measurable functions defined on L^f, then f∨g, f∧g are defined by (f∨g) (x) = sup {f(x), g(x)} and (f∧g) (x) = inf {f(x), g(x)} for any x∈L.

Definition 7: Let E be a lattice then the complement of E is defined as $E^c = \{x \in E^c / x \notin E\}$.

Note 2: $(E^c)^c = E$.

Definition 8: A countable union of lattice measurable sets is called a σ-lattice.

Definition 9: A countable intersection of lattice measurable sets is called a δ-lattice.

Definition 10: Let X and Y be two lattices then their Cartesian product denoted by X×Y is defined as X×Y = {(x, y)/x∈X, y∈Y}.

Definition 11: If A<X, B<Y then A×B<X×Y. Any lattice of the form A×B is called super lattice in X×Y.

Remark 1 (Rudin, 1987): Let (X, S), (Y, T) be lattice measurable spaces. Then S is a lattice σ-algebra in X and T is a lattice σ-algebra in Y.

Definition 12: If A∈S and B∈T then the lattice of the form A×B is called super lattice measurable set.

Definition 13: If $Q = R_1 \vee R_2 \vee ... \vee R_n$ where each R_i is a super lattice measurable set and $R_i \wedge R_j = \phi$ for i≠j, then Q is called elementary lattice. The class of all elementary lattices is denoted by L_E.

Definition 14: S×T is defined to be smallest lattice σ-algebra in X×Y which contains every super lattice measurable set.

Definition 15: If $A_i, B_i \in$ σ(L) such that $A_i < A_{i+1}$, $B_i > B_{i+1}$ for i = 1, 2, 3, ... and

$$A = \bigvee_{i=1}^{\infty} Ai,\ B = \bigvee_{i=1}^{\infty} Bi$$

then A∈ σ(L) and B∈ σ(L), this lattice σ-algebra σ(L) is a monotone class.

Example 1: X×Y is a monotone class.

Definition 16: Let E<X×Y where x∈X, y∈Y, we define x-section lattice of E by E_x = {y/ (x, y)∈E} and y-section lattice of E_y = {x/(x, y)∈E}.

Note 3: E_x<Y and E_y<X.

CHARACTERIZATION OF CLASS OF SUPER LATTICE MEASURABLE SETS

Result 1: The union of two super lattice measurable sets is a super lattice measurable set.

Proof: Let $(A_1 \times B_1)$, $(A_2 \times B_2)$ be two super lattice measurable sets, clearly $A_1, A_2 \in S$ implies $A_1 - A_2 \in S$, $A_1 \wedge A_2 \in S$ and $A_1 \vee A_2 \in S$ (Since S is a lattice σ-algebra). Also

$B_1, B_2 \in T$ implies $B_1\text{-}B_2 \in T$, $B_1 \wedge B_2 \in T$ and $B_1 \vee B_2 \in T$ (Since T is a lattice σ-algebra). Now $(A_1 \times B_1) \vee (A_2 \times B_2) = (A_1 \vee A_2) \times (B_1 \vee B_2)$. By definition of σ-lattice $(A_1 \vee A_2)$, $(B_1 \vee B_2)$ are lattice measurable sets implies $(A_1 \vee A_2) \times (B_1 \vee B_2)$ is a super lattice measurable set. Therefore $(A_1 \times B_1) \vee (A_2 \times B_2)$ is a super lattice measurable set.

Result 2: The intersection of two super lattice measurable sets is a super lattice measurable set.

Proof: Let $(A_1 \times B_1)$, $(A_2 \times B_2)$ be two super lattice measurable sets, clearly $A_1, A_2 \in S$ implies $A_1\text{-}A_2 \in S$, $A_1 \wedge A_2 \in S$ and $A_1 \wedge A_2 \in S$ (Since S is a lattice σ- algebra). Also $B_1, B_2 \in T$ implies $B_1\text{-}B_2 \in T$, $B_1 \wedge B_2 \in T$ and $B_1 \vee B_2 \in T$ (Since T is a lattice σ- algebra). Now $(A_1 \times B_1) \wedge (A_2 \times B_2) = (A_1 \wedge A_2) \times (B_1 \wedge B_2)$. By definition of δ- lattice $(A_1 \wedge A_2)$, $(B_1 \wedge B_2)$ are lattice measurable sets implies $(A_1 \wedge A_2) \times (B_1 \wedge B_2)$ is a super lattice measurable set. Therefore $(A_1 \times B_1) \wedge (A_2 \times B_2)$ is a super lattice measurable set.

Result 3: The difference of two super lattice measurable sets is a super lattice measurable set.

Proof: Let $(A_1 \times B_1)$, $(A_2 \times B_2)$ be two super lattice measurable sets, clearly $A_1, A_2 \in S$ implies $A_1\text{-}A_2 \in S$, $A_1 \wedge A_2 \in S$ and $A_1 \vee A_2 \in S$ (Since S is a lattice σ- algebra). Also $B_1, B_2 \in T$ implies $B_1\text{-}B_2 \in T$, $B_1 \wedge B_2 \in T$ and $B_1 \vee B_2 \in T$ (Since T is a lattice σ-algebra) implies $(A_1\text{-}A_2) \times B_1$ is a super lattice measurable set. $(A_1 \wedge A_2) \times (B_1\text{-}B_2)$ is a super lattice measurable set implies $((A_1\text{-}A_2) \times B_1) \vee ((A_1 \wedge A_2) \times (B_1\text{-}B_2))$ is a super lattice measurable set (By definition of σ-lattice) implies $(A_1 \times B_1)\text{-}(A_2 \times B_2)$ is a super lattice measurable set.

Theorem 1: If $E \in S \times T$, then $E_x \in T$ and $E_y \in S$ for every $x \in X$ and $y \in Y$.

Proof: Let K be the class of all $E \in S \times T$ such that $E_x \in T$ for every $x \in X$ that is $K = \{E \in S \times T / E_x \in T \text{ for every } x \in X\}$. Let $F = A \times B$ be a super lattice measurable set that is $A \in S$, $B \in T$. Also $F_x = B$ if $x \in A$ and $F_x = \phi$ if $x \notin A$ implies $F_x \in T$ for every $x \in X$. Therefore $F \in K$. That is every super lattice measurable set belongs to K. In particular $X \times Y \in K$.

Let $E \in K$. Then $y \in (E^c)_x$ if and only if $(x, y) \in E^c$ if and only if $(x, y) \notin E$ if and only if $y \notin E_x$ if and only if $y \in (E_x)^c$. Therefore $(E^c)_x = (E_x)^c$. Since $E_x \in T$ and since T is a lattice σ- algebra we have $E^c_x \in T$. Therefore $E^c \in K$.

Let $E_i \in K$ ($i = 1, 2, 3, \ldots$) and let:

$$E = \bigvee_{i=1}^{\infty} E_i$$

then $Y \in E_x$ if and only if $(x, y) \in E$ if and only if $(x, y) \in E_c$ for some i if and only if $y \in (E_i)_x$. Therefore:

$$E_x = \bigvee_{i=1}^{\infty} (E_i)_x$$

Since T is a lattice σ- algebra, $\bigvee_{i=1}^{\infty} (E_i)_x \in T$ implies $E_x \in T$. Therefore $E \in T$.

From Eq. 2 and 3 K is a lattice σ-algebra. Since $K < S \times T$ we get $K = S \times T$. Hence for any $E \in S \times T$, $E_x \in T$ for every $x \in X$. In a similar way we can prove $E_y \in S$ for every $y \in Y$.

Lemma 1: To prove $S \times T$ is a monotone class.

Proof: Let $A_i, B_i \in S \times T$, $A_i < A_{i+1}$, $B_i > B_{i+1}$ for $i = 1, 2, 3, \ldots$ and:

$$A = \bigvee_{i=1}^{\infty} Ai,\ B = \bigvee_{i=1}^{\infty} Bi$$

Since $S \times T$ is a lattice σ- algebra implies $A \in S \times T$ and:

$$B^c = \bigvee_{i=1}^{\infty} B_i^c \in S \times T$$

implies $B \in S \times T$. Since $S \times T$ is a lattice σ- algebra. Therefore $S \times T$ is a monotone class.

Lemma 2: To prove L_E is closed under intersection, difference and unions.

Proof: Let $A_1 \times B_1$, $A_2 \times B_2$ be two super lattice measurable sets. Clearly $(A_1 \times B_1) \wedge (A_2 \times B_2) = (A_1 \wedge A_2) \times (B_1 \wedge B_2)$, we get $(A_1 \wedge A_2) \times (B_1 \wedge B_2)$ is super lattice measurable set (Since by definition of δ-lattice and every δ-lattice is lattice measurable). Also $(A_1 \times B_1)\text{-}(A_2 \times B_2) = ((A_1\text{-}A_2) \times B_1) \vee ((A_1 \wedge A_2) \times (B_1\text{-}B_2))$, we get the difference of two super lattice measurable sets is a union of two disjoint super lattice measurable sets. (Since by σ-lattice and every σ-lattice is lattice measurable). [Note that $A_1, A_2 \in S$ implies $A_1\text{-}A_2 \in S$ and $A_1 \wedge A_2 \in S$ since S is a lattice σ-algebra. Also $B_1\text{-}B_2 \in T$, $B_1 \in T$ implies $(A_1\text{-}A_2) \times B_1$, $(A_1 \wedge A_2) \times (B_1\text{-}B_2)$ are super lattice measurable sets and they are disjoint since $(A_1 \wedge A_2) \wedge (A_1\text{-}A_2) = \phi$]. Hence $(A_1 \times B_1)\text{-}(A_2 \times B_2) \in L_E$.

Part 1: Closed under intersection. Let $P, Q \in L_E$ implies $P = R_1 \vee R_2 \vee \ldots\ldots \vee R_n$, $Q = R^1_1 \vee R^1_2 \vee \ldots\ldots \vee R^1_m$ where $R_i \wedge R_j = \phi$ for $i \neq j$, $R^1_i \wedge R^1_j = \phi$ for $i \neq j$ and R_i's$\wedge R_j$'s are super lattice measurable sets (By result3..). Now $P \wedge Q = (R_1 \wedge R^1_1) \vee (R_1 \wedge R^1_2) \vee \ldots \vee (R_1 \wedge R^1_m) \vee \ldots\ldots \vee (R_n \wedge R^1_1) \vee \ldots\ldots \vee (R_n \wedge R^1_m)$. Here each $R_i \wedge R_j$ is super lattice measurable set (By result 3.2) and clearly these are disjoint. Therefore $P \wedge Q \in L_E$.

Part 2: Closed under difference. Now $P-Q = P \wedge Q^c = (R_1 \vee R_2 \vee \ldots\ldots \vee R_n) \wedge (R^1{}_1 \vee R^1{}_2 \vee \ldots\ldots \vee R^1{}_m)^c = (R_1 \vee R_2 \vee \ldots\ldots \vee R_n) \wedge (R^{1c}{}_1 \wedge R^{1c}{}_2 \wedge \ldots\ldots \wedge R^{1c}{}_m) = (R_1 \wedge (R^{1c}{}_1 \wedge R^{1c}{}_2 \wedge \ldots\ldots \wedge R^{1c}{}_m)) \vee R_2 \wedge (R^{1c}{}_1 \wedge R^{1c}{}_2 \wedge \ldots\ldots \wedge R^{1c}{}_m) \vee \ldots\ldots \vee R_n \wedge (R^{1c}{}_1 \wedge R^{1c}{}_2 \wedge \ldots\ldots \wedge R^{1c}{}_m)) = \{([(R_1-R^1{}_1)-R^1{}_2]\ldots\ldots)-R^1{}_m\} \vee \ldots\ldots \vee \{([(R_n-R_{11})-R^1{}_2]\ldots\ldots)-R^1{}_m\}$.

Since the difference of two super lattice measurable sets is the disjoint union of two super lattice measurable set (By result 3), we get right hand side is the disjoint union of super lattice measurable sets. Hence $P-Q \in L_E$.

Part 3: Closed under union. Now $P \vee Q = (P-Q) \vee Q$ and $(P-Q) \wedge Q = \phi$, we get $P \vee Q \in L_E$. Therefore L_E is closed under intersection, difference and unions.

Theorem 2: $S \times T$ is the smallest monotone class which contains all elementary lattices.

Proof: Let $\sigma(L)$ be the smallest monotone class which contains L_E. This can be exists, let M be the family of all monotone class containing L_E. Since $X \times Y \in M$ implies M is non-empty. Let F be the intersection of all members of M that is $F = \wedge M$. Then $L_E < F$. We prove F is monotone class. Let A_i. $B_i \in F$, $A_i < A_{i+1}$, $B_i > B_{i+1}$ for i = 1, 2, 3, ... and

$$A = \bigvee_{i=1}^{\infty} A_i,\ B = \bigvee_{i=1}^{\infty} B_i.$$

Then A_i , B_i belongs to every member of M and since every member of M is a monotone class, A, B belongs to every member of M therefore $A, B \in F$. Therefore F is a monotone class. By lemma1, $S \times T$ is a monotone class also it is obvious that $L_E < S \times T$ and since $\sigma(L)$ be the smallest monotone class which contains L_E we get L:

$$\sigma(L) < S \times T \quad (1)$$

Now for any lattice $P < X \times Y$ define $K(P) = \{Q < X \times Y / P-Q \in \sigma(L),\ Q-P \in \sigma(L) \text{ and } P \vee Q \in \sigma(L)\}$. Clearly $Q \in K(P)$ iff $P-Q$, $Q-P$, $P \vee Q \in \sigma(L)$ iff $Q-P$, $P-Q$, $Q \vee P \in \sigma(L)$ iff $P \in K(Q)$ that I s:

$$Q \in K(P) \text{ iff } P \in K(Q) \quad (2)$$

Let A_i, $B_i \in K(P)$, $A_i < A_{i+1}$, $B_i > B_{i+1}$ for i = 1, 2, 3, ... and:

$$A = \bigvee_{i=1}^{\infty} A_i,\ B = \bigvee_{i=1}^{\infty} B_i.$$

then A_i-P, $P-A_i$, $P \vee A_i$, B_i-P, $P-B_i$, $P \vee B_i$ all belongs to $\sigma(L)$ for every i, also:

$$A-P = \bigvee_{i=1}^{\infty} A_i - \bigvee_{i=1}^{\infty} (A_i-P) \in \sigma(L) \quad (3)$$

(Since $\sigma(L)$ is a monotone class).

$$P-A = P-\bigvee_{i=1}^{\infty} A_i = P \wedge (\bigvee_{i=1}^{\infty} A_i)^c = P \wedge (\bigvee_{i=1}^{\infty} A_i^c) = \bigvee_{i=1}^{\infty} (P \wedge A_i^c) = (P-A_i). \quad (4)$$

Since $P-A_i \in \sigma(L)$ and $P-A_i > P-A_{i+1}$ we get:

$$\bigvee_{i=1}^{\infty} (P-A_i) \in \sigma(L) \quad (5)$$

(Since $\sigma(L)$ is a monotone class) implies $P-A \in \sigma(L)$. Again:

$$P \vee A = P \vee \bigvee_{i=1}^{\infty} Ai = \bigvee_{i=1}^{\infty} (P \vee A_i) \in \sigma(L) \quad (6)$$

(Since each $P \vee A_i \in \sigma(L)$ and $\sigma(L)$ is a monotone class). Hence $A \in K(P)$. Similarly $B \in K(P)$. Therefore K(P) is a monotone class. Fix $P \in L_E$. Let Q be any lattice of L_E. Then $P-Q$, $Q-P$, $P \vee Q \in L_E$ (By lemma2). Hence $P-Q$, $Q-P$, $P \vee Q \in \sigma(L)$ (Since $L_E < \sigma(L)$. Therefore $Q \in K(P)$ that is $Q \in K(P)$ for every $Q \in L_E$ that is $L_E < K(P)$ and by Eq. 3 K(P) is a monotone class. But $\sigma(L)$ is the smallest monotone class containing L_E. Therefore $\sigma(L) < K(P)$. Again fix $Q \in \sigma(L)$ for any $P \in L_E$. By Eq. 5 we have $Q \in K(P)$. Therefore by Eq. 2 $P \in K(Q)$. Therefore by Eq. 4 $L_E \in K(Q)$ and hence by Eq. 5 $\sigma(L) < K(Q)$. Thus $\sigma(L) < K(Q)$ for every $Q \in \sigma(L)$. Let P, $Q \in \sigma(L)$ then $Q \in K(P)$. Therefore $Q-P$, $P-Q$, $P \vee Q \in \sigma(L)$. Therefore $\sigma(L)$ is closed under difference and since:

$$X \times Y \in L_E,\ X \times Y \in \sigma(L)$$

Let $Q \in \sigma(L)$ then $Q^c = (X \times Y) - Q \in \sigma(L)$. By Eq. 6. Let $P_i \in \sigma(L)$ for i = 1, 2, 3, ... Let $P = \vee P_i$ and $Q_n = P_1 \vee P_2 \vee \ldots\ldots \vee P_n$. Since by Eq. 6 $\sigma(L)$ is closed under finite union $Q_n \in \sigma(L)$. Since $Q_n < Q_{n+1}$ and $P = \vee Q_n$, the monotonicity of $\sigma(L)$ shows that $P \in \sigma(L)$. Thus $\sigma(L)$ is a lattice σ-algebra. Also $L_E < \sigma(L) \in S \times T$. But $S \times T$ is the smallest lattice σ-algebra in $X \times Y$ contains every super lattice measurable set. Here $\sigma(L)$ is a lattice σ-algebra containing L_E and hence every super lattice measurable set. Therefore $\sigma(L) = S \times T$.

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