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Characterization of Class of Super Lattice Measurable Sets

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Abstract: This study is an investigation on super lattice measurable sets. It characterizes super lattice, super lattice measurable set, elementary lattice, monotone class and establishes that the union, intersection, difference of two super lattice measurable sets is a super lattice measurable set. Also it ascertains that the class of elementary lattice is closed under union, intersection and difference. Finally, it confirms that the product of lattice $\sigma$-algebra is the smallest monotone class contains all elementary lattices.

Key words: Lattice, measure, $\sigma$-algebra

INTRODUCTION

Gabor (1964) has introduced the concept of product lattice. In the recent past (Royden, 1981) has made an effort on the concept of function lattice. Tanaka (2009) has established a Hahn Decomposition Theorem of Signed Lattice Measure and introduced the concept of lattice $\sigma$-algebra $\sigma(L)$. Recently, Anil Kumar et al. (2011) made a categorization of Class of Measurable Borel Lattices. Also, Anil Kumar et al. (2011) made an investigation on Lattice Boolean Measurable functions and defined the concepts of lattice measurable space, lattice measurable set, $\sigma$-lattice and $\delta$-lattice.

In this study we establish the general framework for the study of the characterization of super lattice measurable sets. Here some concepts in measure theory can be generalized by means of a lattice $\sigma$-algebra $\sigma(L)$. We establish super lattice, super lattice measurable set, elementary lattice, monotone class. We studied the characterization of super lattice measurable sets.

In this study, we establish union, intersection, difference of two super lattice measurable sets. Also we establish that class of elementary lattice is closed under union, intersection and difference. Finally we confirm that the product lattice $\sigma$-algebra is the smallest monotone class contains all elementary lattices.

PRELIMINARIES

Here, we shall briefly review the well-known facts about lattice theory specified by Birkhoff (1967).

$$(L, \land, \lor)$$ is called a lattice if it is enclosed under operations $\land$ and $\lor$ and satisfies, for any elements $x, y, z$, in $L$:

- (L1) commutative law: $x \land y = y \land x$ and $x \lor y = y \lor x$
- (L2) associative law: $x \land (y \land z) = (x \land y) \land z$ and $x \lor (y \lor z) = (x \lor y) \lor z$
- (L3) absorption law: $x \lor (y \land x) = x$ and $\land (y \lor x) = x$

Hereafter, the lattice $(L, \land, \lor)$ will often be written as $L$ for simplicity. A lattice $(L, \land, \lor)$ is called distributive if, for any $x, y, z$ in $L$:

- (L4) distributive law holds: $x \lor (y \land z) = (x \lor y) \land (x \lor z)$ and $x \land (y \lor z) = (x \land y) \lor (x \land z)$

A lattice $L$ is called complete if, for any subset $A$ of $L$, $L$ contains the supremum $\lor A$ and the infimum $\land A$. If $L$ is complete, then $L$ itself includes the maximum and minimum elements which are often denoted by 1 and 0 or I and $\Omega$, respectively.

A distributive lattice is called a Boolean lattice if for any element $x$ in $L$, there exists a unique complement $x'$ such that:

- (L5) the law of excluded middle: $x \lor x' = 1$
- (L6) the law of non-contradiction: $x \land x' = 0$

Let $L$ be a lattice and $\varepsilon: L \rightarrow L$ be an operator. Then $\varepsilon$ is called a lattice complement in $L$ if the following conditions are satisfied:

- (L5) and (L6): $\forall x \in L, x \lor x' = 1$ and $x \land x' = 0$
- (L7) the law of contrapositive: $\forall x, y \in L, x < y$ implies $x' \geq y'$
- (L8) the law of double negation: $\forall x \in L, (x')' = x$

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Throughout this paper, we consider lattices as complete lattices which obey (L1), (L8) except for (L6) the law of non-contradiction. Unless otherwise stated, X is the entire set and L is a lattice of any subsets of X.

**Definition 1:** If a lattice L satisfies the following conditions, then it is called a lattice $\sigma$-Algebra:

- $\forall h \in L$, $h^\bot L$
- If $h_n \in L$ for $n = 1, 2, 3, \ldots$, then $\bigvee_{n=1}^{\infty} h_n \in L$

We denote $\sigma(L)$, as the lattice $\sigma$-Algebra generated by L.

**Note 1:** By definition 1, it is clear that $\sigma(L)$ is closed under finite unions and finite intersections.

**Definition 2:** Let $\sigma(L)$ be a lattice $\sigma$-algebra of sub sets of a set X. A function $\mu: \sigma(L) \rightarrow [0, \infty]$ is called a positive lattice measure defined on $\sigma(L)$ if:

- $\mu(\emptyset) = 0$
- $\mu(\bigvee_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$

where, $\{A_i\}$ is a disjoint countable collection of members of $\sigma(L)$ and $\mu(A) < \infty$ for at least one $A \in \sigma(L)$.

**Definition 3:** The ordered pair $(X, \sigma(L))$ is said to be lattice measurable space.

**Definition 4:** A lattice A is said to be lattice measurable set if A belongs to $\sigma(L)$.

**Definition 5:** A function lattice is a collection $L^1$ of extended real valued functions defined on a lattice $L^1$ with respect to usual partial ordering on functions. That is if $f, g \in L^1$ then $f/v \in L^1$, $f \wedge g \in L^1$.

**Definition 6:** If $f$ and $g$ are extended real valued lattice measurable functions defined on $L^1$, then $f/v, f \wedge g$ are defined by $(f/v)(x) = \sup \{f(x), g(x)\}$ and $(f \wedge g)(x) = \inf \{f(x), g(x)\}$ for any $x \in L^1$.

**Definition 7:** Let $E$ be a lattice then the complement of E is defined as $E^c = \{x \in E/ x \notin E\}$.

**Note 2:** $(E^c)^c = E$.

**Definition 8:** A countable union of lattice measurable sets is called a $\sigma$-lattice.

**Definition 9:** A countable intersection of lattice measurable sets is called a $\delta$-lattice.

**Definition 10:** Let X and Y be two lattices then their Cartesian product denoted by $X \times Y$ is defined as $X \times Y = \{(x, y)/ x \in X, y \in Y\}$.

**Definition 11:** If $A \times X, B \times Y$ then $A \times B \times X \times Y$. Any lattice of the form $A \times B$ is called super lattice in $X \times Y$.

**Remark 1 (Rudin, 1987):** Let $(X, S), (Y, T)$ be lattice measurable spaces. Then S is a lattice $\sigma$-algebra in X and T is a lattice $\sigma$-algebra in Y.

**Definition 12:** If $A \in S$ and $B \in T$ then the lattice of the form $A \times B$ is called super lattice measurable set.

**Definition 13:** If $Q = R_i \cup R_j \cup \ldots \cup R_k$, where each $R_i$ is a super lattice measurable set and $R_i \cap R_j = \emptyset$ for $i \neq j$, then $Q$ is called elementary lattice. The class of all elementary lattices is denoted by $L_E$.

**Definition 14:** $S \times T$ is defined to be smallest lattice $\sigma$-algebra in $X \times Y$ which contains every super lattice measurable set.

**Definition 15:** If $A_i, B_i \in \sigma(L)$ such that $A_i \times A_i, B_i \times B_i$, for $i = 1, 2, 3, \ldots$, and $A = \bigcup_{i=1}^{\infty} A_i, B = \bigcup_{i=1}^{\infty} B_i$ then $A \in \sigma(L)$ and $B \in \sigma(L)$, this lattice $\sigma$-algebra $\sigma(L)$ is a monotone class.

**Example 1:** $X \times Y$ is a monotone class.

**Definition 16:** Let $E \times X \times Y$ where $x \in X, y \in Y$, we define x-section lattice of $E$ by $E_x = \{y/ (x, y) \in E\}$ and y-section lattice of $E_y = \{x/(x, y) \in E\}$.

**Note 3:** $E_x \times Y$ and $E_y \times X$.

**CHARACTERIZATION OF CLASS OF SUPER LATTICE MEASURABLE SETS**

**Result 1:** The union of two super lattice measurable sets is a super lattice measurable set.

**Proof:** Let $(A_i \times B_i), (A_i \times B_i)$ be two super lattice measurable sets, clearly $A_i \times A_i \times S$ implies $A_i \times A_i \times S, A_i \wedge A_i \times S$ and $A_i \vee A_i \times S$. (Since S is a lattice $\sigma$-algebra). Also
Result 2: The intersection of two super lattice measurable sets is a super lattice measurable set.

Proof: Let \((A_i \wedge B_j)\), \((A_i \wedge B_j)\) be two super lattice measurable sets, clearly \(A_i \wedge B_j \subseteq A_i \wedge A_j \subseteq S\) and \(A_i \wedge A_j \subseteq S\) (Since \(S\) is a lattice \(\sigma\)-algebra). Also \(B_i \wedge B_j \subseteq B_i \wedge B_j \subseteq T\) and \(B_i \wedge B_j \subseteq T\) (Since \(T\) is a lattice \(\sigma\)-algebra). Now \((A_i \wedge B_j) \wedge (A_i \wedge B_j) = (A_i \wedge A_j) \wedge (B_i \wedge B_j)\) (By definition of \(\delta\)-lattice \((A_i \wedge A_j)\), \((B_i \wedge B_j)\) are lattice measurable sets because \((A_i \wedge A_j) \wedge (B_i \wedge B_j)\)) is super lattice measurable set. Therefore \((A_i \wedge B_j) \wedge (A_i \wedge B_j)\) is a super lattice measurable set.

Result 3: The difference of two super lattice measurable sets is a super lattice measurable set.

Proof: Let \((A_i \wedge B_j), (A_i \wedge B_j)\) be two super lattice measurable sets, clearly \(A_i \wedge B_j \subseteq A_i \wedge B_j \subseteq S\) implies \(A_i \wedge B_j \subseteq S\) and \(A_i \wedge A_j \subseteq S\) (Since \(S\) is a lattice \(\sigma\)-algebra). Also \(B_i \wedge B_j \subseteq B_i \wedge B_j \subseteq T\) and \(B_i \wedge B_j \subseteq T\) (Since \(T\) is a lattice \(\sigma\)-algebra). \((A_i \wedge B_j) \wedge (A_i \wedge B_j)\) is a super lattice measurable set. \((A_i \wedge A_j) \wedge (B_i \wedge B_j)\) is super lattice measurable set because \((A_i \wedge A_j) \wedge (B_i \wedge B_j)\) is a super lattice measurable set (By definition of \(\sigma\)-lattice) implies \((A_i \wedge B_j) \wedge (A_i \wedge B_j)\) is a super lattice measurable set.

Theorem 1: If \(E \subseteq S \times T\) then \(E \subseteq T\) and \(E \subseteq T\) for every \(x \in X\) and \(y \in Y\).

Proof: Let \(K\) be the class of all \(E \subseteq S \times T\) such that \(E \subseteq T\) for every \(x \in X\). Let \(F = A \times B\) be a super lattice measurable set that is \(A \subseteq S\), \(B \subseteq T\). Also \(F = B\) if \(x \in A\) and \(F = \phi\) if \(x \notin A\) if \(x \notin A\) for every \(x \in X\). Therefore \(F \subseteq K\). That is every super lattice measurable set belongs to \(K\). In particular \(X \times Y \subseteq K\).

Let \(E \subseteq K\). Then \(y \in (E')', \) if and only if \((x, y) \in E'\) if and only if \((x, y) \notin E\) if and only if \((x, y) \in E\) if and only if \(y \in (E')'.\) Therefore \((E')' = (E')'.\) Since \(E \subseteq T\) and \(E' \subseteq T\) is a lattice \(\sigma\)-algebra we have \(E' \subseteq T\). Therefore \(E' \subseteq K\).

Let \(E \subseteq K\) (1, 2, 3, ...), and let:

\[
E = \bigwedge_{i=1}^{n} E_i
\]

then \(y \in E\) if and only if \((x, y) \in E\) if and only if \((x, y) \in E_i\) for some \(i\) if and only if \(y \in (E_i)'\). Therefore:

\[
E_i = \bigwedge_{i=1}^{n} (E_i)_i
\]

Since \(T\) is a lattice \(\sigma\)-algebra, \(E = \bigwedge_{i=1}^{n} (E_i)_i \subseteq E \subseteq T\). Therefore \(E \subseteq T\).

From Eq. 2 and 3 \(K\) is a lattice \(\sigma\)-algebra. Since \(K \subseteq S \times T\) we get \(K = S \times T\). Hence for any \(E \subseteq S \times T\), \(E \subseteq T\) for every \(x \in X\) in a similar way we can prove \(E \subseteq E\) for every \(y \in Y\).

Lemma 1: To prove \(S \times T\) is a monotone class.

Proof: Let \(A_i, B_i \subseteq S \times T\), \(A_i \subseteq A_{i+1}\), \(B_i \subseteq B_{i+1}\) for \(i = 1, 2, 3, \ldots\) and:

\[
A = \bigwedge_{i=1}^{n} A_i, B = \bigwedge_{i=1}^{n} B_i
\]

Since \(S \times T\) is a lattice \(\sigma\)-algebra implies \(A \subseteq S \times T\) and:

\[
B' = \bigwedge_{i=1}^{n} B_i' \subseteq S \times T
\]

implies \(B \subseteq S \times T\). Since \(S \times T\) is a lattice \(\sigma\)-algebra. Therefore \(S \times T\) is a monotone class.

Lemma 2: To prove \(L_{2}\) is closed under intersection, difference and unions.

Proof: Let \(A_i \subseteq B_i\), \(A_i \subseteq B_i\) be two super lattice measurable sets. Clearly \((A_i \subseteq B_i) \wedge (A_i \subseteq B_i) = (A_i \subseteq A_i) \wedge (B_i \subseteq B_i)\) we get \((A_i \subseteq A_i) \wedge (B_i \subseteq B_i)\) is super lattice measurable set (Since by definition of \(\delta\)-lattice and every \(\delta\)-lattice is lattice measurable). Also \((A_i \subseteq B_i) \wedge (A_i \subseteq B_i) = (A_i \subseteq A_i) \wedge (B_i \subseteq B_i)\) we get the difference of two super lattice measurable sets is a union of two disjoint super lattice measurable sets. Since \(S \wedge \sigma\)-lattice and every \(\sigma\)-lattice is lattice measurable. (Note that \(A_i \subseteq A_i \subseteq S\) implies \(A_i \wedge A_i \subseteq S\) and \(A_i \wedge A_i \subseteq S\) since \(S\) is a lattice \(\sigma\)-algebra). Also \(B_i \subseteq B_i \subseteq T\). \(B_i \subseteq B_i \subseteq T\) implies \((A_i \subseteq A_i) \wedge (A_i \subseteq B_i)\) are super lattice measurable sets and they are disjoint since \((A_i \subseteq A_i) \wedge (A_i \subseteq A_i) = \phi\). Hence \((A_i \subseteq B_i) \wedge (A_i \subseteq B_i) \subseteq L_{2}\).

Part 1: Closed under intersection. Let \(P, Q \subseteq L_{2}\) implies:

\[
P = R \wedge R \wedge R \wedge \ldots \wedge R, Q = R \wedge R \wedge R \wedge \ldots \wedge R, \text{ where } R \wedge R = \phi
\]

for \(i \neq j\), \(R_i \wedge R_j = \phi\) for \(i \neq j\) and \(R_i \subseteq R_i \subseteq \ldots \subseteq R_i\) are super lattice measurable sets (By results.). Now:

\[
\begin{align*}
    P \wedge Q &= (R \wedge R) \wedge (R \wedge R) \wedge \ldots \wedge (R \wedge R) \\
    &= (R \wedge R) \wedge (R \wedge R) \wedge \ldots \wedge (R \wedge R) \\
    &= (R \wedge R) \wedge (R \wedge R) \wedge \ldots \wedge (R \wedge R)
\end{align*}
\]

Here each \(R_i \wedge R_i\) is super lattice measurable set (By result 3.2) and clearly these are disjoint. Therefore \(P \wedge Q \subseteq L_{2}\).
Part 2: Closed under difference. Now \( P \wedge Q = P \wedge Q^c = (R_1 \wedge R_2 \wedge \ldots \wedge R_n) \wedge (R_1^c \wedge R_2^c \wedge \ldots \wedge R_n^c) = (R_1 \wedge R_2 \wedge \ldots \wedge R_n) \wedge (R_1^c \wedge R_2^c \wedge \ldots \wedge R_n^c) \).

Since the difference of two super lattice measurable sets is the disjoint union of two super lattice measurable set (By result 3), we get right hand side is the disjoint union of super lattice measurable sets. Hence \( P \wedge Q \in L_E \).

Part 3: Closed under union. Now \( P \vee Q = (P \vee Q) \wedge Q \) and \( P \wedge Q \wedge \phi = \phi \), we get \( P \vee Q \wedge \phi \in E \). Therefore \( E \) is closed under intersection, difference and union.

Theorem 2: \( S \times T \) is the smallest monotone class which contains all elementary lattices.

Proof: Let \( \sigma(L) \) be the smallest monotone class which contains \( L_E \). This can be exists, let \( M \) be the family of all monotone class containing \( L_E \). Since \( X \times Y \in M \) implies \( M \in E \). Let \( F \) be the intersection of all members of \( M \) that is \( F = \bigwedge M \). Then \( L_E \subseteq F \). We prove \( F \) is a monotone class. Let \( A, B \in F \), \( A_i \wedge B_i \in E \) for \( i = 1, 2, 3, \ldots \) and

\[
A = \bigwedge_{i=1} A_i, \; B = \bigwedge_{i=1} B_i
\]

Then \( A_i, B_i \) belongs to every member of \( M \) and since every member of \( M \) is a monotone class, \( A, B \in F \). Therefore \( F \) is a monotone class. By lemma 1, \( S \times T \) is a monotone class also it is obvious that \( L_E \subseteq S \times T \) and since \( \sigma(L) \) be the smallest monotone class which contains \( L_E \) we get:

\[
\sigma(L) \subseteq S \times T \quad (1)
\]

Now for any lattice \( P \subseteq X \times Y \) define \( K(P) = \{ Q \subseteq X \times Y \mid Q \in \sigma(L), Q \setminus P \in \sigma(L) \} \). Clearly \( Q \in K(P) \) if \( P \subseteq Q \) or \( P \subseteq Q \). Let \( P \subseteq Q \) or \( P \subseteq Q \). Then \( Q \subseteq K(P) \). Let \( K(P) \) be a lattice. Then \( Q \subseteq K(P) \).

Theorem 3: \( \sigma(L) \) is closed under union and since:

\[
X \times Y \in L_E, \; X \times Y \in \sigma(L)
\]

Let \( Q \in \sigma(L) \) then \( Q' = (X \times Y) \setminus Q \). By Eq. 6, \( L \subseteq \sigma(L) \) for \( i = 1, 2, 3, \ldots \). Let \( P = \bigvee \), and \( Q_i = P \wedge Q \), \( P \subseteq \sigma(L) \). Since \( Q_i \in \sigma(L) \) and \( P = \bigvee Q_i \), the monotonicity of \( \sigma(L) \) shows that \( P \in \sigma(L) \). Thus \( \sigma(L) \) is a lattice algebra. Also \( L_E \subseteq \sigma(L) \). Therefore \( \sigma(L) \) is the smallest lattice algebra in \( X \times Y \) contains every super lattice measurable set. Here \( \sigma(L) \) is a lattice algebra containing \( L_E \) and hence every super lattice measurable set. Therefore \( \sigma(L) = S \times T \).

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